An Equivalence Relation between Morphological Dynamics and Persistent Homology in 1D

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Abstract. We state in this paper a strong relation existing between Mathematical Morphology and Discrete Morse Theory when we work with 1D Morse functions. Specifically, in Mathematical Morphology, a classic way to extract robust markers for segmentation purposes, is to use the dynamics. On the other hand, in Discrete Morse Theory, a well-known tool to simplify the Morse-Smale complexes representing the topological information of a Morse function is the persistence. We show that pairing by persistence is equivalent to pairing by dynamics. Furthermore, self-duality and injectivity of these pairings are proved.

Keywords: mathematical morphology · discrete Morse theory · dynamics · persistence.

1 Introduction

In *Mathematical Morphology* [14,15,16], *dynamics* [10,11,17] represent a very powerful tool to measure the significance of an extrema in a gray-level image. Thanks to dynamics, we can construct efficient markers of objects belonging to an image which do not depend on the size or on the shape of the object we want to segment (to compute watershed transforms [13,18] and proceed to image segmentation). This contrasts with convolution filters very often used in digital signal processing or morphological filters [14,15,16] where geometrical properties do matter.

Selecting components of high dynamics in an image is a way to filter objects depending on their contrast, whatever the scale of the objects. In *persistent homology* [6,8] well-known in *Computational Topology* [7], we can find the same paradigm: topological features whose *persistence* is high are "true" when the ones whose persistence is low are considered as sampling artifacts, whatever their scale. An example of application of persistence is the filtering of *Morse-Smale complexes* used in *Discrete Morse Theory* [9] where pairs of extrema of low persistence are canceled for simplification purpose. This way, we obtain simplified topological representations of *Morse functions*.

In this paper, we prove that the relation between Mathematical Morphology and Persistent Homology is strong in the sense that pairing by dynamics and pairing by persistence are equivalent (and then dynamics and persistence are equal), at least in 1D, when we work with Morse functions.

The plan of the paper is the following: Section 2 recalls the mathematical background needed in this paper, Section 3 proves the equivalence between pairing by dynamics and pairing by persistence, Section 4 proves some properties of these pairings, and Section 5 concludes the paper.

2 Mathematical background

A 1D Morse function is a function $f : \mathbb{R} \to \mathbb{R}$ which belongs to $\mathcal{C}^2(\mathbb{R})$ and whose second derivative $f''(x^*)$ at each critical point $x^* \in \mathbb{R}$ verifies that $f''(x^*)$ is different from 0. A consequence of this property is that the critical points of a Morse function are isolated.

In this paper, we work with one-dimensional Morse functions $f : \mathbb{R} \to \mathbb{R}$ with the additional property that for any two local extrema x_1 and x_2 of f, $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2))$. In other words, critical values of f are "unique".

Even if it does not seem realistic to assume that the critical values are unique, we can easily obtain this property by perturbing slightly the given function while preserving its topology.

Let us define the *lower threshold sets*: the set $[f \leq \lambda]$ for any $\lambda \in \mathbb{R}$ is defined as the set $\{x \in \mathbb{R}\ ;\ f(x) \leq \lambda\}$. Then, we define the *connected component* of a set $X \subseteq \mathbb{R}$ containing $x \in X$ the greatest interval contained in X and containing x and we denote it $\mathcal{CC}(X, x)$.

We denote as usual $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. For a, b two elements of $\overline{\mathbb{R}}$, $\mathrm{iv}(a, b)$ is defined as the *interval value* $[\min(a, b), \max(a, b)]$. Also, for a given function $f : \mathbb{R} \to \mathbb{R}$ and for $(a, b) \in \overline{\mathbb{R}}$ verifying $a < b$, we denote:

$$\mathrm{Rep}([a, b], f) := \arg \min_{x \in [a,b]} f(x).$$

$\mathrm{Rep}([a, b], f)$ is said to be the *representative* [6] of the interval $[a, b]$ relatively to f. Finally, we denote by $\varepsilon \to 0^+$ the fact that ε tends to 0 with the constraint $\varepsilon > 0$.

2.1 Pairing by dynamics

Let $f : \mathbb{R} \to \mathbb{R}$ be a Morse function with unique critical values. For $x_{\min} \in \mathbb{R}$ a local minimum of f, if there exists at least one absciss $x'_{\min} \in \mathbb{R}$ of f such that $f(x'_{\min}) < f(x_{\min})$, then we define the *dynamics* [11] of $x_{\min}$ by:

$$\mathrm{dyn}(x_{\min}) := \min_{\gamma \in C} \max_{s \in [0,1]} f(\gamma(s)) - f(x_{\min}),$$

where C is the set of paths $\gamma : [0, 1] \to \mathbb{R}$ verifying $\gamma(0) := x_{\min}$ and verifying that there exists some $s \in]0, 1]$ such that $f(\gamma(s)) < f(x_{\min})$.

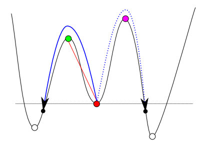

Fig. 1: Example of pairing by dynamics: the absciss $x_{\min}$ of the red point is paired by dynamics relatively to f with the absciss $x_{\max}$ of the green point on its left because the "effort" needed to reach a point of lower height than $f(x_{\min})$ (like the two black points) following the graph of f is minimal on the left.

Let us now define γ^* as a path of C verifying:

$$\max_{s\in[0,1]} f(\gamma^*(s)) - f(x_{\min}) = \min_{\gamma\in C} \max_{s\in[0,1]} f(\gamma(s)) - f(x_{\min}),$$

then we say that this path is *optimal*. The real value $x_{\max}$ *paired by dynamics* to $x_{\min}$ (relatively to f) is characterized by:

$$x_{\max} := \gamma^*(s^*),$$

with $f(\gamma^*(s^*)) = \max_{s\in[0,1]} f(\gamma^*(s))$ and $\gamma^*(s^*)$ is a local maximum of f. We obtain then:

$$f(x_{\max}) - f(x_{\min}) = \mathrm{dyn}(x_{\min}).$$

Note that the local maximum $x_{\max}$ of f does not depend on the path γ^* (see Figure 1), and its value is unique (by hypothesis on f), which shows that in some way $x_{\max}$ and $x_{\min}$ are "naturally" paired by dynamics.

2.2 Pairing by persistence

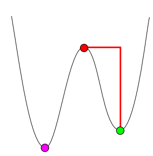

Fig. 2: Example of pairing by persistence: the absciss $x_{\max}$ of the local maximum in red is paired by persistence relatively to f with the absciss of the local minimum in green since its image by f is greater than the image by f on the right local minima drawn in pink.

Let $f : \mathbb{R} \to \mathbb{R}$ be a Morse function with unique critical values, and let $x_{\max}$ be a local maximum of f. Let us recall the 1D procedure [6] which pairs (relatively to f) local maxima to local minima (see Algorithm 1). Roughly speaking, the representatives $x^-_{\min}$ and $x^+_{\min}$ are the abscisses where connected components of

Algorithm 1: Pairing by persistence of $x_{\max}$.

$x_{\min} := \emptyset$;
$[x^-_{\max}, x^+_{\max}] := \mathcal{CC}([f \leq f(x_{\max})], x_{\max})$;
if $x^-_{\max} > -\infty \parallel x^+_{\max} < +\infty$ **then**
  $x^-_{\min} := \mathrm{Rep}([x^-_{\max}, x_{\max}], f)$;
  $x^+_{\min} := \mathrm{Rep}([x_{\max}, x^+_{\max}], f)$;
  if $x^-_{\max} > -\infty$ && $x^+_{\max} < +\infty$ **then**
    $x_{\min} := \arg\max_{x \in \{x^-_{\min}, x^+_{\min}\}} f(x)$;
  if $x^-_{\max} > -\infty$ && $x^+_{\max} = +\infty$ **then**
    $x_{\min} := x^-_{\min}$;
  if $x^-_{\max} = -\infty$ && $x^+_{\max} < +\infty$ **then**
    $x_{\min} := x^+_{\min}$;
return $x_{\min}$;

respectively $[f \leq (f(x^-_{\min})]$ and $[f \leq (f(x^+_{\min})]$ "emerge" (see Figure 2), when $x_{\max}$ is the absciss where two connected components of $[f < f(x_{\max})]$ "merge" into a single component of $[f \leq f(x_{\max})]$. Pairing by persistence associates then $x_{\max}$ to the value $x_{\min}$ belonging to $\{x^-_{\min}, x^+_{\min}\}$ which maximizes $f(x_{\min})$. The *persistence* of $x_{\max}$ relatively to f is then equal to $\mathrm{per}(x_{\max}) := f(x_{\max}) - f(x_{\min})$.

3 Pairings by dynamics and by persistence are equivalent in 1D

In this section, we prove that under some constraints, pairings by dynamics and by persistence are equivalent in the 1D case.

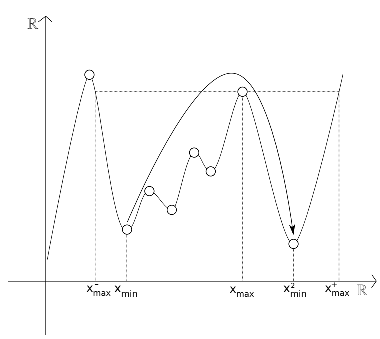

Fig. 3: A Morse function where the local extrema $x_{\min}$ and $x_{\max}$ are paired by dynamics.

Proposition 1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a Morse function with a finite number of local extrema and unique critical values. Now let us assume that a local minimum $x_{\min} \in \mathbb{R}$ of f is paired with a local maximum $x_{\max}$ of f by dynamics. We assume without constraints that $x_{\min} < x_{\max}$. Also, we denote by $(x^-_{\max}, x^+_{\max}) \in \overline{\mathbb{R}}^2$ the two values verifying:*

$$[x^-_{\max}, x^+_{\max}] = \mathcal{CC}([f \leq f(x_{\max})], x_{\max}).$$

Then the following properties are true:

(P1) $x_{\min} = \mathrm{Rep}([x^-_{\max}, x_{\max}], f)$,

(P2) With $x^2_{\min} := \mathrm{Rep}([x_{\max}, x^+_{\max}], f)$, *then* $f(x^2_{\min}) < f(x_{\min})$,

(P3) $x_{\max}$ *and* $x_{\min}$ *are paired by persistence.*

Proof: Figure 3 depicts an example of Morse function where $x_{\min}$ and $x_{\max}$ are paired by dynamics.

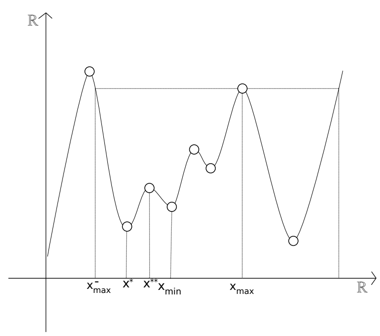

Fig. 4: Proof of $(P1)$.

Let us first remark that $x^-_{\max}$ is finite since $x_{\min}$ is paired with $x_{\max}$ by dynamics relatively to f with $x_{\min} < x_{\max}$.

Now, let us prove $(P1)$; we proceed by *reductio ad absurdum*. When $x_{\min}$ is not the absolute minimum of f on the interval $[x^-_{\max}, x_{\max}]$, then there exists $x^* := \arg\min_{x \in [x^-_{\max}, x_{\max}]} f(x)$ which is different from $x_{\min}$ (see Figure 4) which verifies $f(x^*) < f(x_{\min})$ (x^* and $x_{\min}$ being distinct local extrema of f, their images by f are not equal). Then, because the path joining $x_{\min}$ and x^* belongs to C, we have:

$$\mathrm{dyn}(x_{\min}) \leq \max\{f(x) - f(x_{\min})\ ;\ x \in \mathrm{iv}(x^*, x_{\min})\}.$$

Let us call $x^{**} := \arg\max_{x \in [\mathrm{iv}(x_{\min}, x^*)]} f(x)$, we can deduce that $f(x^{**}) < f(x_{\max})$ since $x^{**} \in \mathrm{iv}(x^*, x_{\min}) \subseteq]x^-_{\max}, x_{\max}[$. This way,

$$\mathrm{dyn}(x_{\min}) \leq f(x^{**}) - f(x_{\min}),$$

which is lower than $f(x_{\max}) - f(x_{\min})$; this is a contradiction since $x_{\min}$ and $x_{\max}$ are paired by dynamics. $(P1)$ is then proven.

Let $x^2_{\min}$ be the representative of $[x_{\max}, x^+_{\max}]$ relatively to f. Two cases are then possible:

- When $x^+_{\max} = +\infty$, it implies that $f(+\infty) = -\infty$ because f is a Morse function, and then $x^2_{\min} = +\infty$, which implies that $f(x^2_{\min}) = -\infty$. Then $f(x^2_{\min}) < f(x_{\min})$.

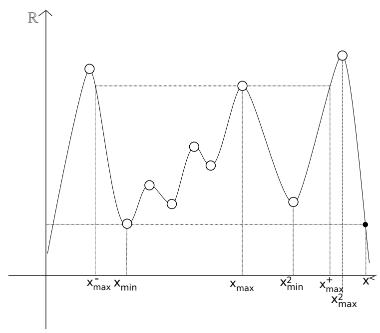

Fig. 5: Proof of $(P2)$ in the case where $x_{\max}$ is finite.

- When $x^+_{\max}$ is finite, let us assume that $f(x^2_{\min}) > f(x_{\min})$. Note that we cannot have equality of $f(x^2_{\min})$ and $f(x_{\min})$ since $x_{\min}$ and $x^2_{\min}$ are both local extrema of f. Then we obtain Figure 5. Since with $x \in [x_{\max}, x^+_{\max}]$, we have $f(x) > f(x_{\min})$, and because $x_{\min}$ is paired with $x_{\max}$ by dynamics with $x_{\min} < x_{\max}$, then there exists a value x on the right of $x_{\min}$ where $f(x)$ is lower than $f(x_{\min})$. In other words, there exists:

$$x^< := \inf\{x \in [x_{\max}, +\infty] \; ; \; f(x) < f(x_{\min})\}$$

such that for any $\varepsilon \to 0^+$, $f(x^< + \varepsilon) < f(x_{\min})$. Since $x^< > x^+_{\max}$, every path γ joining $x_{\min}$ to $x^<$ go through a local maximum $x^2_{\max}$ defined by

$$x^2_{\max} := \arg \max_{x \in]x^+_{\max}, x^<[} f(x)$$

which verifies $f(x^2_{\max}) > f(x^+_{\max})$ (otherwise, $x^2_{\max}$ would belong to the interval $[x_{\max}, x^+_{\max}]$ by definition of $x^+_{\max}$). Then the dynamics of $x_{\min}$ is greater than or equal to $f(x^2_{\max}) - f(x_{\min})$ which is greater than $f(x_{\max}) - f(x_{\min})$. We obtain a contradiction. One more time, $f(x^2_{\min}) < f(x_{\min})$.

The proof of $(P2)$ is done.

Thanks to $(P1)$ and $(P2)$, we obtain directly $(P3)$ by applying the algorithm of pairing by persistence since $f(x_{\min}) > f(x^2_{\min})$ with $x_{\min}$ the representative of $[x^-_{\max}, x_{\max}]$ and $x^2_{\min}$ the representative of $[x_{\max}, x^+_{\max}]$. □

Proposition 2 *Let $f : \mathbb{R} \to \mathbb{R}$ be a Morse function with a finite number of local extrema and unique critical values. Now let us assume that a local minimum $x_{\min} \in \mathbb{R}$ of f is paired with a local maximum $x_{\max}$ of f by persistence. We assume without constraints that $x_{\min} < x_{\max}$. Then, $x_{\max}$ and $x_{\min}$ are paired by dynamics.*

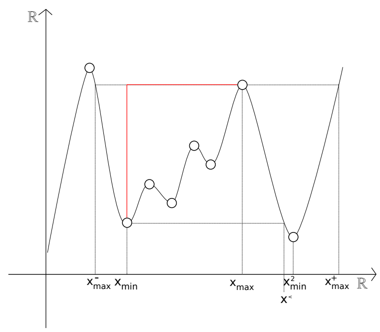

Fig. 6: A Morse function $f : \mathbb{R} \to \mathbb{R}$ where the local extrema $x_{\min}$ and $x_{\max}$ are paired by persistence relatively to f.

Proof: We denote by $(x^-_{\max}, x^+_{\max}) \in \overline{\mathbb{R}}^2$ the two values verifying:

$$[x^-_{\max}, x^+_{\max}] = \mathcal{CC}([f \le f(x_{\max})], x_{\max}).$$

Since $x_{\min}$ is paired by persistence to $x_{\max}$ with $x_{\min} < x_{\max}$ (see Figure 6), then:

$$x_{\min} = \mathrm{Rep}([x^-_{\max}, x_{\max}], f) \in \mathbb{R},$$

and there exists $x^2_{\min} \in \overline{\mathbb{R}}$ such that $x^2_{\min} := \arg\min_{x \in [x_{\max}, x^+_{\max}]} f(x)$ verifies $f(x^2_{\min}) < f(x_{\min})$.

Thanks to this last inequality, we know that the path defined as:

$$\gamma : \lambda \in [0,1] \to \gamma(\lambda) := (1-\lambda)x_{\min} + \lambda x^2_{\min}$$

belongs to the set of paths C defining the dynamics of $x_{\min}$ (see Section 2). Then,

$$\mathrm{dyn}(x_{\min}) \le \max\{f(x) - f(x_{\min})\ ;\ x \in \gamma([0,1])\},$$

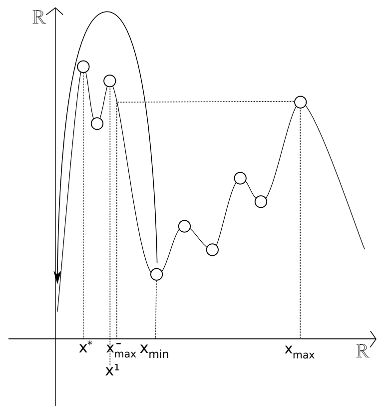

Fig. 7: The proof that it is impossible to obtain a local maximum $x^* < x_{\min}$ paired with $x_{\min}$ by dynamics when $x_{\min}$ is paired with $x_{\max} > x_{\min}$ by persistence.

which is lower than or equal to $f(x_{\max}) - f(x_{\min})$ since f is maximal at $x_{\max}$ on $[x^-_{\max}, x^+_{\max}]$. Then we have the following property:

$$\mathrm{dyn}(x_{\min}) \leq f(x_{\max}) - f(x_{\min}). \quad (P1)$$

Because $f(x^2_{\min}) < f(x_{\min})$, we know that there exists some local maximum of f which is paired with $x_{\min}$ by dynamics. However we do not know whether the absciss of this local maximum is lower than or greater than $x_{\min}$. Then, let us assume that there exists a local maximum $x^* < x_{\min}$ (lower case) which is associated to $x_{\min}$ by dynamics. We denote this property (H) and we depict it in Figure 7. This would imply that $x^* < x^-_{\max}$ since f is greater than or equal to $f(x_{\min})$ on $[x^-_{\max}, x_{\min}]$. The consequence would be $f(x^*) > f(x_{\max})$, since the local maximum x^1 of f of maximal absciss in $[x^*, x^-_{\max}]$ verifies $f(x^*) \geq f(x^1) > f(x_{\max})$, and then $\mathrm{dyn}(x_{\min}) = f(x^*) - f(x_{\min}) > f(x_{\max}) - f(x_{\min})$ which contradicts (P). (H) is then false. In other words, we are in the upper case: the local maximum paired by dynamics to $x_{\min}$ belongs to $]x_{\min}, +\infty[$, let us call this property $(P2)$.

Now let us define (see again Figure 6):

$$x^< := \inf\{x > x_{\min} \ ; \ f(x) < f(x_{\min})\},$$

and let us remark that $x^< > x_{\max}$ (because $x_{\min}$ is the representative of f on $[x^-_{\max}, x_{\max}]$). Since we know by $(P2)$ that a local maximum $x > x_{\min}$ of f is paired by dynamics with $x_{\min}$, then the image of every optimal path belonging to C contains $\{x^<\}$, and then $[x_{\min}, x^<]$. Indeed, an optimal path in C whose image would not contain $\{x^<\}$ would then contain an absciss $x < x^-_{\max}$ and then we would obtain $\mathrm{dyn}(x_{\min}) > f(x_{\max}) - f(x_{\min})$, which contradicts $(P1)$.

However, the maximal value of f on $[x_{\min}, x^{<}]$ is equal to $f(x_{\max})$, then $\mathrm{dyn}(x_{\min}) = f(x_{\max}) - f(x_{\min})$. The only local maximum of f whose value is $f(x_{\max})$ is $x_{\max}$, then $x_{\max}$ is paired with $x_{\min}$ by dynamics relatively to f. □

Theorem 1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a Morse function with a finite number of local extrema and unique critical values. A local minimum $x_{\min} \in \mathbb{R}$ of f is paired by dynamics to a local maximum $x_{\max} \in \mathbb{R}$ of f iff $x_{\max}$ is paired by persistence to $x_{\min}$. In other words, pairings by dynamics and by persistence lead to the same result. Furthermore, we obtain* $\mathrm{per}(x_{\max}) = \mathrm{dyn}(x_{\min})$.

Proof: This theorem results from Propositions 1 and 2. □

4 Properties of these pairings

Let us observe and prove some properties relative to the pairings studied in this paper.

4.1 Self-duality

Let us prove that pairings by dynamics and by persistence are *self-dual* on a 1D Morse function $f : \mathbb{R} \to \mathbb{R}$, that is, the result is the same whatever if we work with f or its *dual* $f^- : \mathbb{R} \to \mathbb{R} : x \to f^-(x) := -f(x)$.

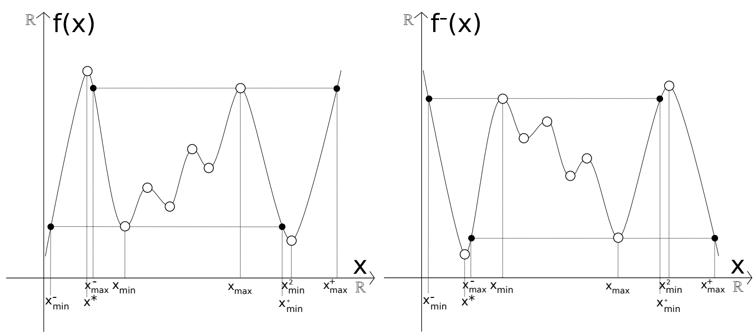

Fig. 8: Proof of self-duality of these pairings.

Proposition 3 *Let $f : \mathbb{R} \to \mathbb{R}$ be a Morse function with a finite number of local extrema and unique critical values. Then the pairing by dynamics (resp. by persistence) of f and of f^- lead to the same result. In other words, these pairings are self-dual.*

Proof: We assume that two finite real values $x_{\min}$ and $x_{\max}$ are paired by persistence relatively to f with $x_{\min} < x_{\max}$. Let us define $(x^-_{\max}, x^+_{\max}) \in \overline{\mathbb{R}}^2$ such that:

$$[x^-_{\max}, x^+_{\max}] = \mathcal{CC}([f \leq f(x_{\max})], x_{\max}),$$

and we also define $(x^-_{\min}, x^+_{\min}) \in \overline{\mathbb{R}}^2$ such that:

$$[x^-_{\min}, x^+_{\min}] = \mathcal{CC}([f^- \leq f^-(x_{\min})], x_{\min}).$$

We can observe by noticing that:

$$x_{\min} = \mathrm{Rep}([x^-_{\max}, x_{\max}], f)$$

(since $x_{\min} < x_{\max}$) and by defining:

$$x^2_{\min} := \mathrm{Rep}([x_{\max}, x^+_{\max}], f)$$

that $f(x_{\min}) > f(x^2_{\min})$ (see Figure 8).

First, let us observe that $x^-_{\max}$ is finite (otherwise, $f(x_{\min}) = -\infty$ which is impossible because $f(x_{\min}) > f(x^2_{\min})$).

Secund, let us prove that $x_{\max}$ is the representative of f^- on $[x_{\min}, x^+_{\min}]$. For any $x \in [x_{\min}, x_{\max}[\cup]x_{\max}, x^2_{\min}]$, the value $f(x)$ is lower than $f(x_{\max})$ because $x_{\min} \in]x^-_{\max}, x_{\max}[$ and $x^2_{\min} \in]x_{\max}, x^+_{\max}[$. Because $f^-(x_{\min}) < f^-(x^2_{\min})$, $x^+_{\min} < x^2_{\min}$ (the case $x^2_{\min} < x^-_{\min}$ is impossible since $x^2_{\min} > x_{\max}$). Also, we have $x^+_{\min} > x_{\max}$ because for any $x \in]x_{\min}, x_{\max}]$, $f(x) > f(x_{\min})$ ($x_{\min}$ is the representative of f on $[x^-_{\max}, x_{\max}]$). Then $x^+_{\min} \in]x_{\max}, x^2_{\min}[$. Then, for any $x \in [x_{\min}, x_{\max}[\cup]x_{\max}, x^+_{\min}]$, we have $f(x) < f(x_{\max})$, and the consequence is that $x_{\max}$ is the representative of f^- on $[x_{\min}, x^+_{\min}]$.

Third, let us define $x^* := \mathrm{Rep}([x^-_{\min}, x_{\min}], f^-)$, and let us prove that $f^-(x^*) < f^-(x_{\max})$. Two cases are possible: either f^- does not admit a local minimum of absciss lower than $x^-_{\max}$ and then $f^-(-\infty) = -\infty$ which implies $x^* = -\infty$ and $f^-(x^*) = -\infty$, or f^- admits a local minimum x lower than $x^-_{\max}$ such that $f^-(x) < f^-(x^-_{\max}) = f^-(x_{\max})$. In both cases, $f^-(x^*) < f^-(x_{\max})$.

Since $x_{\max}$ is the representative of f^- on $[x_{\min}, x^+_{\min}]$, x^* is the representative of f^- on $[x^-_{\min}, x_{\min}]$, and $f^-(x^*) < f^-(x_{\max})$, then $x_{\max}$ is paired with $x_{\min}$ by persistence relatively to f^-.

By Theorem 1, we can conclude that both pairings by persistence and by dynamics are self-dual. □

4.2 Injectivity

Let us prove that the pairings that are studied here are injective.

Proposition 4 *Let $f : \mathbb{R} \to \mathbb{R}$ be a Morse function with a finite number of local extrema and unique critical values. Let $P_{\mathrm{dyn}} : \mathbb{R} \to \mathbb{R}$ the real function which gives for a local minimum $x_{\min}$ of f the local maximum $x_{\max}$ of f paired to $x_{\min}$ by dynamics. Then, P_{dyn} is injective (see Figure 9).*

Proof: Let us assume that $P_{\mathrm{dyn}}(x_{\min}) = P_{\mathrm{dyn}}(x^2_{\min}) = x_{\max}$ with $x_{\min}, x^2_{\min}$ and $x_{\max}$ three real values. Then by Theorem 1, we know that $x_{\max}$ is paired with $x_{\min}$ and $x^2_{\min}$ by persistence, which means that $x_{\min} = x^2_{\min}$. □

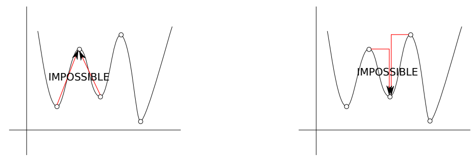

Fig. 9: Pairings by dynamics (on the left side) and by persistence (on the right side) are injective.

Proposition 5 *Let $f : \mathbb{R} \to \mathbb{R}$ be a Morse function with a finite number of local extrema and unique critical values. Let $P_{\text{per}} : \mathbb{R} \to \mathbb{R}$ the real function which gives for a local maximum $x_{\max}$ of f the local minimum $x_{\min}$ of f paired to $x_{\max}$ by persistence. Then, P_{per} is injective (see Figure 9).*

Proof: Let us assume that $P_{\text{per}}(x_{\max}) = P_{\text{per}}(x^2_{\max}) = x_{\min}$ with $x_{\max}, x^2_{\max}$ and $x_{\min}$ three real values. Then by Theorem 1, we know that $x_{\min}$ is paired with $x_{\max}$ and $x^2_{\max}$ by dynamics, which means that $x_{\max} = x^2_{\max}$. □

5 Conclusion

In this paper, we prove the equivalence between pairing by dynamics and pairing by persistence for 1D Morse functions and also their self-duality and their injectivity. As future work, we plan to study their relation in the n-D case, $n \geq 2$. Another interesting issue is to explore how ideas steaming from Discrete Morse Theory can infuse Mathematical Morphology. Conversely, since the watershed is clearly linked to the topology of the surfaces [3,4,12], it is definitely worthwile to search how such ideas can contribute to (Discrete) Morse Theory. This can be done along the same lines as what is proposed in [1,2,5].

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