A Study of Well-composedness in n-D

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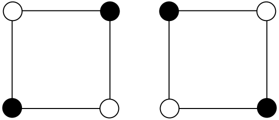
Our quest

Digital topology has topological issues on cubical grids.

These topological issues results from critical configurations:



We are looking for a new representation of signals with **no** topological issues.

Outline

1. Cubical grids in digital topology lead to topological issues
2. Usual solutions to get rid of topological issues on cubical grids
3. How to make a self-dual representation in n-D without topological issues
4. Theoretical Results and Applications
5. Conclusion

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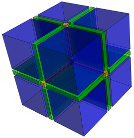
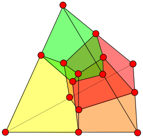
Our choice (1/2)

Simplicial complexes

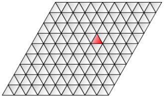
Cubical complexes



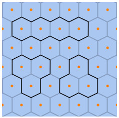
Polyhedral complexes



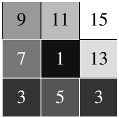
Triangular tilings



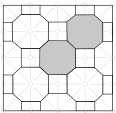
Hexagonal tilings



Cubical tilings/grids



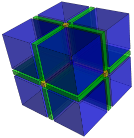
Khalimsky tilings



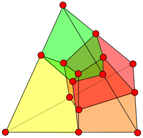
Our choice (1/2)

Simplicial complexes

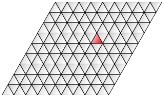
Cubical complexes



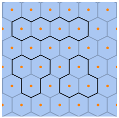
Polyhedral complexes



Triangular tilings



Hexagonal tilings



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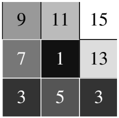

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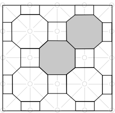


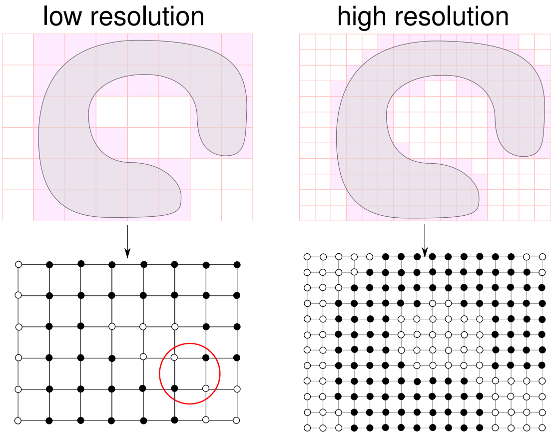
Our choice (2/2)

Cubical signals

- many sensors are cubical
- they are easy to process
- they are easy to store
- ...

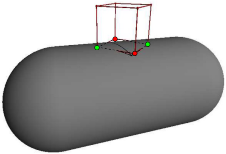
How to get rid of critical configurations in 2D

2D digitization by intersection:

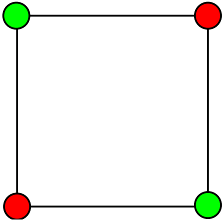


$\rightsquigarrow$ there exists a small enough ρ in 2D.

Any 3D digitization leads to critical configurations



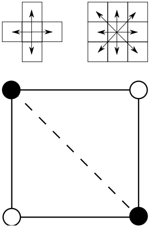
$\Rightarrow$



$\rightsquigarrow$ even regular objects lead to critical configurations in 3D+.

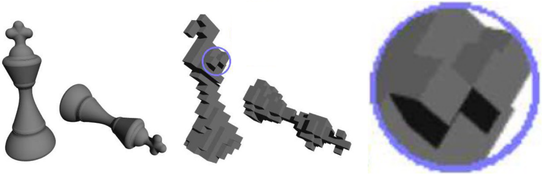
Critical configurations lead to topological issues

discrete topological issues



$\rightsquigarrow$ object counting?

continuous topological issues



manifoldness **not** preserved ("pinch")

Cross-section topology

Threshold sets/binarizations of $u : \mathcal{D} \to \mathbb{Z}$:

$$\forall \lambda \in \mathbb{R},\ [u \geq \lambda] = \{x \in \mathcal{D}\ ;\ u(x) \geq \lambda\},$$
$$\forall \lambda \in \mathbb{R},\ [u < \lambda] = \{x \in \mathcal{D}\ ;\ u(x) < \lambda\}.$$

$\rightsquigarrow$ extension from set operators to graylevel operators ("stacking method").

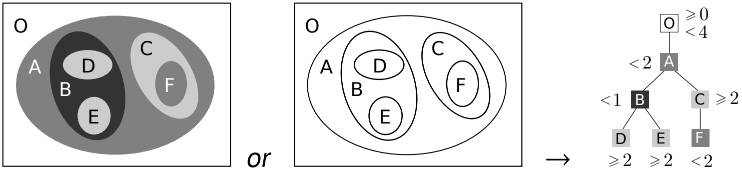
The Tree of Shape of an image

[Monasse & Guichard 2000, Caselles & Monasse 2009]:

- Shapes:

$$\mathcal{US} = \{\mathrm{Sat}(\Gamma)\ ;\ \Gamma \in \mathcal{CC}([u \geq \lambda], \lambda \in \mathbb{R})\},$$
$$\mathcal{LS} = \{\mathrm{Sat}(\Gamma)\ ;\ \Gamma \in \mathcal{CC}([u < \lambda]), \lambda \in \mathbb{R}\},$$

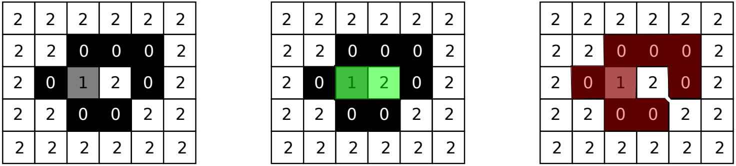
- Shape boundaries = level lines,
- To compute of the tree of shapes (ToS)...



...a necessary condition is: level lines shall be Jordan curves.

Ill-definedness of the ToS on cubical grids

ToS with the same connectivity for lower/upper shapes [Géraud *et al.* 2013]:



We have "intersecting $\not\Leftrightarrow$ nested" $\Rightarrow$ the ToS does **not** exist.

Outline

Solutions to get rid of topological issues

Many solutions exist:

- topological reparations
- interpolations
- mixed methods

Their motivations:

- no “pinches” in the boundary (manifoldness)
- no connectivity ambiguity (determinism)
- both at the same time

Topological reparations in $\mathbb{Z}^n$

Methodology: "remove" critical configurations.

Problem: "propagation" of the critical configurations.

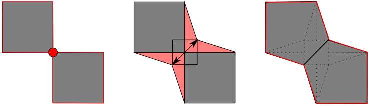
- [Latecki *et al.* 1998/2000] (2D, binary),
  $\rightsquigarrow$ minimal number of modifications (case-by-case study).
- [Siqueira *et al.* 2005/2008] (3D, binary).
  $\rightsquigarrow \frac{3}{2} \times \mathrm{Card}\,(CCs)$ modifications (randomized method).

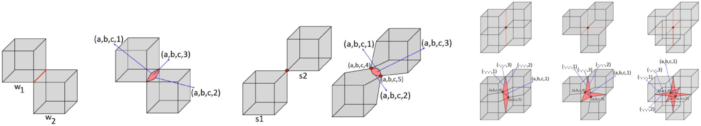
However, modifying the data destroys the topology of the set/ binary image.

Topological reparation of cubical complexes

[Gonzalez-Diaz *et al.* 2011]: the topological reparation of cubical complexes in a homotopy equivalent polyhedral complex.

Application: (co)homology computation and recognition tasks.





However, the new structure is **not** cubical.

Interpolations with no topological issues (1/2)

- [Rosenfeld *et al.* 1998] (2D): image magnification + C.C. elimination (simple deformations)

  Property: topology preserving (adjacency tree).

- [Latecki *et al.* 2000] (2D): resolution doubling + $0 \rightarrow 1$

  Property: sets of black/white/boundary points are WC.

- [Stelldinger & Latecki 2006] (3D): “Majority Interpolation” (“counting process”)

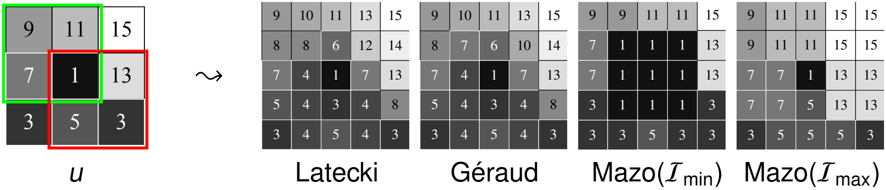
$\rightsquigarrow$ this techniques work on sets, not on graylevel images.

Interpolations with no topological issues (2/2)

[Latecki *et al.* 2000]: 2D, mean/median method (self-dual),

[Géraud *et al.* 2015]: 2D, median method (self-dual),

[Mazo *et al.* 2012]: *n*-D, min-/max-based interpolations (**not** self-dual),



Strong property: they "preserve" the topology of the initial image ("no new extrema").

State-of-the-Art

	2D	3D	n-D	graylevel	self-dual	cubical	topo.-pr.
Latecki *et al.* 98	🟢	🔴	🔴	🔴	🔴	🟢	🔴
Siqueira *et al.* 2005	🟢	🟢	🔴	🔴	🔴	🟢	🔴
Gonzalez-Diàz *et al.* 2011	🟢	🟢	🔴	🔴	🔴	🔴	🟢
Rosenfeld *et al.* 98	🟢	🔴	🔴	🔴	🔴	🟢	🟢
Latecki *et al.* 2000 (1)	🟢	🔴	🔴	🔴	🔴	🟢	🔴
Stelldinger *et al.* 2006	🟢	🟢	🔴	🔴	🔴	🟢	🟢
Latecki *et al.* 2000 (2)	🟢	🔴	🔴	🟢	🟢	🟢	🟢
Géraud *et al.* 2015	🟢	🔴	🔴	🟢	🟢	🟢	🟢
Mazo *et al.* 2012	🟢	🟢	🟢	🟢	🔴	🟢	🟢

Outline

About self-duality

In practice, no contrast is known *a priori*:

- different objects of various contrasts in a same image
- more complex: nested objects

$\rightsquigarrow$ need for a contrast-invariant representation.

Necessary properties of the new representation

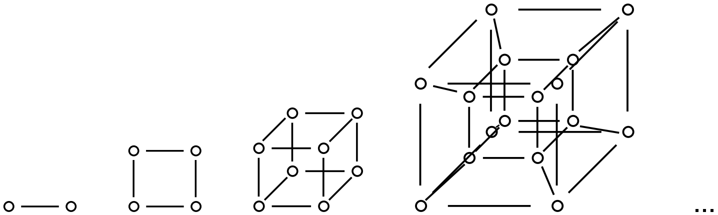
Usual cubical signals present topological issues

$\rightsquigarrow$ a new representation is needed:

- n-dimensionality ($n \geq 2$),
- self-duality,
- no new extrema (in-between),
- no topological issues (no critical configurations).

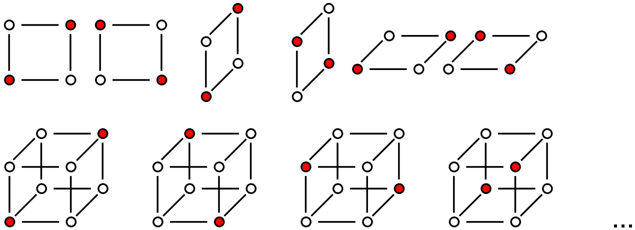
A generalization of DWCness to *n*-D

n-D blocks:



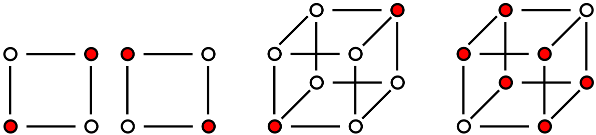
A generalization of DWCness to n-D

Antagonists:



A generalization of DWCness to n-D

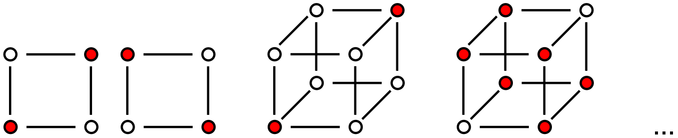
Critical configurations:



...

A generalization of DWCness to n-D

Critical configurations:



Definition ([Boutry *et al.* ISMM 2015])

A digital set $X \subset \mathbb{Z}^n$, $n \geq 2$, is said (digitally) well-composed (DWC) iff it does not contain any critical configuration.

Well-composedness for images

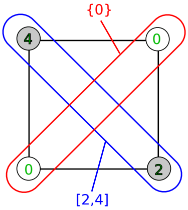
Definition ([Boutry *et al.* ISMM 2015])

A digital image $u : \mathcal{D} \subset \mathbb{Z}^n \to \mathbb{Z}$, $n \geq 2$, is said DWC iff its threshold sets are DWC.

Theorem ([Boutry *et al.* ISMM 2015])

An image $u : \mathcal{D} \to \mathbb{R}$ is DWC iff $\forall p, p' \in \mathcal{D}$ s.t. $p' = \mathrm{antag}_S(p)$:

$$\mathrm{intvl}(u(p), u(p')) \cap \mathrm{Span}\{u(q)\ ;\ q \in S \setminus \{p, p'\}\} \neq \emptyset.$$



$\rightsquigarrow$ no need to check the DWCness of each threshold set.

Computation of a local self-dual DWC n-D interpolation? (1/2)

DWC = local phenomenon (local $2n$-connectivity)

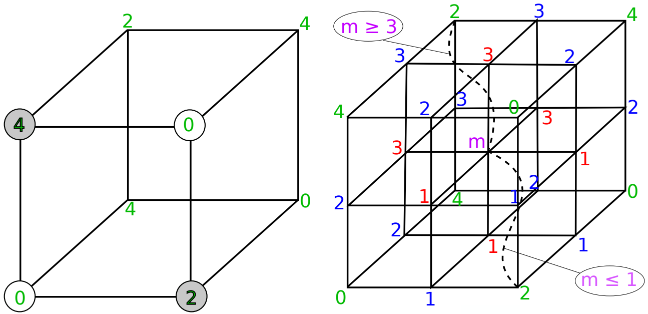
$\rightsquigarrow$ a local interpolation should be adapted ...

$\rightsquigarrow$ usual properties of a local DWC interpolation:

- locality,
- DWC,
- ordered,
- in-between,
- self-duality,
- translation-/$\pi/2$-rotation-invariance.

Computation of a local self-dual DWC *n*-D interpolation? (2/2)

[Boutry *et al.* DGCI 2014]:

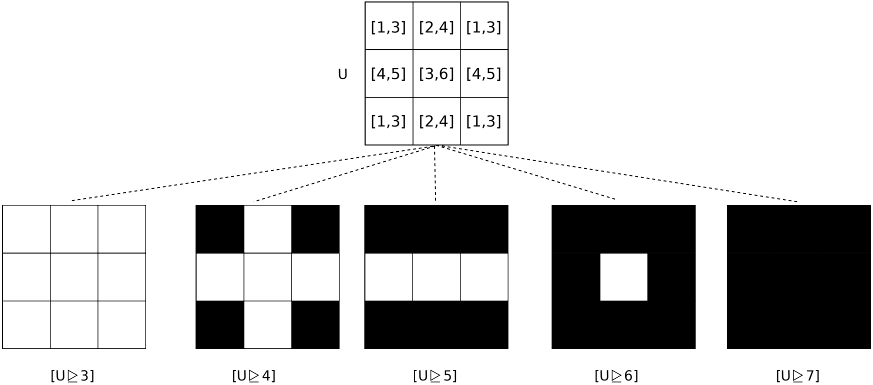


No self-dual local interpolation can make images DWC in *n*-D ($n \geq 3$).

Threshold sets of Interval-valued maps

Let $U : \mathcal{D} \to \mathbb{I}_{\mathbb{R}}$ be an interval-valued map. We define its threshold sets s.t. $\forall \lambda \in \mathbb{R}$:

- $[U \rhd \lambda] \; = \; \{x \in \mathcal{D} \, ; \; \forall v \in U(x), v > \lambda\}$,
- $[U \lhd \lambda] \; = \; \{x \in \mathcal{D} \, ; \; \forall v \in U(x), v < \lambda\}$,
- $[U \unrhd \lambda] \; = \; \mathcal{D} \setminus [U \lhd \lambda]$,
- $[U \unlhd \lambda] \; = \; \mathcal{D} \setminus [U \rhd \lambda]$.



DWC Interval-valued maps

Definition ([Boutry *et al.* 2015])

$U : \mathcal{D} \rightarrow \mathbb{I}_{\mathbb{R}}$ is said DWC iff its threshold sets are DWC.

Proposition ([Boutry *et al.* 2015])

U is DWC iff $\lfloor U \rfloor$ and $\lceil U \rceil$ are both DWC.

Origin of the front-propagation algorithm

[Géraud *et al.* 2013] $\rightsquigarrow$ computation of the tree of shape:

$$u \xrightarrow{\text{immersion}} U \xrightarrow{\text{sort}} (u^\flat, \mathcal{R}) \xrightarrow{\text{union-find}} \mathcal{T}(u^\flat) \xrightarrow{\text{emersion}} \mathcal{T}(u).$$

$\rightsquigarrow$ the sorting step “flattens” U into a temporary image $u^\flat$ (“front-propagation”)

The front-propagation algorithm

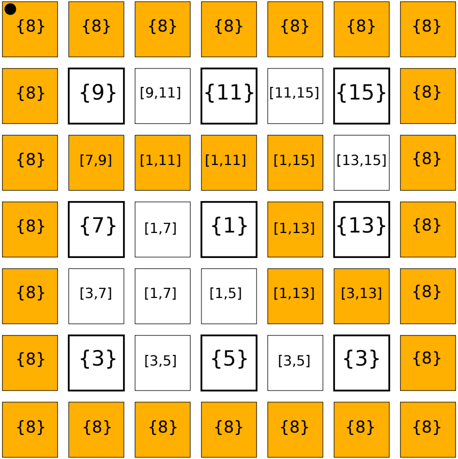
```
Input: U (interval-valued);
Output: u♭ (single-valued);
begin
    forall the h do
        deja_vu(h) ← false;
    PUSH(Q[ℓ∞], p∞);
    deja_vu(p∞) ← true;
    ℓ ← ℓ∞ while Q is not empty do
        h ← PRIORITY_POP(Q, ℓ);
        u♭(h) ← ℓ;
        forall the n ∈ N_2n(h) such as deja_vu(n) = false do
            PRIORITY_PUSH(Q, n, U, ℓ);
            deja_vu(n) ← true;
```

Front-propagation preserves DWCness (1/3)

{8}	{8}	{8}	{8}	{8}	{8}	{8}
{8}	{9}	[9,11]	{11}	[11,15]	{15}	{8}
{8}	[7,9]	[1,11]	[1,11]	[1,15]	[13,15]	{8}
{8}	{7}	[1,7]	{1}	[1,13]	{13}	{8}
{8}	[3,7]	[1,7]	[1,5]	[1,13]	[3,13]	{8}
{8}	{3}	[3,5]	{5}	[3,5]	{3}	{8}
{8}	{8}	{8}	{8}	{8}	{8}	{8}

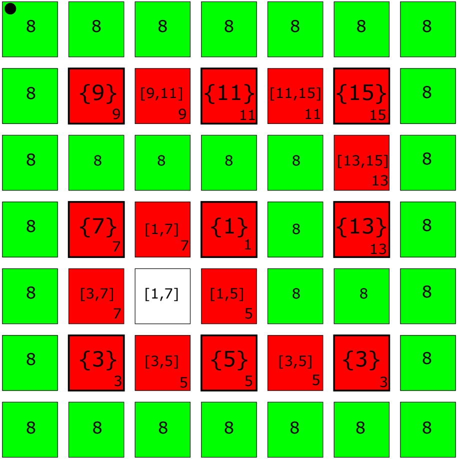
Let us start with a DWC interval-valued map U.

Front-propagation preserves DWCness (1/3)



Propagation of value $\ell = 8$.

Front-propagation preserves DWCness (1/3)



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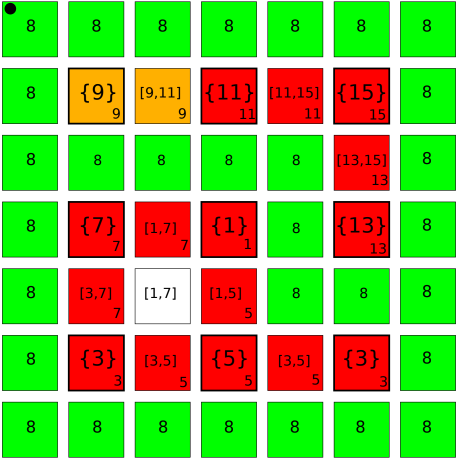
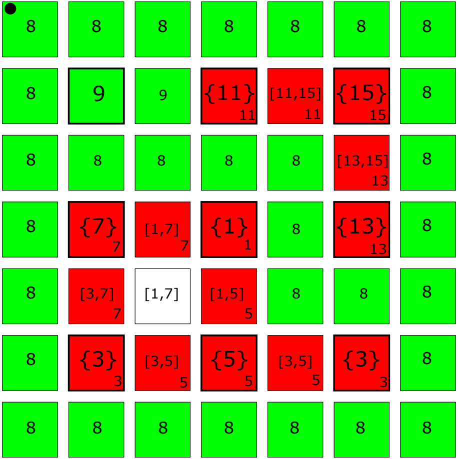


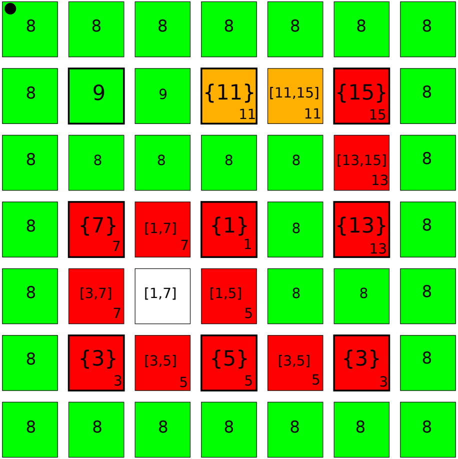
Propagation of value $\ell = 9$.

Front-propagation preserves DWCness (1/3)



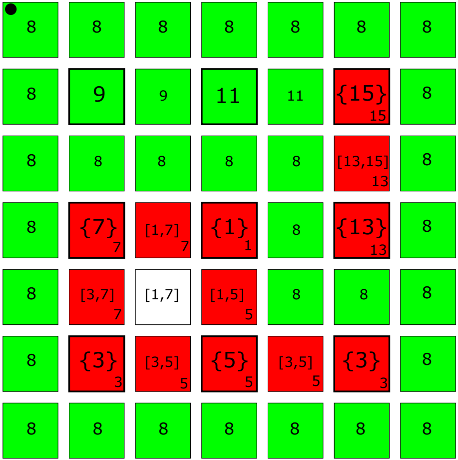
Propagation of value $\ell = 9$.

Front-propagation preserves DWCness (1/3)



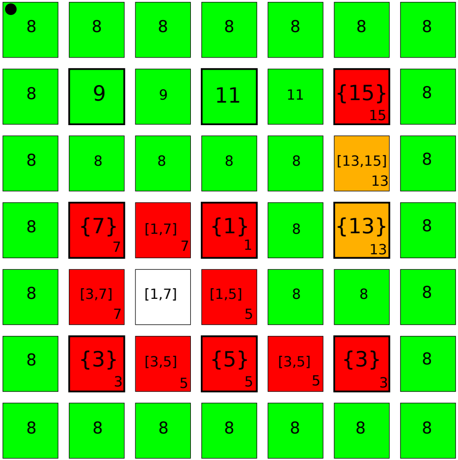
Propagation of value $\ell = 11$.

Front-propagation preserves DWCness (1/3)



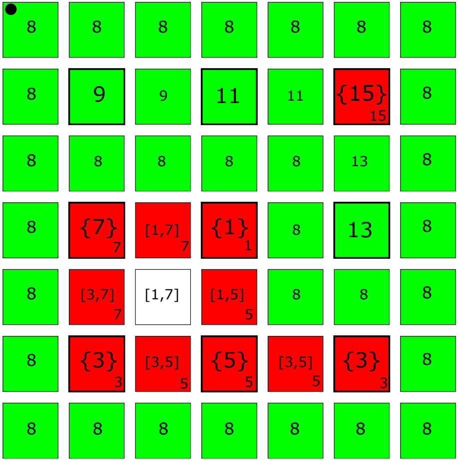
Propagation of value $\ell = 11$.

Front-propagation preserves DWCness (1/3)



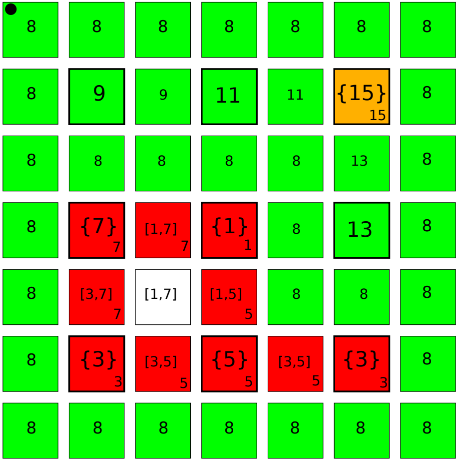
Propagation of value $\ell = 13$.

Front-propagation preserves DWCness (1/3)



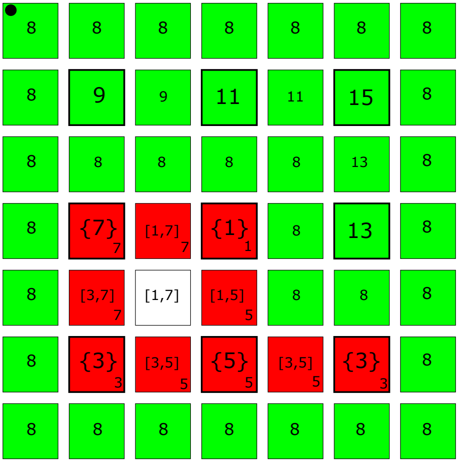
Propagation of value $\ell = 13$.

Front-propagation preserves DWCness (1/3)



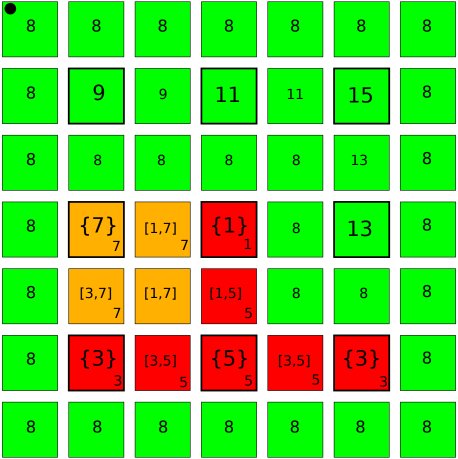
Propagation of value $\ell = 15$.

Front-propagation preserves DWCness (1/3)



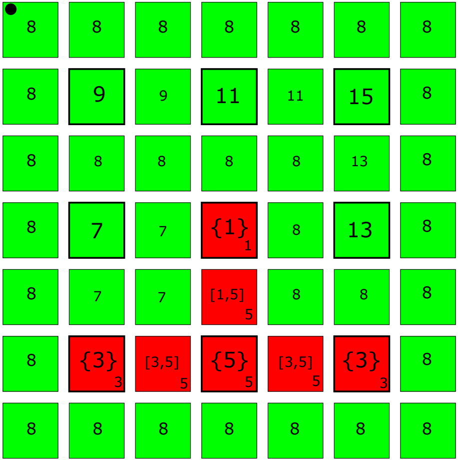
Propagation of value $\ell = 15$.

Front-propagation preserves DWCness (1/3)



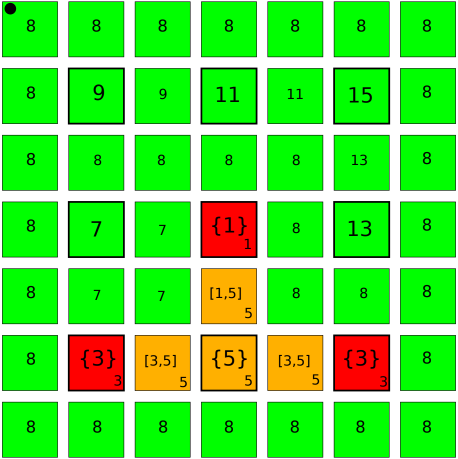
Propagation of value $\ell = 7$.

Front-propagation preserves DWCness (1/3)



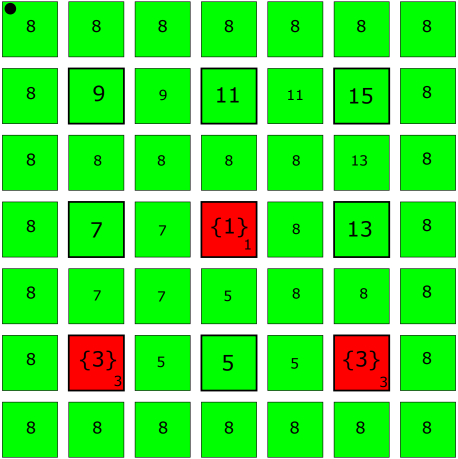
Propagation of value $\ell = 7$.

Front-propagation preserves DWCness (1/3)



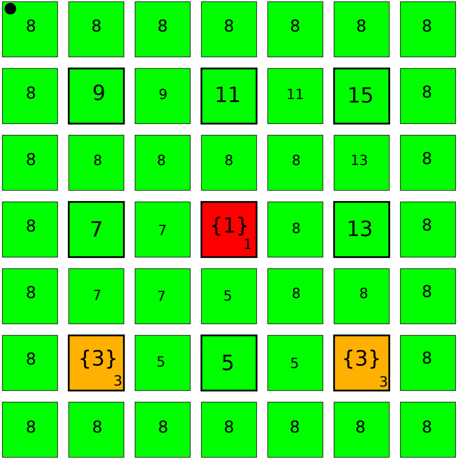
Propagation of value $\ell = 5$.

Front-propagation preserves DWCness (1/3)



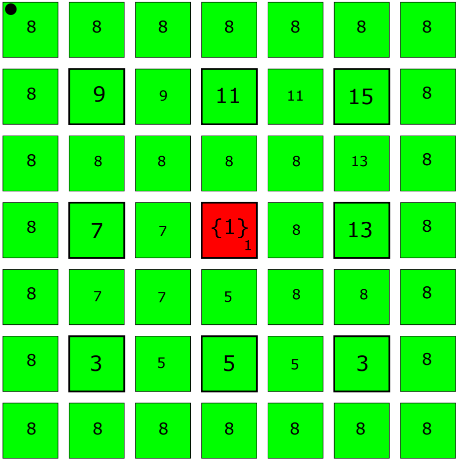
Propagation of value $\ell = 5$.

Front-propagation preserves DWCness (1/3)



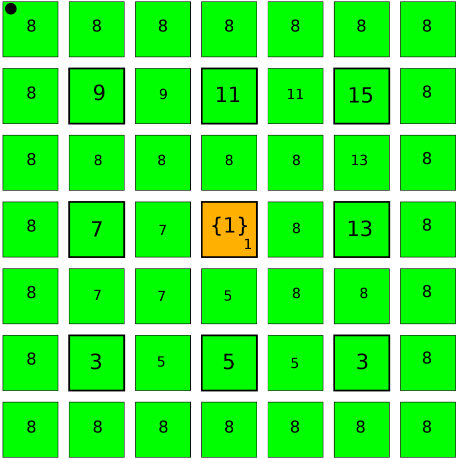
Propagation of value $\ell = 3$.

Front-propagation preserves DWCness (1/3)



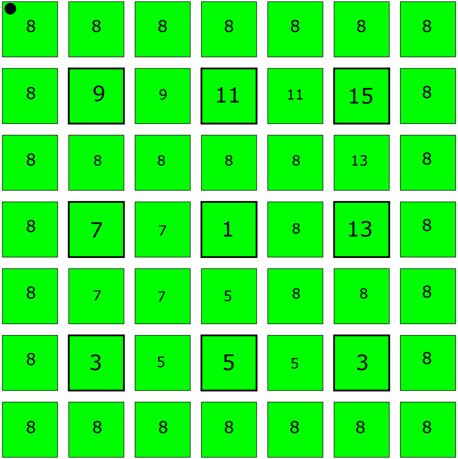
Propagation of value $\ell = 3$.

Front-propagation preserves DWCness (1/3)



Propagation of value $\ell = 1$.

Front-propagation preserves DWCness (1/3)



Result: $u^\flat$ is DWC.

Front-propagation preserves DWCness (2/3)

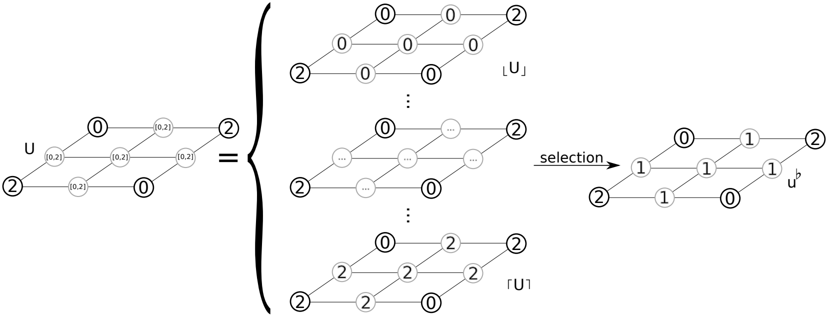
[Boutry *et al.* ISMM 2015]:

$$\forall\ U \text{ DWC},\ u^\flat = \mathfrak{F}\mathfrak{P}(U) \text{ is DWC.}$$

Note: we proved it in *n*-D ($n \geq 2$).

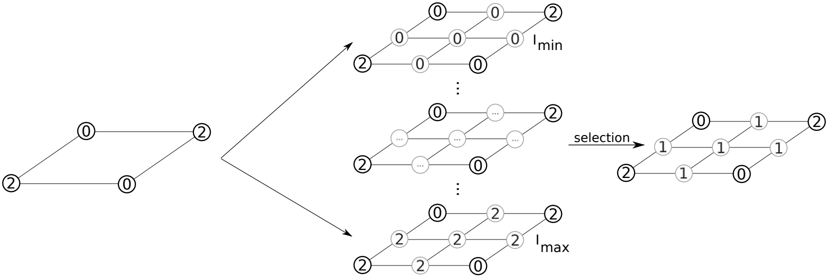
Front-propagation preserves DWCness (3/3)

Intuition: $\mathfrak{FP}$ chooses in a set of images one which is "regular" (DWC).



Our self-dual DWC n-D interpolation

[Boutry *et al.* ISMM 2015]:



$$u \overset{\mathcal{I}_{\text{Span}}}{\longrightarrow} U \overset{\mathfrak{FP}}{\longrightarrow} u^{\flat}.$$

Properties of $u^\flat$ (1/2)

- $u^\flat$ is Alexandrov-well-composed (AWC),

  $\rightsquigarrow$ boundaries in $\mathbb{H}^n$ are discrete surfaces,

- $u^\flat$ is Continuously well-composed (CWC),

  $\rightsquigarrow$ boundaries in $\mathbb{R}^n$ are manifolds,

- $u^\flat$ is well-composed based on the Equivalence of connectivities (EWC),

  $\rightsquigarrow$ same components whatever the connectivity,

- the ToS of $u^\flat$ exists and is connectivity-invariant.

Properties of $u^\flat$ (2/2)

	2D	3D	n-D	graylevel	self-dual	cubical	topo.-pr.
Latecki *et al.* 98	🟢	🔴	🔴	🔴	🔴	🟢	🔴
Siqueira *et al.* 2005	🟢	🟢	🔴	🔴	🔴	🟢	🔴
Gonzalez-Diàz *et al.* 2011	🟢	🟢	🔴	🔴	🔴	🔴	🟢
Rosenfeld *et al.* 98	🟢	🔴	🔴	🔴	🔴	🟢	🟢
Latecki *et al.* 2000 (1)	🟢	🔴	🔴	🔴	🔴	🟢	🔴
Stelldinger *et al.* 2006	🟢	🟢	🔴	🔴	🔴	🟢	🟢
Latecki *et al.* 2000 (2)	🟢	🔴	🔴	🟢	🟢	🟢	🟢
Géraud *et al.* 2015	🟢	🔴	🔴	🟢	🟢	🟢	🟢
Mazo *et al.* 2012	🟢	🟢	🟢	🟢	🔴	🟢	🟢
Boutry *et al.* 2015 ($u^\flat$)	🟢	🟢	🟢	🟢	🟢	🟢	🟢

$\leadsto$ all goals have been reached!

Take-home message

We developed a **new representation** on cubical grids which is:

- self-dual,
- n-D,
- with no topological issues (DWC),
- topology-preserving (in-between interpolation),
- deterministic,
- $\pi/2$-rotation-/translation-invariant,
- in linear time,
- ...

Bonus: many powerful topological properties.

Outline

Theoretical result: “Pure” self-duality

Self-duality equation [Géraud *et al.* 2015]:

$$\mathrm{ToS}_{c_a,c_b}(-u) = \mathrm{ToS}_{c_b,c_a}(u).$$

$\rightsquigarrow$ we had to switch the connectivities.

$u^\flat$ DWC $\Rightarrow$ we obtain “pure” self-duality:

$$\boxed{\mathrm{ToS}(-u^\flat) = \mathrm{ToS}(u^\flat).}$$

Note: **any** self-dual operator become “purely” self-dual on $u^\flat$.

Applications: DWC Laplacian (1/2)

Zero-crossings of the Laplacian ≡ boundaries of objects (image processing).

Remark: a hierarchical representation of the Z.-C.'s could be useful:

- shape recognition,
- text detection,
- ...

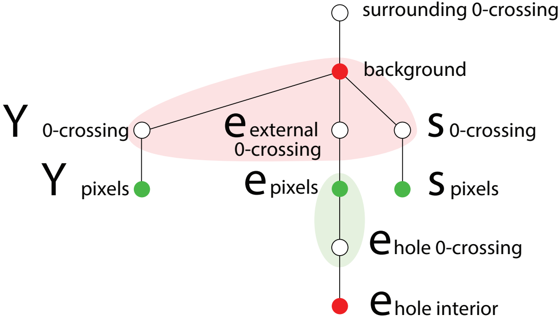
BUT boundaries must be Jordan curves/surfaces:

[Huynh *et al.* 2016] $\rightsquigarrow \text{ToS} \circ \text{Sign} \circ \mathcal{I}_{DWC} \circ \mathcal{L}.$

Applications: DWC Laplacian (2/2)

⇒

⇒



Outline

What we did not speak about (1/2)

The different “flavors” of well-composedness
and their relationship on cubical grids

2D:	**EWC** **[Latecki 1995]**	$\Leftrightarrow$	DWC	$\Leftrightarrow$	AWC	$\Leftrightarrow$	CWC
3D:	EWC	$\Leftarrow$	DWC	$\Leftrightarrow$	AWC	$\Leftrightarrow$	**CWC** **[Latecki 1997]**
*n*D:	**EWC** **[Boutry *et al.* 2015]**	$\Leftarrow$	**DWC** **[Boutry *et al.* 2015]**	$\overset{\text{HAL}}{\Leftrightarrow}$	**AWC** **[Najman *et al.* 2013]**	$\overset{\text{Conj.}}{\Leftrightarrow}$	**CWC** **[Latecki *et al.* 2000]**

What we did not speak about (2/2)

- n-D topological reparation of graylevel images [Boutry *et al.* ICIP 2015],
- n-D reformulation of DWCness for sets ($2n$-connectivity),
- hierarchical subdivision on orders,
- bordered discrete surfaces in polyhedral complexes,
- AWC interpolation(s) on polyhedral complexes.

Our self-dual DWC interpolation is “central”

COMPUTATION
Interpolation (Imm+FP)
Linear time
Easy to program

Our
DWC
self-dual
representation
on cubical grids

DWC-PRESERVING TRANSFORMS
Shapings (grain filters, ...)
Geodesic Dilation/Erosion
Monotone plannings

VISUALIZATION
No crack in the extracted isosurface (M.C.)
No pinch in the bdCA of threshold sets

APPLICATIONS
Tree of shapes well-defined
Text Detection (DWC Laplacian Sign ToS)
Pure self-duality of self-dual MM operators
Faster Euler number computation
No irreducible thickness problem (thinning)
Coherent topological maps (watershed)

TOPOLOGICAL PROPERTIES
No topological paradoxes / no ambiguities
---> Results are connectivity invariant
---> (2n,2n)-connectivity (faster algorithms)
Boundaries are discrete surfaces (AWCness)
--->Separation property
Equivalence of connectivities (EWCness)
---> unicity of the ToS
Boundaries are manifolds (CWCness)
---> Surface Parameterization (coordinate charts)
---> Calculus on Manifolds (Integrals)
Continuity (set-valued sense)
---> intermediate value theorem

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Thank you for your attention

Thierry Géraud, Yongchao Xu, Edwin Carlinet, and Nicolas Boutry.
Introducing the dahu pseudo-distance (submitted).
In *International Symposium on Mathematical Morphology and Its Applications to Signal and Image Processing*, 2017.

Nicolas Boutry, Thierry Géraud, and Laurent Najman.
Digitally well-composed sets and functions on the n-D cubical grid (in preparation).
In *Journal of Mathematical Imaging and Vision*, 2017.

Nicolas Boutry, Laurent Najman, and Thierry Géraud.
About the equivalence between AWCness and DWCness.
Research report, LIGM/LRDE, October 2016.

Nicolas Boutry, Thierry Géraud, and Laurent Najman.
How to make n-D functions digitally well-composed in a self-dual way.
In *International Symposium on Mathematical Morphology and Its Applications to Signal and Image Processing*, pages 561–572. Springer, 2015.

Nicolas Boutry, Thierry Géraud, and Laurent Najman.
How to make n-D images well-composed without interpolation.
In *Image Processing (ICIP), 2015 IEEE International Conference on*, pages 2149–2153. IEEE, 2015.

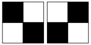
Nicolas Boutry, Thierry Géraud, and Laurent Najman.
Une généralisation du *bien-composé* à la dimension n.
Communication at Journée du Groupe de Travail de Géometrie Discrète (GT GeoDis, Reims Image 2014), November 2014.

Nicolas Boutry, Thierry Géraud, and Laurent Najman.
On making n-D images well-composed by a self-dual local interpolation.
In *International Conference on Discrete Geometry for Computer Imagery*, pages 320–331. Springer, 2014.

Context: rigid transformations

Topological properties should be preserved under rigid tranformations (continuous VS discrete):

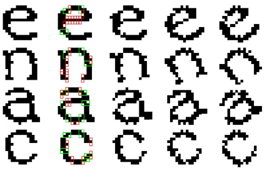
- “well-composedness”? (no ambiguity)



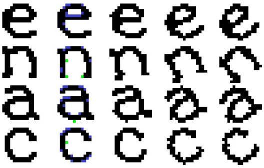
- adjacency tree?

Methodology [Ngo *et al.* 2013]): (simply) forbid some critical patterns.

(a) $\mathscr{C}_1$ (b) $\mathscr{C}_2$ (c) $\mathscr{C}_3$



(a) (b) (c) (d) (e)



(a) (b) (c) (d) (e)

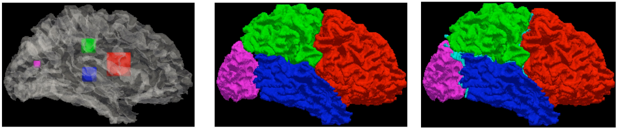
Context: well-composed segmentations

[Tustison *et al.* 2011]: front-propagation method s.t.:

- adds only simple points,
- does not create any C.C. in the expanded seeds.

⇒ topology- and WCness-preserving FP method.

⇒ boundary of the final segmentation is a manifold (glamorous glue by Jordan arcs).



Context: thin topological maps of grayscale images

[Marchadier *et al.* 2004]:
"A discrete image I is the digitization of a piecewise continuous function f."

Methodology:
(1) Computation of the gradient (made WC) of I,
(2) WC thinning ⇒ WC irreducible image,

Note: No ambiguity ⇒ well-defined crest network.

(3) Case-by-case study → coherent topological map (representing f).

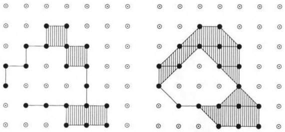
Context: Euler characteristic

- $\xi(\emptyset) = 0$,
- $\xi(S) = 1$ if S non-empty and convex,
- $\xi(S_1 \cup S_2) = \xi(S_1) + \xi(S_2) - \xi(S_1 \cap S_2)$.

$S \subset \mathbb{R}^3$ polyhedral $\Rightarrow \xi = \eta_0 - \eta_1 + \eta_2 - \eta_3$ ($\forall$ triangulation).

$\Rightarrow \xi = b_0 - b_1 + b_2$ (topological invariant)

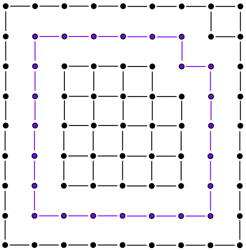
$\rightsquigarrow$ License Plates Recognition tasks, Object Counting, ...



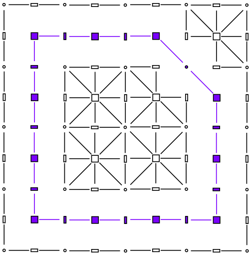
BUT depends on the connectivity:
No critical configuration $\Rightarrow \xi_{(4,8)} = \xi_{(8,4)} \Rightarrow \xi$ well-defined
Bonus of WCness: in 2D, ξ is locally computable (and then faster).

Context: Well-composed Jordan Curves

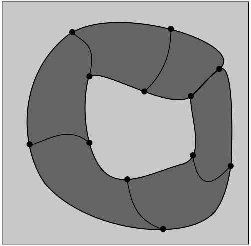
In $\mathbb{Z}^2$



In $\mathbb{H}^2$



[Wang & Bhattacharya 1997]



Jordan curve theorem holds for WC curves.