

A Generic Approach to Quantitative Verification

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Titre: Une approche générique à la vérification quantitative

Mots clés: vérification quantitative, compositionnalité, incrémentalité, robustesse

Résumé: Ce mémoire porte sur la vérification quantitative, c'est-à-dire la vérification des propriétés quantitatives des systèmes quantitatifs. Ces systèmes se retrouvent dans de nombreuses applications, et leur vérification quantitative est importante, mais aussi assez complexe. En particulier, étant donné que la plupart des systèmes trouvés dans les applications sont plutôt larges, il est alors essentiel que les méthodes soient compositionnelles et incrémentielles.

Afin d'assurer la robustesse de la vérification, nous remplaçons les réponses booléennes de la vérification standard par des distances. Selon le contexte de l'application, de nombreux types de distances différentes sont utilisées dans la vérification quantitative. Par conséquent, il est nécessaire d'avoir une théorie générale des distances de systèmes qui puisse s'abstraire des distances concrètes, et de développer une vérification quantitative qui est indépendante de la distance. Nous sommes de l'avis que dans une théorie de la vérification quantitative, les aspects quantitatifs devraient être traités, tout autant que les aspects

qualitatifs, comme des éléments d'entrée d'un problème de vérification.

Dans ce travail, nous développons de la sorte une théorie générale de la vérification quantitative. Nous supposons comme entrée une distance entre traces, ou exécutions, puis utilisons la théorie des jeux à objectifs quantitatifs pour définir des distances entre systèmes quantitatifs. Différentes versions du jeu de bisimulation (quantitatif) donnent lieu à différents types de distances : distance de bisimulation, distance de simulation, distance d'équivalence de trace, etc., permettant de construire une généralisation quantitative du spectre temps linéaire—temps de branchement de van Glabbeek.

Nous étendons notre théorie générale de la vérification quantitative à une théorie des spécifications quantitatives. Pour cela nous utilisons des systèmes de transitions modaux, et nous développons les propriétés quantitatives des opérateurs usuels pour les théories de spécifications. Tout cela est indépendant de la distance concrète entre les traces utilisée.

Title: A Generic Approach to Quantitative Verification

Keywords: quantitative verification, compositionality, incrementality, robustness

Abstract: This thesis is concerned with quantitative verification, that is, the verification of quantitative properties of quantitative systems. These systems are found in numerous applications, and their quantitative verification is important, but also rather challenging. In particular, given that most systems found in applications are rather big, compositionality and incrementality of verification methods are essential.

In order to ensure robustness of verification, we replace the Boolean yes-no answers of standard verification with distances. Depending on the application context, many different types of distances are being employed in quantitative verification. Consequently, there is a need for a general theory of system distances which abstracts away from the concrete distances and develops quantitative verification at a level independent of the distance. It is our view that in a theory of quantitative verification, the quantitative aspects should be treated just as much as input to a verification

problem as the qualitative aspects are.

In this work we develop such a general theory of quantitative verification. We assume as input a distance between traces, or executions, and then employ the theory of games with quantitative objectives to define distances between quantitative systems. Different versions of the quantitative bisimulation game give rise to different types of distances, viz. bisimulation distance, simulation distance, trace equivalence distance, etc., enabling us to construct a quantitative generalization of van Glabbeek's linear-time—branchingtime spectrum.

We also extend our general theory of quantitative verification to a theory of quantitative specifications. For this we use modal transition systems, and we develop the quantitative properties of the usual operators for behavioral specification theories. All this is independent of the concrete distance between traces which is utilized.

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1 Introduction

This thesis is concerned with quantitative verification, that is, the verification of quantitative properties of quantitative systems. These systems are found in numerous applications, and their quantitative verification is important, but also rather challenging. In particular, given that most systems found in applications are rather big, compositionality and incrementality are essential. That is, quantitative verification should be applied as much as possible to subsystems and at as high a level as possible, and then verified partial specifications should be composed and refined into an implementation.

Much work has been done in the area of compositional and incremental design, but robust quantitative frameworks are lacking. This thesis presents work published between 2009 and 2020 by the author and various co-authors which attempts to introduce such a framework. Much remains to be done, in particular in applications to real-time and hybrid systems, but we believe that the foundations laid out here will be useful in this endeavor.

1.1 Motivation

1.1.1 Quantitative Verification

Motivated by applications in real-time systems, hybrid systems, embedded systems, and other areas, formal verification has seen a trend towards modeling and analyzing systems which contain quantitative information. Quantitative information can thus be a variety of things: probabilities, time, tank pressure, energy intake, etc.

A number of quantitative models have been developed: probabilistic automata [SL94]; stochastic process algebras [Hil96]; timed automata [AD94]; hybrid automata [ACH+95]; timed variants of Petri nets [MF76, Han93]; continuous-time Markov chains [Ste94]; etc. Similarly, there is a number of specification formalisms for expressing quantitative properties: timed computation tree logic [HNSY94]; probabilistic computation tree logic [HJ94]; metric temporal logic [Koy90]; stochastic continuous logic [ASSB00]; etc. Quantitative model checking, the verification of quantitative properties for quantitative systems, has also seen rapid development: for probabilistic systems in PRISM [KNP02] and PEPA [GH94]; for real-time systems in Uppaal [LPY97],

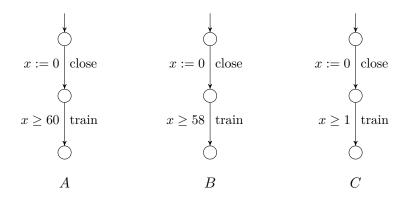


Figure 1.1: Three timed automata modeling a train crossing.

RED [WME93], TAPAAL [BJS09] and Romeo [GLMR05]; and for hybrid systems in HyTech [HHWT97], SpaceEx [FGD⁺11] and HySAT [FH07], to name but a few.

Quantitative model checking has, however, a problem of *robustness*. When the answers to model checking problems are Boolean—either a system meets its specification or it does not—then small perturbations in the system's parameters may invalidate the result. This means that, from a model checking point of view, small, perhaps unimportant, deviations in quantities are indistinguishable from larger ones which may be critical.

As an example, Figure 1.1 shows three simple timed-automaton models of a train crossing, each modeling that once the gates are closed, some time will pass before the train arrives. Now assume that the specification of the system is

The gates have to be closed 60 seconds before the train arrives.

Model A does guarantee this property, hence satisfies the specification. Model B only guarantees that the gates are closed 58 seconds before the train arrives, and in model C, only one second may pass between the gates closing and the train.

Neither model B or C satisfy the specification, so this is the result which a model checker like for example Uppaal would output. What this does not tell us, however, is that model C is dangerously far away from the specification, whereas model B only violates it slightly and may be acceptable given other engineering constraints, or may be more easily amenable to satisfy the specification than model C.

In order to address the robustness problem, our approach is to replace the Boolean yes-no answers of standard verification with *distances*. That is, the Boolean co-domain of model checking is replaced by the non-negative real numbers. In this setting, the Boolean true corresponds to a distance of zero, and false corresponds to any non-zero number, so that quantitative model checking can now tell us not only that a specification is violated, but also how much it is violated, or how far the system is from corresponding to its specification.

In the example of Figure 1.1 and for a simple definition of system distances, the distance from A to our specification would be 0, whereas the distances from B and C to the specification would be 2 and 59, respectively. The precise interpretation of distance values will be application-dependent; but in any case, it is clear that C is much farther away from the specification than B is.

1.1.2 Specification Theories

One of the major current challenges to rigorous design of software systems is that these systems are becoming increasingly complex and difficult to reason about [Sif11]. As an example, an integrated communication system in a modern airplane can have more than 10^{900} distinct states [BBB+10], and state-of-the-art tools offer no possibility to reason about, and model check, the system as a whole. One promising approach to overcome such problems is the one of compositional and incremental design. Here the reasoning is done as much as possible at higher specification levels rather than with implementations; partial specifications are proven correct and then composed and refined until one arrives at an implementation model. Practical experience indicates that this is a viable approach [Str, SPE].

Specifications of system requirements are high-level finite abstractions of possibly infinite sets of implementations. A model of a system is considered an implementation of a given specification if the behavior defined by the implementation is implied by the description provided by the specification.

Any practical specification formalism comes equipped with a number of operations which permit compositional and incremental reasoning. The first of these is a *refinement* relation which allows to successively distill specifications into more detailed ones and eventually into implementations. In an implementation, all optional behavior defined in the specification has been decided upon in compliance with the specification. Also needed is an operation of *logical conjunction* which allows to combine specifications so that the systems which refine the conjunction of two specifications are precisely the ones which satisfy both of them. Refinement and conjunction together permit incremental reasoning as specifications are successively refined and conjoined.

For compositional reasoning, one needs another operation of *structural composition* which allows to infer specifications from sub-specifications of independent requirements, mimicking at the implementation level for example the interaction of components in a distributed system. A partial inverse of this operation is given by a *quotient* operation which allows to synthesize a specification of missing components from an overall specification and an implementation which realizes a part of that specification.

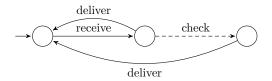


Figure 1.2: Modal transition system modeling a simple email system, with an optional behavior: Once an email is received it may e.g., be scanned for containing viruses, or automatically decrypted, before it is delivered to the receiver.

Over the years, there have been a series of advances on specification theories [dAH05, CdAHM02, DLL+10, Del10, LT89, Nym08, Thr11]. The predominant approaches are based on modal logics and process algebras but have the drawback that they cannot naturally embed both logical and structural composition within the same formalism [Lar89]. Hence such formalisms do not permit to reason incrementally through refinement.

In order to leverage these problems, the concept of modal transition systems was introduced [Lar89]. In short, modal transition systems are labeled transition systems equipped with two types of transitions: must transitions which are mandatory for any implementation, and may transitions which are optional for implementations. It is well established that modal transition systems match all the requirements of a reasonable specification theory, and much progress has been made in this area, see for example [Nym08, GLS08, GHJ01, GLLS05] or [AHL⁺08] for an overview. Also, practical experience shows that the formalism is expressive enough to handle complex industrial problems [Str, SPE].

As an example, consider the modal transition system shown in Figure 1.2 which models the requirements of a simple email system in which emails are first received and then delivered. Before delivering the email, the system may check or process the email, for example for en- or decryption, filtering of spam emails, or generating automatic answers using an auto-reply feature (see also [Hal00]). Must transitions, representing obligatory behavior, are drawn as solid arrows, whereas may transitions, modeling optional behavior, are shown as dashed arrows: hence any implementation of this email system specification must be able to receive and deliver email, and it may also be able to check arriving email before delivering it. No other behavior is allowed.

Implementations can also be represented within the modal transition system formalism, simply as specifications without may transitions. Here, any implementation choice has been resolved, so that implementations are (isomorphic to) plain labeled transition systems. Formally, for a labeled transition system to be an implementation of a given specification, we require that the states of the two objects are related by a refinement relation with the property that all behavior required by the specification has been implemented, and

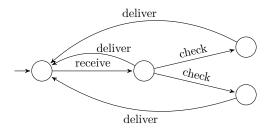


Figure 1.3: An implementation of the simple email system in Figure 1.2 in which we explicitly model two distinct types of email pre-processing.

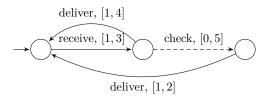


Figure 1.4: Specification of a simple email system, with integer intervals modeling time constraints for performing the corresponding actions.

that any implementation behavior is permitted in the specification. Figure 1.3 shows an implementation of our email specification with two different checks, leading to distinct processing states.

1.1.3 Quantitative Specification Theories

In recent work [JLS12, BJL⁺12a, BJL⁺12b, BKL⁺12], modal transition systems have been extended by adding richer information to the usual discrete label set of transition systems, permitting to reason about *quantitative* aspects of models and specifications. These quantitative labels can be used to model and analyze for example timing behavior [HMP05, DLL⁺10], resource usage [RLS06, BJL⁺12b], or energy consumption [BFLM11, FJLS11].

In particular, [JLS12] extends modal transition systems with integer intervals and introduces corresponding extensions of the above operations which observe the added quantitative information, and [BJL⁺12a] generalizes this theory to general *structured labels*. Both theories are, however, *fragile* in the sense that they rely on Boolean notions of satisfaction and refinement: as refinement either holds or does not, they are unable to *quantify* the impact of small variations in quantities.

An example of a quantitative specification is shown in Figure 1.4. The intuition is that any concrete implementation must be able to receive and deliver email, within one to three and one to four time units, respectively; but it also may be able to check incoming email, e.g., for viruses, before delivering

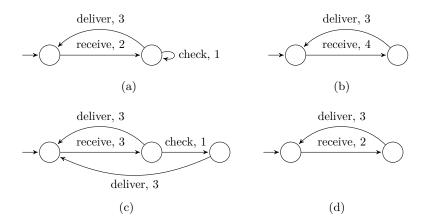


Figure 1.5: Four implementations of the simple email system in Figure 1.4.

it. No other behavior is permitted.

Figure 1.5 shows four different implementation candidates for the specification of Figure 1.4. The first candidate, in Figure 1.5(a), has an error in the discrete structure: after receiving an email, it may check the email indefinitely. Hence it does not satisfy the specification. The second candidate, in Figure 1.5(b), is also problematic: not implementing the checking part of the specification is entirely permissible, but it takes too long to receive email. Thus, if the timing constraints are abstracted away, it is a perfectly good implementation; but the quantitative timing constraints are off. The implementation candidate in Figure 1.5(c) has similar problems, as it takes too long to deliver emails after checking them. The transition system in Figure 1.5(d) is, finally, a true implementation of the specification.

An important observation is, now, that even though the systems in Figs. 1.5(b) and 1.5(c) strictly are not implementations of the email system specification, they conform much better to it than the system in Figure 1.5(a). Intuitively, they "almost" comply with the specification; given some other engineering constraints, they might indeed be considered "good enough" compared to the specification. It is, then, this "almost" and "good enough" which we shall attempt to formalize in this work.

Our point of view is, more generally speaking, that any quantitative specification formalism falls short with a Boolean notion of satisfaction and refinement. If the specification formalism is intended to model quantitative properties, then it is of little use to know that a proposed implementation does not precisely adhere to a specification; much more useful information is obtained by knowing how well it implements the specification, or how far it is deviating. Of course, the answer to this "how far" question might be ∞ , due to discrete errors as in Figure 1.5(a); but in case it is finite, useful knowledge may be gained, for example as to how much more implementation

effort is needed, or whether one can satisfy oneself with this slightly imperfect implementation. Our approach will hence again be to replace satisfaction and refinement relations by satisfaction and refinement distances.

1.1.4 Related Work

The distance-based approach to quantitative verification has been developed the furthest for probabilistic and stochastic systems. Panangaden and Desharnais *et.al.* have worked with distances for Markov processes in [DGJP04, FPP05,DGJP99,DLT08,DJGP02,Pan09,LMP12,BBLM13] and other papers, and van Breugel and Worrell *et.al.* have developed distances for probabilistic transition systems in [vBW05,vBW01,vBW06]. De Alfaro and Stoelinga *et.al.* have worked on distances between probabilistic systems and specifications in [dAFH+05,dAHM03,dAMRS07,CdAF+06,CdAMR10,dAMRS08,dAFS04] and other papers.

For real-time and hybrid systems, some explicit work on distances is available in [HMP05, CHP11, QFD11]. Otherwise, distances have been used in approaches to robust verification [LLTW11,BLM+11], and Girard *et.al.* have developed a theory of approximate bisimulation for robust control [ZG09,GP07].

Also general work on distances for quantitative systems where the precise meaning of the quantities remains unspecified has been done. Van Breugel has developed a general theory of behavioral pseudometrics [vB01, BvBR98, vB96, vB05]. Henzinger *et.al.* have employed distances in a software engineering context in [ČHR12, ČHR10] and for abstraction refinement and synthesis in [ČHR13, ČH11, ČCH+11, ČCHR14, CdAF+06].

Common to all the above distance-based approaches is that they introduce distances between systems, or between systems and specifications, and then employ these for approximate or quantitative verification. However, depending on the application context, a plethora of different distances are being used, motivating the need for a general theory. This is a point of view which is also argued in $[\check{C}HR13, CdAF^+06]$.

To be more specific, most of the above approaches can be classified according to the way they measure distances between *executions*, or system traces. The perhaps easiest such way is the *point-wise* distance, which measures the greatest individual distance between corresponding points in the traces. Theory for this specific distance has been developed in [dAFS09, dAFH+05, dAFS04, BLM+11] and other papers. Sometimes *discounting* is applied to diminish the influence of individual distances far in the future, for example in [dAFS09, dAFH+05, dAFS04].

Another distance which has been used is the *accumulating* one, which sums individual distances along executions. Two major types have been considered here: the *discounted* accumulating distance *e.g.*, in [ČHR10, dAHM03, AT11,dAFH+05] and the *limit-average* accumulating distance *e.g.*, in [ČHR10,

AT11]. Both are well-known from the theory of discounted and mean-payoff games [EM79, ZP96].

For real-time systems, a useful distance is the *maximum-lead* distance of [HMP05] which measures the maximum difference between accumulated time delays along traces. For hybrid systems, things are more complicated, as distances between hybrid traces have to take into account both spatial and timing differences, see for example [QFD11, Gir10, ZG09, GP07].

1.1.5 A General Theory of Quantitative Verification

Depending on the application context, many different types of distances are being employed in quantitative verification. Consequently, there is a need for a general theory of system distances which abstracts away from the concrete distances and develops quantitative verification at a level independent of the distance. It is our view that in a theory of quantitative verification, the quantitative aspects should be treated just as much as input to a verification problem as the qualitative aspects are.

In this work we develop such a general theory of quantitative verification. We assume as input a distance between traces, or executions, and then employ the theory of games with quantitative objectives to define distances between quantitative systems. Different versions of the (quantitative) bisimulation game give rise to different types of distances, viz. bisimulation distance, simulation distance, trace equivalence distance, etc., enabling us to construct a quantitative generalization of the linear-time-branching-time spectrum.

We also extend our general theory of quantitative verification to a theory of quantitative specifications. For this we use modal transition systems, and we develop the quantitative properties of the usual operators for behavioral specification theories. All this is independent of the concrete distance between traces which is utilized.

1.2 Contributions

In the following chapters we present work based on eight papers, published between 2009 and 2020 by the author of this thesis with different co-authors, on quantitative verification and quantitative specification theories. The first three, Chapters 2 to 4, are each concerned with properties of three specific system distances: the point-wise distance, the discounted accumulating distance, and the maximum-lead distance. The next Chapter 5 develops our general theory of quantitative verification and shows basic properties. Chapters 6 and 7 then extend this theory to specification theories, first for the discounted accumulating distance in Chapter 6 and then for the general setting in Chapter 7. In Chapter 8 we take a break from the quantitative setting in order to introduce an extension of modal transition systems and show that the so-obtained specification theory is closely related to other popular specification

formalisms. The final Chapter 9 extends these results to general quantitative and develops their properties.

Compared to their sources, all chapters have been heavily redacted in order to correct errors, unify notation, and smoothen the presentation. Any remaining errors are the sole responsibility of the author of this thesis.

1.2.1 Geometric Preliminaries

Before we can give an overview of our contributions, we recall a few standard notions from geometry and topology which we will use throughout. Let $\mathbb{R}_{\geq 0} \cup \{\infty\}$ denote the extended non-negative reals.

A hemimetric on a set X is a function $d: X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ which satisfies d(x,x) = 0 and $d(x,y) + d(y,z) \geq d(x,z)$ (the triangle inequality) for all $x,y,z \in X$. The hemimetric is said to be symmetric if also d(x,y) = d(y,x) for all $x,y \in X$; it is said to be separating if d(x,y) = 0 implies x = y.

A symmetric hemimetric is generally called a *pseudometric*, and a hemimetric which is both symmetric and separating is simply a *metric*. The tuple (X, d) is called a (hemi/pseudo)metric *space*.

Note that our hemimetrics are extended in that they can take the value ∞ . This is convenient for several reasons, cf. [Law73], one of them being that it allows for a disjoint union, or coproduct, of hemimetric spaces: the disjoint union of (X_1, d_1) and (X_2, d_2) is the hemimetric space $(X_1, d_1) \uplus (X_1, d_2) = (X_1 \uplus X_2, d)$ where points from different components are infinitely far away from each other, i.e., with d defined by

$$d(x,y) = \begin{cases} d_1(x,y) & \text{if } x,y \in X_1, \\ d_2(x,y) & \text{if } x,y \in X_2, \\ \infty & \text{otherwise.} \end{cases}$$

The product of two hemimetric spaces (X_1, d_1) and (X_2, d_2) is the hemimetric space $(X_1, d_1) \times (X_2, d_2) = (X_1 \times X_2, d)$ with d given by $d((x_1, x_2), (y_1, y_2)) = \max(d_1(x_1, y_1), d_2(x_2, y_2))$.

The symmetrization of a hemimetric d on X is the symmetric hemimetric $\bar{d}: X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ defined by $\bar{d}(x,y) = \max(d(x,y),d(y,x))$; this is the smallest among all pseudometrics d' on X for which $d \leq d'$. The topology generated by a hemimetric d on X is defined to be the same as the one generated by its symmetrization \bar{d} ; it has as open sets all unions of open balls $B(x;r) = \{y \in X \mid \bar{d}(x,y) < r\}$, for $x \in X$ and r > 0.

A continuous function $f: X \to X$ on a pseudometric space (X, d) is called a *contraction* if there exists $0 \le \alpha < 1$ (its *Lipschitz constant*) such that $d(f(x), f(y)) \le \alpha d(x, y)$ for all $x, y \in X$.

Two pseudometrics d_1 , d_2 on X are said to be

- topologically equivalent provided that for all $x \in X$ and all $\varepsilon \in \mathbb{R}_+$, there exists $\delta \in \mathbb{R}_+$ such that $d_1(x,y) < \delta$ implies $d_2(x,y) < \varepsilon$ and $d_2(x,y) < \delta$ implies $d_1(x,y) < \varepsilon$ for all $y \in X$,
- Lipschitz equivalent if there exist $m, M \in \mathbb{R}_+$ such that $md_1(x, y) \leq d_2(x, y) \leq Md_1(x, y)$ for all $x, y \in X$.

Hemimetrics are topologically or Lipschitz equivalent if their symmetrizations are.

Topological equivalence is the same as asking the identity function id: $(X, d_1) \to (X, d_2)$ to be a homeomorphism, and Lipschitz equivalence implies topological equivalence.

Topological equivalence of d_1 and d_2 is also the same as requiring the topologies generated by d_1 and d_2 to coincide. Topological equivalence hence preserves topological notions such as convergence of sequences: If a sequence (x_j) of points in X converges in one pseudometric, then it also converges in the other. As a consequence, topological equivalence of hemimetrics d_1 and d_2 implies that for all $x, y \in X$, $d_1(x, y) = 0$ if, and only if, $d_2(x, y) = 0$.

Topological equivalence is the weakest of the common notions of equivalence for metrics; it does not preserve geometric properties such as distances or angles. We are hence mainly interested in topological equivalence as a tool for showing *negative* properties; we will later prove a number of results on topological *inequivalence* of hemimetrics which imply that any other reasonable metric equivalence, such as Lipschitz equivalence, also fails for these cases.

The Hausdorff hemimetric associated with a hemimetric $d: X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is the function $d^{\mathrm{H}}: 2^X \times 2^X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ given for subsets $A, B \subseteq X$ by

$$d^{\mathrm{H}}(A,B) = \sup_{x \in A} \inf_{y \in B} d(x,y).$$

This is a well-known construction for metric spaces, cf. [Mun00, AB07]; there it is usually symmetrized and defined only for closed subsets, in which case it is a metric. The following alternative formulation follows straight from the definition:

1.1 Proposition. For a hemimetric d on X, $A, B \subseteq X$, and $\varepsilon \in \mathbb{R}_+$, we have $d(A, B) \leq \varepsilon$ if and only if for any $x \in A$ there exists $y \in B$ for which $d(x, y) \leq \varepsilon$.

A sequence (x_j) in a metric space X is a Cauchy sequence if it holds that for all $\varepsilon > 0$ there exist $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for all $n, m \ge N$. X is said to be complete if every Cauchy sequence in X converges in X.

Finally, we recall the Banach fixed-point theorem: Any contraction on a complete metric space has precisely one fixed point.

1.2.2 Chapter 2, "Quantitative Analysis of Weighted Transition Systems"

In Chapter 2 we introduce the point-wise, accumulating and maximum-lead trace distances, all in a discounted version which allows to diminish the influence of future differences. In a notation which is simpler than the one used in Chapter 2 and follows the one of later chapters, these are given as follows. Let \mathbb{K} be a set of symbols together with an extended metric $d_{\mathbb{K}} : \mathbb{K} \times \mathbb{K} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$. A trace $\sigma = (\sigma_0, \sigma_1, \dots)$ is an infinite sequence of symbols in \mathbb{K} . Let $\lambda \in \mathbb{R}_{\geq 0}$ with $\lambda \leq 1$ be a discounting factor and σ and τ traces.

• The point-wise trace distance between σ and τ is

$$d_{\bullet}^{\mathsf{T}}(\sigma,\tau) = \sup_{i \geq 0} \lambda^{i} d_{\mathbb{K}}(\sigma_{i},\tau_{i}).$$

• The accumulating trace distance between σ and τ is

$$d_+^{\mathsf{T}}(\sigma,\tau) = \sum_{i\geq 0} \lambda^i d_{\mathbb{K}}(\sigma_i,\tau_i).$$

• The maximum-lead trace distance between σ and τ is

$$d_{\pm}^{\mathsf{T}}(\sigma,\tau) = \sup_{i \ge 0} \lambda^i \Big| \sum_{0 \le j \le i} (\sigma_i - \tau_i) \Big|.$$

Note that the definition of the last distance requires extra structure of addition and subtraction on \mathbb{K} ; generally this is only used for $\mathbb{K} = \mathbb{Z}$ or $\mathbb{K} = \mathbb{R}$.

We then use these trace distances to define point-wise, accumulating and maximum-lead *linear* distances between states in weighted transition systems. If d^{T} is any of the above trace distances, then the linear distance between two states s and t of a transition system is defined to be

$$d^{\mathsf{L}}(s,t) = \sup_{\sigma \in \mathsf{Tr}(s)} \inf_{\tau \in \mathsf{Tr}(t)} d^{\mathsf{T}}(\sigma,\tau),$$

where $\mathsf{Tr}(s)$ denotes the set of traces emanating from s, similarly for $\mathsf{Tr}(t)$. This is thus the Hausdorff distance from $\mathsf{Tr}(s)$ to $\mathsf{Tr}(t)$; note that the definition is independent of which particular trace distance is used.

1.2 Example. We show a computation of the different distances between the states s_1 and t_1 in the transition system in Figure 1.6. Edges without specified weight have weight 0, and the discounting factor is $\lambda = .90$.

It is easy to see that supremum trace distance is obtained for the path from s_1 which always turns left at s_2 , *i.e.*, takes the transition $s_2 \xrightarrow{11} s_4$, and then for the point-wise and accumulating trace distances, that the matching

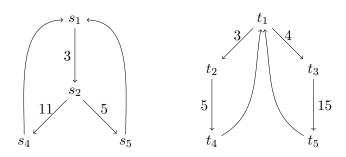


Figure 1.6: Two example transition systems

trace from t_1 giving infimum trace distance in turn is obtained for the path which always takes the transition $t_1 \stackrel{4}{\rightarrow} t_3$. Hence we can compute

$$d_{\bullet}^{\mathsf{L}}(s_1, t_1) = \sup_{i} \left\{ \max(1, 4 \cdot .90) \cdot .90^{3i} \right\} = 3.60,$$

$$d_{+}^{\mathsf{L}}(s_1, t_1) = \sum_{i} (1 + 4 \cdot .90) \cdot .90^{3i} \approx 17.0.$$

For maximum-lead trace distance the situation is more involved. It can be shown that for this distance, an infimum trace τ from t_1 follows the path which takes $t_1 \stackrel{4}{\to} t_3$, followed by $t_1 \stackrel{3}{\to} t_2$ three times, and then repeats $t_1 \stackrel{4}{\to} t_3$ indefinitely. Using this trace, we obtain

$$d_+^{\mathsf{L}}(s_1, t_1) = 13 \cdot .90^{10} \approx 4.53$$
.

The definition of branching distance is *not* independent of which trace distance is being used. We only give the definitions for the first two of our example trace distances. They are defined as least fixed points to the following equations:

$$d_{\bullet}^{\mathsf{B}}(s,t) = \sup_{s \xrightarrow{x} s'} \inf_{t \xrightarrow{y} t'} \max \left(d_{\mathbb{K}}(x,y), \lambda d_{\bullet}^{\mathsf{B}}(s',t') \right)$$
$$d_{+}^{\mathsf{B}}(s,t) = \sup_{s \xrightarrow{x} s'} \inf_{t \xrightarrow{y} t'} d_{\mathbb{K}}(x,y) + \lambda d_{\bullet}^{\mathsf{B}}(s',t')$$

1.3 Example. Continuing the previous example, repeated application of the definition yields the following fixed-point equation for $d^{\mathsf{B}}_{\bullet}(s_1, t_1)$ (note that there

is only one transition from s_1 , t_2 and t_3 , respectively):

$$d_{\bullet}^{\mathsf{B}}(s_{1}, t_{1}) = \inf \begin{cases} \max \left(|3 - 3|, .90 \, d_{\bullet}^{\mathsf{B}}(s_{2}, t_{2}) \right) \\ \max \left(|3 - 4|, .90 \, d_{\bullet}^{\mathsf{B}}(s_{2}, t_{3}) \right) \end{cases}$$

$$= \inf \begin{cases} \max \left(0, .90 \, |11 - 5|, .90^{2} d_{\bullet}^{\mathsf{B}}(s_{4}, t_{4}), .90 \, |5 - 5|, .90^{2} d_{\bullet}^{\mathsf{B}}(s_{5}, t_{4}) \right) \\ \max \left(1, .90 \, |11 - 15|, .90^{2} d_{\bullet}^{\mathsf{B}}(s_{4}, t_{5}), .90 \, |5 - 15|, .90^{2} d_{\bullet}^{\mathsf{B}}(s_{5}, t_{5}) \right) \end{cases}$$

$$= \inf \begin{cases} \max \left(5.4, .90^{3} d_{\bullet}^{\mathsf{B}}(s_{1}, t_{1}) \right) \\ \max \left(9, .90^{3} d_{\bullet}^{\mathsf{B}}(s_{1}, t_{1}) \right) \end{cases}$$

which has least fixed point $d_{\bullet}^{\mathsf{B}}(s_1, t_1) = 5.4$. For the accumulating distance, we calculate:

$$d_{+}^{\mathsf{B}}(s_{1}, t_{1}) = \inf \begin{cases} |3 - 3| + .90 \, d_{+}^{\mathsf{B}}(s_{2}, t_{2}) \\ |3 - 4| + .90 \, d_{+}^{\mathsf{B}}(s_{2}, t_{3}) \end{cases}$$

$$= \inf \begin{cases} .90 \sup \begin{cases} |11 - 5| + .90 \, d_{+}^{\mathsf{B}}(s_{4}, t_{4}) \\ |5 - 5| + .90 \, d_{+}^{\mathsf{B}}(s_{5}, t_{4}) \end{cases}$$

$$1 + .90 \sup \begin{cases} |11 - 15| + .90 \, d_{+}^{\mathsf{B}}(s_{4}, t_{5}) \\ |5 - 15| + .90 \, d_{+}^{\mathsf{B}}(s_{5}, t_{5}) \end{cases}$$

$$= \inf \begin{cases} .90 \sup \begin{cases} 6 + .90^{2} d_{+}^{\mathsf{B}}(s_{1}, t_{1}) \\ .90^{2} d_{+}^{\mathsf{B}}(s_{1}, t_{1}) \end{cases}$$

$$1 + .90 \sup \begin{cases} 4 + .90^{2} d_{+}^{\mathsf{B}}(s_{1}, t_{1}) \\ 10 + .90^{2} d_{+}^{\mathsf{B}}(s_{1}, t_{1}) \end{cases}$$

$$= \inf (.90(6 + .90^{2} d_{+}^{\mathsf{B}}(s_{1}, t_{1})), 1 + .90(10 + .90^{2} d_{+}^{\mathsf{B}}(s_{1}, t_{1})))$$

$$= 5.4 + .90^{3} d_{+}^{\mathsf{B}}(s_{1}, t_{1})$$

Hence $d_{+}^{\mathsf{B}}(s_{1}, t_{1}) \approx 19.9.$

Linear distances generalize *trace inclusion* for transition systems, whereas branching distances generalize *simulation*. We show that the linear distance between two states is always bounded above by the corresponding branching distance (Theorem 2.13), a generalization of the fact that simulation implies trace inclusion.

We also show that the point-wise linear and the point-wise branching distances are *topologically inequivalent*, that is, one may be zero while the other is infinite. This is a quantitative generalization of the fact that trace inclusion and simulation are not equivalent. We show the same topological inequivalence for the accumulating and maximum-lead distances (Theorem 2.14).

When discounting is applied, then the point-wise, accumulating and maximum-lead linear distances are *Lipschitz equivalent* (Theorem 2.17); similarly,

the three branching distances are Lipschitz equivalent (Theorem 2.18). Without discounting, the distances are topologically inequivalent. Lipschitz equivalence means that one distance is bounded by the other, multiplied by a scaling factor; hence properties of one distance may be transferred to the other.

Chapter 2 is based on work by the author's PhD student Claus Thrane, Kim G. Larsen, and the author, which has been presented at the 20th Nordic Workshop on Programming Theory (NWPT) [TFL08] and subsequently published in the Journal of Logic and Algebraic Programming (now the Journal of Logical and Algebraic Methods in Programming) in 2010 [TFL10].

1.2.3 Chapter 3, "A Quantitative Characterization of Weighted Kripke Structures in Temporal Logic"

In Chapter 3 we consider the discounted point-wise and accumulating distances and introduce corresponding semantics for weighted CTL. In these semantics, the evaluation of a formula in a state is not a Boolean true or false, but instead a non-negative real number (or infinity) which, intuitively, characterizes how well the state satisfies the formula. Our syntax for WCTL extends the one of CTL [CE81] as follows:

$$\Phi ::= p \mid \neg p \mid \Phi_1 \land \Phi_2 \mid \Phi_1 \lor \Phi_2 \mid \mathsf{E}\Psi \mid \mathsf{A}\Psi$$

$$\Psi ::= \mathsf{X}_c \Phi \mid \mathsf{G}_c \Phi \mid \mathsf{F}_c \Phi \mid [\Phi_1 \mathsf{U}_c \Phi_2]$$

Here, as usual, Φ generates state formulae whereas Ψ generates path formulae, p is an atomic proposition, and $c \in \mathbb{R}_{\geq 0}$ is any non-negative real number.

Semantically, formulae are interpreted in states of a Kripke structure with labels in \mathbb{K} , and the result of such an interpretation is a non-negative real number. First, the semantics of state formulae is given as follows:

In the last two formulae, $\llbracket \psi \rrbracket(\sigma)$ is the semantics of the trace σ with respect to ψ , which depends on whether the point-wise or the accumulating distance

is used. For example, the point-wise path semantics is given as follows:

Here $\sigma^k = (\sigma_k, \sigma_{k+1}, \dots)$ denotes the k-shift of the trace $\sigma = (\sigma_0, \sigma_1, \dots)$.

We then show in Theorems 3.11 and 3.13 that with these semantics, WCTL is *adequate* for the corresponding bisimulation distances. That is, the bisimulation distance between two states is precisely the supremum, over all WCTL formulae, of the absolute value of the difference of the formula's evaluation in these two states.

We also show, in Theorems 3.17 and 3.18, that with the corresponding semantics, WCTL is *expressive* for the discounted point-wise and accumulating distances. This means that given a state in a Kripke structure, there exists a WCTL formula which characterizes the state in the sense that the bisimulation distance to any other state is precisely the evaluation of the formula in that state.

Our notions of adequacy and expressiveness are standard quantitative generalizations of Hennessy and Milner's definitions from [HM85].

Chapter 3 is based on work by the author's PhD student Claus Thrane, Kim G. Larsen, and the author, which has been presented at the 5th Conference on Mathematical and Engineering Methods in Computer Science (MEMICS; best paper award) [FLT09] and subsequently published in the Journal of Computing and Informatics [FLT10].

1.2.4 Chapter 4, "Metrics for Weighted Transition Systems: Axiomatization"

In Chapter 4 we develop sound and complete axiomatizations of the point-wise and the discounted accumulating distances for finite and for regular weighted processes. In this context, a finite weighted process is given using the grammar

$$E ::= \mathbf{0} \mid n.E \mid E + E \mid \qquad n \in \mathbb{K}$$
,

where \mathbb{K} is a finite set of weights, with a metric $d_{\mathbb{K}}$, and $\mathbf{0}$ denotes the empty process. A regular weighted process is given using the grammar

$$E ::= \mathbf{U} \mid X \mid n.E \mid E + E \mid \mu X.E \qquad n \in \mathbb{K}, X \in V,$$

where V is a set of variables, U is the universal process, and $\mu X.E$ denotes a minimal fixed point.

$$(A1) \frac{}{\vdash [\mathbf{0}, E] \bowtie r} \quad 0 \bowtie r \qquad (A2) \frac{}{\vdash [n.E, \mathbf{0}] \bowtie r} \quad \infty \bowtie r$$

$$(R1^{\bullet}) \frac{\vdash [E, F] \bowtie r_1}{\vdash [n.E, m.F] \bowtie r} \quad \max(d_{\mathbb{K}}(n, m), \lambda r_1) \bowtie r$$

$$(R1^{+}) \frac{\vdash [E, F] \bowtie r_1}{\vdash [n.E, m.F] \bowtie r} \quad d_{\mathbb{K}}(n, m) + \lambda r_1 \bowtie r$$

$$(R2) \frac{\vdash [E_1, F] \bowtie r_1 \quad \vdash [E_2, F] \bowtie r_2}{\vdash [E_1 + E_2, F] \bowtie r} \quad \max(r_1, r_2) \bowtie r$$

$$(R3) \frac{\vdash [n.E, F_1] \bowtie r_1 \quad \vdash [n.E, F_2] \bowtie r_2}{\vdash [n.E, F_1 + F_2] \bowtie r} \quad \min(r_1, r_2) \bowtie r$$

Figure 1.7: Axiomatization of point-wise and discounted accumulating distance for finite weighted processes

We then give axiomatizations of the point-wise and the discounted accumulating distances for finite weighted processes, as shown in Figure 1.7. These differ only in one proof rule: for the point-wise distance, rule $(R1^{\bullet})$ applies, for the discounted accumulating distance, rule $(R1^{+})$. We show the axiomatizations to be sound and complete in Theorems 4.8, 4.9 and 4.10.

For regular weighted processes, we develop similar axiomatizations, which we then show to be sound and ε -complete in Theorems 4.12, 4.16 and 4.17. Here, ε -completeness means that a distance of d can be proven within an interval $[d - \varepsilon, d + \varepsilon]$, for any positive real ε .

Chapter 4 is based on work by the author's PhD student Claus Thrane, Kim G. Larsen, and the author, which has been published in Theoretical Computer Science [LFT11].

1.2.5 Chapter 5, "The Quantitative Linear-Time-Branching-Time Spectrum"

Chapter 5 presents a generalization of the work in Chapter 2 along several directions. Instead of developing theory separately for different trace distances, we treat the trace distance as an input and develop a general theory of linear and branching distances pertaining to a given, but unspecified, trace distance.

Let again \mathbb{K} be a set of symbols, and denote by $\mathbb{K}^{\infty} = \mathbb{K}^* \cup \mathbb{K}^{\omega}$ the set of finite and infinite traces in \mathbb{K} . A *trace distance* is, then, a function $d: \mathbb{K}^{\infty} \times \mathbb{K}^{\infty} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ which satisfies $d(\sigma, \sigma) = 0$ and $d(\sigma, \tau) + d(\tau, \chi) \geq d(\sigma, \chi)$ for all $\sigma, \tau, \chi \in \mathbb{K}^{\infty}$, and additionally, $d(\sigma, \tau) = \infty$ if σ and τ have different length.

Given such a general trace distance d, we can define the linear distance between two states s and t of a transition system by

$$d^{1\text{-trace}}(s,t) = \sup_{\sigma \in \mathsf{Tr}(s)} \inf_{\tau \in \mathsf{Tr}(t)} d(\sigma,\tau)$$

as before. As this generalizes the standard trace inclusion preorder, we now call this the (1-nested) trace inclusion distance.

Using a quantitative Ehrenfeucht-Fraïssé game, we can then define a corresponding (1-nested) simulation distance $d^{1-\text{sim}}$ and show that $d^{1-\text{trace}}(s,t) \leq d^{1-\text{sim}}(s,t)$ for all states s, t. Similarly, we can define the (1-nested) trace equivalence distance between s and t by

$$d^{\text{1-trace-eq}}(s,t) = \max \Big(\sup_{\sigma \in \mathsf{Tr}(s)} \inf_{\tau \in \mathsf{Tr}(t)} d(\sigma,\tau), \sup_{\tau \in \mathsf{Tr}(t)} \inf_{\sigma \in \mathsf{Tr}(s)} d(\sigma,\tau) \Big)$$

and use a different quantitative Ehrenfeucht-Fraïssé game to define the bisimulation distance d^{bisim} , with the property that $d^{1-\text{trace-eq}}(s,t) \leq d^{\text{bisim}}(s,t)$ for all s,t.

In Chapter 5, we generalize these considerations to define linear and branching distances for most of the preorders and equivalences in van Glabbeek's linear-time-branching-time spectrum [vG01]. Hence we can define nested simulation distances, ready simulation distances, possible-futures distances, readiness distances, and others, all parameterized by the given-but-unspecified trace distance. The resulting quantitative linear-time-branching-time spectrum is depicted in Figure 1.8.

We also show that if the trace distance has a recursive characterization in a lattice L above $\mathbb{R}_{\geq 0} \cup \{\infty\}$, then all distances in the quantitative linear-time-branching-time spectrum have a fixed-point characterization over L. As an example, if $d: \mathbb{K}^{\infty} \times \mathbb{K}^{\infty} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is the point-wise distance, then

$$d(\sigma, \tau) = F(\sigma_0, \tau_0, d(\sigma^1, \tau^1))$$

for all $\sigma, \tau \in \mathbb{K}^{\infty}$ (recall that σ_0 denotes the head of σ and σ^1 its tail), where $F : \mathbb{K} \times \mathbb{K} \times (\mathbb{R}_{\geq 0} \cup \{\infty\}) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is given by $F(x, y, \alpha) = \max(d_{\mathbb{K}}(x, y), \lambda \alpha)$. (In this case, the lattice $L = \mathbb{R}_{\geq 0} \cup \{\infty\}$.)

The simulation distance is then the least fixed point to the equations

$$d^{1-\text{sim}}(s,t) = \sup_{s \xrightarrow{x} s'} \inf_{t \xrightarrow{y} t'} F(x,y,d^{1-\text{sim}}(s',t')).$$

If, instead, d is the discounted accumulating distance, then the above equations hold for F replaced by $F(x, y, \alpha) = d_{\mathbb{K}}(x, y) + \lambda \alpha$.

Chapter 5 is based on work by the author's PhD student Claus Thrane, Kim G. Larsen, Axel Legay, and the author, which has been presented at the 9th Workshop on Quantitative Aspects of Programming Languages (QAPL) [FTL11] and the 31st IARCS Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS) [FLT11] and subsequently published in Theoretical Computer Science [FL14b].

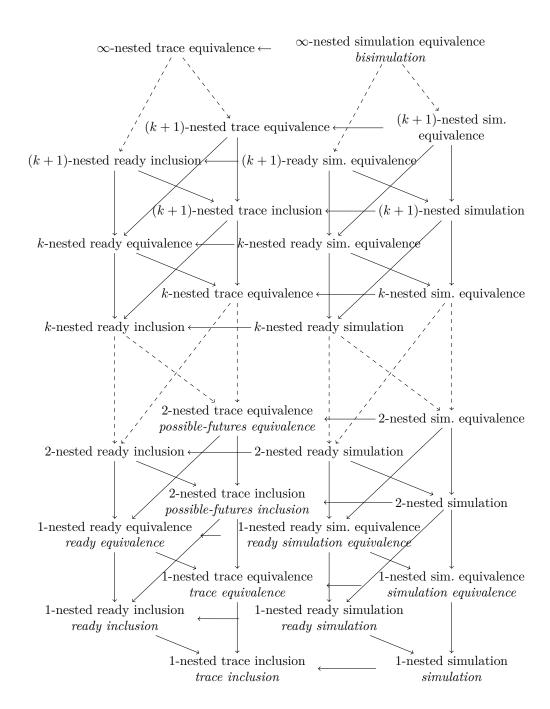


Figure 1.8: The quantitative linear-time-branching-time spectrum. The nodes are different system distances, and an edge $d_1 \longrightarrow d_2$ or $d_1 \dashrightarrow d_2$ indicates that $d_1(s,t) \ge d_2(s,t)$ for all states s, t, and that d_1 and d_2 in general are topologically inequivalent.

1.2.6 Chapter 6, "Weighted Modal Transition Systems"

Chapter 6 presents a lifting of our work on quantitative verification to quantitative specification theories. Fundamental to specification theories is the refinement relation which permits to successively refine specifications until an implementation is reached. Here implementations are the models with which previous chapters were concerned, i.e., transition systems. In the context of quantitative verification, we have in previous chapters replaced equivalence relations and preorders between models by linear and branching distances. Similarly in spirit, we replace in this chapter the refinement relation with a refinement distance, in order to be able to reason quantitatively about quantitative specifications.

In Chapter 6 we treat a special case of quantitative specification theory, using models which are transition systems whose transitions are labeled with symbols from a discrete alphabet Σ and with integers. We also use one particular distance, the discounted accumulating one. In the following Chapter 7, we generalize this setting to arbitrary models and specifications and arbitrary distances.

In the specifications of Chapter 6, integer weights are relaxed to *integer intervals* and, as usual in modal specifications, transitions can be of type *must* or of type *may*. Hence we define a *weighted modal transition system* (WMTS) to be a structure $S = (S, s^0, \dashrightarrow, \longrightarrow)$ consisting of a set of states S with an initial state $s^0 \in S$ and transition relations $\longrightarrow, \dashrightarrow \subseteq S \times \operatorname{Spec} \times S$ such that for every $(s, k, s') \in \longrightarrow$ there is $(s, \ell, s') \in \dashrightarrow$ where $k \leq \ell$. Here

$$\mathsf{Spec} = \Sigma \times \{ [x,y] \mid x \in \mathbb{Z} \cup \{-\infty\}, y \in \mathbb{Z} \cup \{\infty\}, x \leq y \}$$

is the set of (weighted) *specification labels*, and the partial order \leq on Spec is defined by $(a, I) \leq (a', I')$ if a = a' and $I \subseteq I'$.

A WMTS S as above is an *implementation* if $\longrightarrow = -- \rightarrow \subseteq S \times \mathsf{Imp} \times S$, where

$$\mathsf{Imp} = \Sigma \times \{ [x, x] \mid x \in \mathbb{Z} \} \approx \Sigma \times \mathbb{Z}$$

is the set of (weighted) *implementation labels* in Spec: the minimal elements of Spec with respect to \leq .

Now in a standard modal refinement $S_1 \leq_{\mathsf{m}} S_2$, must-transitions in S_2 must be preserved in S_1 , whereas may-transitions in S_1 must correspond to may-transitions in S_2 . Using the accumulating distance with a discounting factor $\lambda < 1$, and our work in Chapter 5, we extend this to a modal refinement distance which is defined as follows. First, a distance on specification labels is introduced by $d_{\mathsf{Spec}}((a,I),(a',I')) = \infty$ if $a \neq a'$ and

$$\begin{split} d_{\mathsf{Spec}}\big((a,[x,y]),(a,[x',y'])\big) &= \sup_{z \in [x,y]} \inf_{z' \in [x',y']} |z-z'| \\ &= \max(x'-x,y-y',0) \,. \end{split}$$

The modal refinement distance between the states of weighted modal transition systems $S_1 = (S_1, s_1^0, -\rightarrow_1, \longrightarrow_1)$ and $S_2 = (S_2, s_2^0, -\rightarrow_2, \longrightarrow_2)$ is then defined to be the least fixed point of the equations

$$d_{\mathsf{m}}(s_1, s_2) = \max \begin{cases} \sup_{\substack{k_1 - \dots + 1 \\ s_1 - \dots + 1 t_1}} \inf_{\substack{k_2 \\ s_2 - \dots + 2 t_2}} d_{\mathsf{Spec}}(k_1, k_2) + \lambda d_{\mathsf{m}}(t_1, t_2), \\ \sup_{\substack{k_2 \\ s_2 + \dots + 2 t_2}} \inf_{\substack{k_1 \\ s_1 + \dots + 1 \\ s_2 + \dots + 1 t_1}} d_{\mathsf{Spec}}(k_1, k_2) + \lambda d_{\mathsf{m}}(t_1, t_2), \end{cases}$$

and then $d_{\mathsf{m}}(\mathcal{S}_1, \mathcal{S}_2) = d_{\mathsf{m}}(s_1^0, s_2^0)$.

We show in Theorem 6.14 that the modal refinement distance bounds the so-called thorough refinement distance: for any implementation $\mathcal{I}_1 \leq_{\mathsf{m}} \mathcal{S}_1$, there is an implementation $\mathcal{I}_2 \leq_{\mathsf{m}} \mathcal{S}_2$ such that $d(\mathcal{I}_1, \mathcal{I}_2) \leq d_{\mathsf{m}}(\mathcal{S}_1, \mathcal{S}_2)$. Hence the modal refinement distance between two specifications can serve as an overapproximation of how far respective implementations can deviate from each other.

Modal specifications come equipped with a logical operation of conjunction and with structural operations of composition and quotient. Conjunction is the greatest lower bound in the modal refinement preorder. We show that such a conjunction exists in our formalism, but that it does not satisfy a natural quantitative generalization of the greatest lower bound property; in fact, Theorem 6.24 shows that there is no operation \wedge on WMTS which satisfies that for any $\varepsilon \geq 0$, there exist $\varepsilon_1, \varepsilon_2 \geq 0$ such that whenever $d_{\mathsf{m}}(S, S_1) \leq \varepsilon_1$ and $d_{\mathsf{m}}(S, S_2) \leq \varepsilon_2$ for some WMTS S, S_1, S_2 , then $d_{\mathsf{m}}(S, S_1 \wedge S_2) \leq \varepsilon$. Conjunction is thus, in this sense, discontinuous; we shall see in the following Chapter 7 that this is a fundamental problem with any quantitative specification theory.

For structural composition, we use CSP-style synchronization on labels and addition of intervals. That is, synchronization $(a, I) \oplus (a', I')$ on Spec is undefined if $a \neq a'$, and otherwise

$$(a, [x, y]) \oplus (a, [x', y']) = (a, [x + x', y + y']).$$

Using this label operation, we show in Theorem 6.27 that there is a structural composition operator \parallel for WMTS which satisfies $d_{\mathsf{m}}(\mathcal{S}_1 \| \mathcal{S}_3, \mathcal{S}_2 \| \mathcal{S}_4) \leq d_{\mathsf{m}}(\mathcal{S}_1, \mathcal{S}_2) + d_{\mathsf{m}}(\mathcal{S}_3, \mathcal{S}_4)$ for all WMTS $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4$. This property of independent implementability ensures that composition preserves distances. We also show in Theorem 6.29 that structural composition admits a partial inverse, a quotient operation / such that $d_{\mathsf{m}}(\mathcal{S}_3, \mathcal{S}_1/\mathcal{S}_2) = d_{\mathsf{m}}(\mathcal{S}_2 \| \mathcal{S}_3, \mathcal{S}_1)$ for all WMTS $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ whenever \mathcal{S}_2 is deterministic. The quotient operation can hence be used to synthesize partial specifications also in this quantitative context.

Chapter 6 is based on work by Sebastian S. Bauer, Line Juhl, Claus Thrane, Kim G. Larsen, Axel Legay, and the author, which has been presented at the 36th International Symposium on Mathematical Foundations of Computer Science (MFCS) [BFJ⁺11] and subsequently published in Formal Methods in System Design [BFJ⁺13].

1.2.7 Chapter 7, "General Quantitative Specification Theories with Modal Transition Systems"

In Chapter 7 we develop a general setting for quantitative specification theories. Combining the work in Chapters 5 and 6, we work in a setting of modal transition systems which are labeled with elements in a partially ordered set Spec of specification labels. The set of implementation labels is then $\mathsf{Imp} = \{k \in \mathsf{Spec} \mid k' \preccurlyeq k \Rightarrow k' = k\}$, and implementations are Imp -labeled transition systems.

We assume given an abstract trace distance $d: \mathsf{Spec}^\infty \times \mathsf{Spec}^\infty \to \mathbb{L}$, where $\mathbb{L} = (\mathbb{R}_{\geq 0} \cup \{\infty\})^M$, for an arbitrary set M, is the lattice of functions from M to $\mathbb{R}_{\geq 0} \cup \{\infty\}$. We also assume that there exists a *distance iterator* function $F: \mathsf{Spec} \times \mathsf{Spec} \times \mathbb{L} \to \mathbb{L}$ such that $d(\sigma, \tau) = F(\sigma_0, \tau_0, d(\sigma^1, \tau^1))$, similarly to the recursive characterization developed in Chapter 5.

We can then introduce an abstract modal refinement distance between the states of two such structured modal transition systems (or SMTS) $\mathcal{S} = (S, s_0, \dashrightarrow_S, \longrightarrow_S)$ and $\mathcal{T} = (T, t_0, \dashrightarrow_T, \longrightarrow_T)$ to be the least fixed point, in \mathbb{L} , to the equations

$$d_{\mathsf{m}}(s,t) = \max \begin{cases} \sup_{s \xrightarrow{k}_{- \to S} s'} \inf_{t \xrightarrow{\ell}_{- \to T} t'} F(k,\ell,d_{\mathsf{m}}(s',t')), \\ \sup_{t \xrightarrow{\ell}_{- T} t'} \inf_{s \xrightarrow{k}_{- S} s'} F(k,\ell,d_{\mathsf{m}}(s',t')), \end{cases}$$

and let $d_{\mathsf{m}}(\mathcal{S}, \mathcal{T}) = d_{\mathsf{m}}(s_0, t_0)$.

For conjunction of SMTS, we introduce a property of conjunctive boundedness on labels in Spec which implies, cf. Theorem 7.33, that conjunction of SMTS is uniformly bounded in the sense that the modal refinement distance from an SMTS \mathcal{U} to a conjunction $\mathcal{S} \wedge \mathcal{T}$ is bounded above by a uniform function of the distances $d_{\mathsf{m}}(\mathcal{U}, \mathcal{S})$ and $d_{\mathsf{m}}(\mathcal{U}, \mathcal{T})$. Unfortunately, it turns out that common label conjunction operators are not conjunctively bounded, hence we propose another property of relaxed conjunctive boundedness which does hold for common label operators, and show in Theorem 7.35 that it implies a similar property for SMTS conjunctions.

For structural composition, we generalize the work in Chapter 6 by introducing an abstract notion of (partial) label composition \oplus on Spec. Assuming that this operator is recursively *uniformly bounded* in the sense that there exists a function $P: \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ such that

$$F(k \oplus k', \ell \oplus \ell', P(\alpha, \alpha')) \sqsubseteq_{\mathbb{L}} P(F(k, \ell, \alpha), F(k', \ell', \alpha'))$$

for all $k, \ell, k', \ell' \in \mathsf{Spec}$ and $\alpha, \alpha' \in \mathbb{L}$ for which $k \oplus k'$ and $\ell \oplus \ell'$ are defined, we can then show in Theorem 7.23 that independent implementability holds, viz.

$$d_{\mathsf{m}}(\mathcal{S}||\mathcal{S}',\mathcal{T}||\mathcal{T}') \sqsubseteq_{\mathbb{L}} P(d_{\mathsf{m}}(\mathcal{S},\mathcal{T}),d_{\mathsf{m}}(\mathcal{S}',\mathcal{T}'))$$

for all SMTS \mathcal{S} , \mathcal{T} , \mathcal{S}' , \mathcal{T}' . In examples, we expose several different label composition operators and show that they are uniformly bounded. We show that the quotient operator / from Chapter 6 has a similar generalization to SMTS.

We also show in Chapter 7 that quantitative refinement admits a *logical characterization*, generalizing the work in Chapter 3. We use standard Hennessy-Milner logic, with formulae generated by the syntax

$$\phi, \phi_1, \phi_2 := \mathbf{tt} \mid \mathbf{ff} \mid \langle \ell \rangle \phi \mid [\ell] \phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \qquad (\ell \in \mathsf{Spec})$$

and with quantitative semantics $S \to \mathbb{L}$, for an SMTS $\mathcal{S} = (S, s_0, \dashrightarrow_S, \longrightarrow_S)$ given as follows:

$$\begin{split} \llbracket \mathbf{tt} \rrbracket s &= \bot & \llbracket \mathbf{ff} \rrbracket s &= \top \\ \llbracket (\phi_1 \land \phi_2) \rrbracket s &= \max(\llbracket \phi_1 \rrbracket s, \llbracket \phi \rrbracket_2 s) & \llbracket (\phi_1 \lor \phi_2) \rrbracket s &= \min(\llbracket \phi_1 \rrbracket s, \llbracket \phi_2 \rrbracket s) \\ \llbracket \langle \ell \rangle \phi \rrbracket s &= \inf \{ F(k, \ell, \llbracket \phi \rrbracket t) \mid s \xrightarrow{k} t, d(k, \ell) \neq \top_{\mathbb{L}} \} \\ \llbracket [\ell] \phi \rrbracket s &= \sup \{ F(k, \ell, \llbracket \phi \rrbracket t) \mid s \xrightarrow{k} t, d(k, \ell) \neq \top_{\mathbb{L}} \} \end{split}$$

Writing $\llbracket \phi \rrbracket \mathcal{S} = \llbracket \phi \rrbracket s_0$, we can then show in Theorem 7.41 that this logic is quantitatively sound for the modal refinement distance, in the sense that $\llbracket \phi \rrbracket \mathcal{S} \sqsubseteq_{\mathbb{L}} \llbracket \phi \rrbracket \mathcal{T} \oplus_{\mathbb{L}} d_{\mathsf{m}}(\mathcal{S}, \mathcal{T})$ for all formulae ϕ and all SMTS \mathcal{S} , \mathcal{T} . For disjunction-free formulae, we can show a complementary completeness result in Theorem 7.42, namely that $\llbracket \phi \rrbracket \mathcal{S} = \sup_{\mathcal{I} \in \llbracket \mathcal{S} \rrbracket} \llbracket \phi \rrbracket \mathcal{I}$ for all disjunction-free ϕ and all SMTS \mathcal{S} , where $\llbracket \mathcal{S} \rrbracket$ denotes the set of implementations of \mathcal{S} .

Chapter 7 is based on work by Sebastian S. Bauer, Claus Thrane, Axel Legay, and the author, which has been presented at the 7th International Computer Science Symposium in Russia (CSR) [BFLT12] and subsequently published in Acta Informatica [FL14a].

1.2.8 Chapter 8, "Logical vs. Behavioral Specifications"

DMTS hence generalize MTS in that they allow for multiple (or zero) initial states and permit must transitions to branch to a disjunction of destination states. As an example, Figure 1.9 shows a DMTS expressing the CTL property

$$AG(req \Rightarrow AX(work AW grant))$$

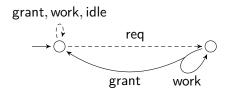


Figure 1.9: DMTS corresponding to the CTL property $AG(req \Rightarrow AX(work AW grant))$

(here "AW" denotes the weak-until operator): "at all time points after executing req, no idle nor further requests but only work is allowed until grant is executed". The same property may be expressed as a recursive system of equations in Hennessy-Milner logic [Lar90b] as

$$X = [\mathsf{grant}, \mathsf{idle}, \mathsf{work}] X \wedge [\mathsf{req}] Y$$

$$Y = (\langle \mathsf{work} \rangle Y \vee \langle \mathsf{grant} \rangle X) \wedge [\mathsf{idle}, \mathsf{req}] \mathbf{ff}$$

where the solution is given by the maximal fixed point.

In Chapter 8 we exhibit an equivalence between DMTS and Hennessy-Milner logic with maximal fixed points (the $modal\ \nu\text{-}calculus$) and also with a non-deterministic extension of the $acceptance\ automata$ of [Hen85, Rac08]. This allows one to freely switch between formalisms and, more importantly, to generalize the logical and structural operations on specifications and expose their algebraic properties.

We thus show in Theorem 8.23 that DMTS (and hence also acceptance automata and the modal ν -calculus) admit notions of conjunction and disjunction which are greatest lower, respectively least upper bounds, in the modal refinement order. That is, DMTS form a bounded distributive lattice up to modal equivalence.

We also generalize structural composition and quotient to DMTS and further introduce quotients S_1/S_2 also for the cases where S_2 is not deterministic. Theorem 8.33 then shows that quotient is a *residual* to structural composition, with defining property

$$S_1 || S_2 \leq_{\mathsf{m}} S_3 \implies S_2 \leq_{\mathsf{m}} S_3 / S_1$$

as before, but now without any restrictions on the involved specifications.

Combining the four operations, DMTS form a *commutative residuated lat*tice [JT02] up to modal equivalence. As an example, this immediately entails the following properties which may be used in a calculus of specifications:

$$\begin{split} \mathcal{S}_1 \| (\mathcal{S}_2/\mathcal{S}_3) & \leq_{\mathsf{m}} (\mathcal{S}_1 \| \mathcal{S}_2) / \mathcal{S}_3 & \mathcal{S}_1/\mathcal{S}_2 & \leq_{\mathsf{m}} (\mathcal{S}_1 \| \mathcal{S}_3) / (\mathcal{S}_2 \| \mathcal{S}_3) \\ (\mathcal{S}_1/\mathcal{S}_2) \| (\mathcal{S}_2/\mathcal{S}_3) & \leq_{\mathsf{m}} \mathcal{S}_1/\mathcal{S}_3 & (\mathcal{S}_1/\mathcal{S}_2) / \mathcal{S}_3 & \equiv_{\mathsf{m}} (\mathcal{S}_1/\mathcal{S}_3) / \mathcal{S}_2 \\ \mathcal{S}_1/(\mathcal{S}_2 \| \mathcal{S}_3) & \equiv_{\mathsf{m}} (\mathcal{S}_1/\mathcal{S}_2) / \mathcal{S}_3 & \mathcal{S} \| (\mathcal{S}/\mathcal{S}) & \equiv_{\mathsf{m}} \mathcal{S} \end{split}$$

Chapter 8 is based on work by Nikola Beneš, Jan Křetínský, Axel Legay, Louis-Marie Traonouez, and the author, which has been presented at the 24th International Conference on Concurrency Theory (CONCUR) [BDF⁺13] and at the 11th International Colloquium on Theoretical Aspects of Computing (ICTAC) [FLT14b] and subsequently published in Information and Computation [BFK⁺20].

1.2.9 Chapter 9, "Compositionality for Quantitative Specifications"

The final Chapter 9 combines the work of Chapters 7 and 8. It introduces general quantitative specification theories based on disjunctive modal transition systems [LX90b] and (non-deterministic) acceptance automata [Hen85, Rac08] on the one hand and abstract trace distances on the other hand.

As in Chapter 7, specification labels are partially ordered by a label refinement relation \leq , and implementation labels are those specification labels which cannot be further refined. We also assume partial conjunction and synchronization operators on labels and work with specification-labeled disjunctive modal transition systems and acceptance automata.

Also as in Chapter 7, we assume given a recursively specified distance on specification traces, which takes values in a *commutative quantale*: a complete lattice \mathbb{L} together with a commutative operation $\oplus_{\mathbb{L}}$ which distributes over arbitrary suprema. We then generalize the translations between DMTS, acceptance automata, and the modal ν -calculus from Chapter 8 to our general setting and show in Theorem 9.20 that they respect modal refinement distances: denoting the translations by da, dn etc.,

$$\begin{split} &d_{\mathsf{m}}(\mathcal{D}_1,\mathcal{D}_2) = d_{\mathsf{m}}(\mathit{da}(\mathcal{D}_1),\mathit{da}(\mathcal{D}_2)),\\ &d_{\mathsf{m}}(\mathcal{A}_1,\mathcal{A}_2) = d_{\mathsf{m}}(\mathit{ad}(\mathcal{A}_1),\mathit{ad}(\mathcal{A}_2)),\\ &d_{\mathsf{m}}(\mathcal{D}_1,\mathcal{D}_2) = d_{\mathsf{m}}(\mathit{dn}(\mathcal{D}_1),\mathit{dn}(\mathcal{D}_2)),\\ &d_{\mathsf{m}}(\mathcal{N}_1,\mathcal{N}_2) = d_{\mathsf{m}}(\mathit{nd}(\mathcal{N}_1),\mathit{nd}(\mathcal{N}_2)). \end{split}$$

We then turn to the quantitative properties of the operations and show in Theorem 9.26 that disjunction is quantitatively sound and complete in the sense that $d_{\mathsf{m}}(\mathcal{S}_1 \vee \mathcal{S}_2, \mathcal{S}_3) = \max(d_{\mathsf{m}}(\mathcal{S}_1, \mathcal{S}_3), d_{\mathsf{m}}(\mathcal{S}_2, \mathcal{S}_3))$ for all specifications \mathcal{S}_1 , \mathcal{S}_2 and \mathcal{S}_3 . Conjunction on the other hand is only quantitatively sound, for the same reasons as exposed in Chapter 6. Assuming a uniform bound on label synchronization, we again derive a quantitative version of independent implementability in Theorem 9.28. We also show in Theorem 9.29 that with our new generalized definition of DMTS quotient, it holds that $d_{\mathsf{m}}(\mathcal{S}_1 || \mathcal{S}_2, \mathcal{S}_3) = d_{\mathsf{m}}(\mathcal{S}_2, \mathcal{S}_3/\mathcal{S}_1)$ for all specifications \mathcal{S}_1 , \mathcal{S}_2 and \mathcal{S}_3 .

Chapter 9 is based on work by Jan Křetínský, Axel Legay, Louis-Marie Traonouez, and the author, which has been presented at the 11th International Symposium on Formal Aspects of Component Software (FACS) [FKLT14] and subsequently published in Soft Computing [FKLT18].

1.3 Applications

Our theory of quantitative specification and verification has found applications in robustness of real-time systems, feature interactions in software product lines, compatibility of service interfaces, text separation, and other areas. We present four such applications here.

1.3.1 A Robust Specification Theory for Modal Event-Clock Automata

The paper [FL12], written by Axel Legay and the author and presented at the Fourth Workshop on Foundations of Interface Technologies, contains an application of the general quantitative framework of this thesis in the area of real-time specifications. We define a notion of robustness for the modal event-clock specifications (MECS) of [BLPR09, BLPR12].

We propose a new version of refinement for MECS which is adequate to reason on MECS in a robust manner. We then proceed to exhibit the properties of the standard operations of specification theories: conjunction, structural composition and quotient, with respect to this quantitative refinement. We show that structural composition and quotient have properties which are useful generalizations of their standard Boolean properties, hence they can be employed for robust reasoning on MECS. Conjunction, on the other hand, is generally not robust, but together with the new operator of quantitative widening can be used in a robust manner.

MECS are modal transition systems in which may- and must-transitions are labeled with symbols from a set Σ and annotated with constraints which are used to enable or disable transitions depending on the values of real variables. In the language of Section 1.2.7, their semantics is given as SMTS over the set

$$\mathsf{Spec} = (\Sigma \times \{[0,0]\}) \cup (\{\delta\} \times \mathbb{I}) \subseteq (\Sigma \cup \{\delta\}) \times \mathbb{I}$$

of specification labels. Here $\mathbb{I} = \{[x,y] \mid x \in \mathbb{R}_{\geq 0}, y \in \mathbb{R}_{\geq 0} \cup \{\infty\}, x \leq y\}$ is the set of closed extended non-negative real intervals, and $\delta \notin \Sigma$ denotes a special symbol which signifies passage of time.

The partial order on Spec is given by $(a,[l,r]) \leq (a',[l',r'])$ iff a=a', $l \geq l',$ and $r \leq r'$ (hence $[l,r] \subseteq [l',r']$). Thus the implementation labels are $\mathsf{Imp} = \Sigma \times \{0\} \cup \{\delta\} \times \mathbb{R}_{\geq 0}$, so that implementations are usual timed transition systems with discrete transitions $s \xrightarrow{a,0} s'$ and delay transitions $s \xrightarrow{\delta,d} s'$.

We use the maximum-lead distance to measure differences between timed traces. For structural composition, we employ CSP-style label synchronization and intersection of timing intervals; hence in a composition, the timing constraints are conjunctions of the components' constraints. We then show that structural composition is bounded and that conjunction is relaxed bounded; the quotient operator is similarly well-behaved.

1.3.2 Measuring Global Similarity between Texts

The paper [FBC⁺14], written by Fabrizio Biondi, Kevin Corre, Cyrille Jégourel, Simon Kongshøj, Axel Legay, and the author and presented at the Second International Conference on Statistical Language and Speech Processing, contains an application of some of the theory presented here to a problem in statistical natural-language processing. We introduce a new type of distance between texts and show that it can be used to separate different classes in corpuses of scientific papers.

We measure the similarity of two texts using a discounted accumulating distance. Given two texts $A = (a_1, a_2, \ldots, a_{N_A})$ and $B = (b_1, b_2, \ldots, b_{N_B})$, seen as finite sequences of words (and hence stripped of punctuation), we first define an indicator function $\delta_{i,j}$, for $i, j \geq 0$, by

$$\delta_{i,j} = \begin{cases} 0 & \text{if } i \leq N_A, j \leq N_B \text{ and } a_i = b_j, \\ 1 & \text{otherwise,} \end{cases}$$

and then

$$d(i,j,\lambda) = \sum_{k=0}^{\infty} \lambda^k \delta_{i+k,j+k} ,$$

for a discounting factor $\lambda \in \mathbb{R}_{\geq 0}$ with $\lambda < 1$. This measures how much the texts A and B "look alike" when starting with the tokens a_i in A and b_j in B. This position match distance is then summarized and symmetrized as follows:

$$d'(A, B, \lambda) = \frac{1}{N_A} \sum_{i=1}^{N_A} \min_{j=1,\dots,N_B} d(i, j, \lambda)$$
$$d(A, B, \lambda) = \max(d'(A, B, \lambda), d'(B, A, \lambda))$$

We have implemented this computation and then used this implementation to statistically separate different types of scientific papers. In a first experiment, we successfully separate 42 scientific papers from 8 automatically generated "fake" scientific papers (using the tool SCIgen¹). With very high discounting, we also achieve a classification where papers which share authors or are otherwise similar are classified as such. In a second experiment, we compare 97 scientific papers with 100 "fake" ones generated by different methods. Also here we achieve a complete classification. For high discounting factors, our classifications are better than those achieved by other work using bag-of-words distances.

1.3.3 Measuring Behavior Interactions between Product-Line Features

The paper [AFL15], written by Joanne M. Atlee, Axel Legay and the author and presented at the 3rd IEEE/ACM FME Workshop on Formal Methods

¹http://pdos.csail.mit.edu/scigen/

in Software Engineering, suggests a new method for measuring the degree to which features interact in software product lines.

The paper first introduces a distance between labeled transition systems which is similar to the (undiscounted) accumulating simulation distance, except that every pair of states is only treated once. That is, the function computing d(s,s') tries to match every transition $s \xrightarrow{a} t$ in the first system S with a transition $s' \xrightarrow{a} t'$ in the second system S'. If no such exists, a missing behavior is detected and 1 is added to the score; if there are transitions $s' \xrightarrow{a} t'$, then distance is recursively computed for the pair t,t' with the best match. Once a pair of states has been checked for behavior mismatches in this way, it is added to a Passed list of states which need not be checked again.

We model software product lines using featured transition systems, which are transition systems in which transitions are conditioned on the presence or absence of distinct features. A product is then simply a set of features, and a product p has a behavior interaction with a feature f in a given featured transition system S if the projection onto p of S and the projection onto p of the projection onto $p \cup \{f\}$ of S are not bisimilar.

We then generalize this notion to a behavior interaction distance, using the above distance between transition systems. We give evidence that this is a useful notion to assess the degree of feature interactions and show that it can be efficiently computed on the given featured transition system, without resorting to the projections.

1.3.4 Compatibility Flooding: Measuring Interaction of Behavioral Models

The paper [OFLS17], written by Meriem Ouederni, Axel Legay, Gwen Salaün, and the author and presented at the 32nd ACM SIGAPP Symposium on Applied Computing, deals with compatibility verification of service interfaces, focusing on the interaction protocol level.

Checking the compatibility of interaction protocols is a tedious and hard task, even though it is of utmost importance to avoid run-time errors, e.g., dead-lock situations or unmatched messages. Most of the existing approaches return a "True" or "False" result to detect whether services are compatible or not, but for many issues such a Boolean answer is not very helpful. In real world situations, there will seldom be a perfect match, and when service protocols are not compatible, it is useful to differentiate between services that are slightly incompatible and those that are totally incompatible. Our paper aims at quantifying the compatibility degree of service interfaces, taking a semantic point of view.

Incompatibilities are measured between transition systems modeling service interfaces, using a version of discounted accumulating bisimulation distance where differences are propagated both forward and backwards. The distance takes into account the compatibility of parameters and labels and is defined for two different scenarios, one in which all sent and received messages

must be matched, and an asymmetric one where one of the components may send and receive other messages which are irrelevant for the composition.

1.4 Conclusion and Perspectives

We have developed a general theory of quantitative verification and quantitative specification theories. The theory is independent of how precisely quantitative differences are measured and applicable to a large class of distances used in practice. The quantative specification formalism introduced in the last Chapter 9 is also rather robust, admitting translations between several different specification formalisms, and has good algebraic and geometric properties.

On a theoretical level, the above is motivation to concern oneself with the question what precisely *is* a specification theory. While there is some agreement to this at the qualitative / Boolean level, it is not clear how to extend this to the quantitative world. This question is important not only theoretically, but also in applications, given that the algebraic properties of a formalism determine how precisely it can be used in practice.

Somewhat related to the question above is the problem of how to treat *silent* or spontaneous transitions. In applications it is common to model uncertainty or ambiguity with silent transitions, and these are rather well-understood in the qualitative setting; but again it is unclear how to lift them to the quantitative world.

Further, and taking a more applied view, it is somewhat problematic that all the formalisms treated here are based on discrete transition systems. When considering applications in real-time or hybrid systems, discreteness is not sufficient and some treatment of continuous time is required. There is some work on specification theories for real-time systems, but for hybrid systems these are lacking, and in any case it is unclear how to relate them to the quantitative specification theories we have exposed here.

Below we treat the questions and problems above in some more detail and try to show some avenues for further work on these subjects.

1.4.1 Specification Theories

The work presented here has led to more fundamental questions as to what precisely is, or should be, a specification theory. This is what we set out to answer, together with Axel Legay, in [FL17], presented at the 43rd International Conference on Current Trends in Theory and Practice of Computer Science (SOFSEM) and subsequently published in the Journal of Logical and Algebraic Methods in Programming [FL20b], and in [FL20a], to be presented at the 2021 ISoLA Symposium.

We propose here that a specification theory for a set of models \mathfrak{M} consists of the following ingredients:

- a set \mathfrak{S} of specifications;
- a mapping $\chi: \mathfrak{M} \to \mathfrak{S}$; and
- a refinement preorder \leq on \mathfrak{S} which is an equivalence relation on the image of χ in \mathfrak{S} .

It then follows that for all $M \in \mathfrak{M}$, $\chi(M)$ is the *characteristic formula* [Pnu85] for M.

Logical operations on specifications are then obtained by asserting that \mathfrak{S} forms a bounded distributive lattice up to \equiv , the equivalence on \mathfrak{S} defined by $S_1 \equiv S_2$ iff $S_1 \leq S_2$ and $S_2 \leq S_1$. Structural composition and quotient are defined by an extra operation \parallel on specifications which turns \mathfrak{S} into a (bounded distributed) commutative residuated lattice. This puts specification theories into a well-understood algebraic context, see for example [JT02], which also appears in linear logic [Gir87] and other areas.

It is an open question how to transfer this algebraic point of view to the quantitative setting. It is clear that the refinement order above should be replaced by a hemimetric d on \mathfrak{S} , and also that d should be symmetric on the image of χ in \mathfrak{S} ; but we do not know how to correctly introduce characteristic formulae into this setting.

1.4.2 Silent Transitions

Another open question is how to deal with *silent transitions* in the quantitative setting. Van Glabbeek defines a linear-time—branching-time spectrum for "processes with silent moves" in [vG93], but it is unclear how to translate this into a game framework in order to replicate the work contained in Chapter 5.

Silent transitions are obtained whenever two processes synchronize on internal actions, so they are important from an application point of view. Recent advances on coalgebraic approaches to silent transitions [Bre15] and on codensity games [KKH⁺19] appear to provide a way forward.

1.4.3 Applications

Much work is to be done in order to apply our work to real-time, hybrid, or embedded systems. Specification theories for real-time and probabilistic systems do exist, see below, but they all have problems with robustness. For hybrid systems, no work on compositional specification theories seems to be available. Generally speaking, the problem with real-time and hybrid systems is that time itself provides an implicit synchronization mechanism, so compositionality is difficult to achieve for real-time and hybrid systems.

Modal event-clock specifications

We have already mentioned modal event-clock specifications (MECS) in Section 1.3.1. Introduced in [BLPR09, BLPR12], these form a specification theory for event-clock automata [AFH99], a determinizable subclass of timed automata [AD94], under timed bisimilarity. Models and specifications are assume to be deterministic, thus $S_1 \leq S_2$ iff $ModS_1 \subseteq ModS_2$ in this case.

In [BLPR12] it is shown that MECS admit a conjunction, thus forming a meet-semilattice up to \equiv . The authors also introduce composition and quotient; but computation of quotient incurs an exponential blow-up. Using the maximum-lead distance to measure differences between timed traces, we have shown in [FL12] how to develop a framework for robust quantitative reasoning.

Timed input/output automata

[DLL⁺15,DLL⁺12a] introduce a specification theory based on a variant of the timed input/output automata (TIOA) of [KLSV10, KLSV03]. Both models and specifications are TIOA which are action-deterministic and input-enabled; but models are further restricted using conditions of output urgency and independent progress. The equivalence on models being specified is timed bisimilarity.

In $[DLL^+15]$ it is shown that TIOA admit a conjunction. The paper also introduces a composition operation and a quotient, but the quotient is only shown to satisfy the property that

$$S_1 || \mathcal{M} \leq S_3 \Leftrightarrow \mathcal{M} \leq S_3 / S_1$$

for all specifications S_1 , S_3 and all *models* \mathcal{M} . No robust quantitative specification theories for TIOA are available.

Abstract probabilistic automata

Abstract probabilistic automata (APA), introduced in [DKL⁺13, DFLL14], form a specification theory for probabilistic automata [SL95] under probabilistic bisimilarity. They build on earlier models of interval Markov chains (IMC) [DLL⁺12b], see also [BDF⁺18, DLP16] for a related line of work.

In [DKL⁺13] it is shown that APA admit a conjunction, but that IMC do not. Also a composition is introduced in [DKL⁺13], and it is shown that composing two APA with interval constraints (hence, IMCs) may yield an APA with *polynomial* constraints (not an IMC); but APA with polynomial constraints are closed under composition. No robust quantitative specification theories for APA are available.

1.5 About the Author

Ulrich (Uli) Fahrenberg holds a PhD in mathematics from Aalborg University, Denmark. For his thesis, which he defended in 2005, he worked in algebraic topology and its applications in concurrency theory. His work was supervised by Lisbeth Fajstrup and Martin Raussen, and his thesis bore the title "Higher-Dimensional Automata from a Topological Viewpoint".

After his PhD, Fahrenberg started a career in computer science as an assistant professor at Aalborg University. During this time, he worked with Kim G. Larsen and others on weighted timed automata and quantitative verification. The work reported in this thesis was started together with Kim G. Larsen and Fahrenberg's PhD student Claus Thrane while Fahrenberg was at this position. In 2010, Fahrenberg passed his University Teacher Education for Assistant Professors, the prerequisite for holding a permanent post at a Danish university.

From 2010 to 2016, Fahrenberg has worked as a postdoc at Inria Rennes, France, in the group of Axel Legay. During this time, he deepened his work in quantitative analysis and verification and started his work in quantitative specification theories. Between 2016 and 2021 Fahrenberg was a researcher at the computer science lab at École polytechnique in Palaiseau, France, and since 2021 he is associate professor at EPITA Rennes. His current research centers on the theory of concurrent, distributed, and hybrid systems.

Since 2001, Fahrenberg has published 96 scientific contributions, among which 24 papers in peer-reviewed international journals and 47 papers in peer-reviewed international conference or workshop proceedings. He has been a member of numerous program committees, and since 2016 he is a reviewer for AMS Mathematical Reviews. He is a member of the Steering Committee of the RAMiCS international conferences and has been PC co-chair of RAMiCS-2020 and RAMiCS-2021.

Fahrenberg has been co-supervisor for one PhD student and three Masters students. He has supervised two PhD students' internships and two Masters students' internships and also taught a number of courses both in mathematics and computer science.

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2 Quantitative Analysis of Weighted Transition Systems¹

This chapter introduces a notion of weighted transition system (WTS), essentially an extension of the standard concept of (labeled) transition system [Plo81] which has been used to introduce operational semantics for a wide range of systems. It then proceeds to define what is meant by linear and branching distances between such WTS, and introduces three examples of such distances: the point-wise, accumulated, and maximum-lead distances. Finally, it is shown that linear distances are bounded by branching distances, and some results on topological inequivalence are provided.

2.1 Weighted transition systems

The intention of WTS is to describe a system's behavior as well as quantitative properties in terms of transition weights. Recall that a transition system is a quadruple (S, s_0, Σ, R) consisting of a set S of states with initial state $s_0 \in S$, a finite set Σ of labels, and a set of transitions $R \subseteq S \times \Sigma \times S$.

- **2.1 Definition.** A weighted transition system is a tuple (S, s_0, Σ, R, w) , where
 - (S, s_0, Σ, R) is a transition system, and
 - $w: R \to \mathbb{R}_{\geq 0}$ assigns weights to transitions.

We write $s \xrightarrow{\alpha,w} s'$ whenever $(s,\alpha,s') \in R$ and $w(s,\alpha,s') = w$, and $s \not\to if$ there is no transition (s,α,s') in R for any α and s'.

We lift the standard notions of path and trace to WTS:

2.2 Definition. Let $S = (S, s_0, \Sigma, R, w)$ be a WTS and $s \in S$. A path from s in S is a (possibly infinite) sequence $((s_0, \alpha_0, s_1), (s_1\alpha_1, s_2), \ldots)$ of transitions $(s_i, \alpha_i, s_{i+1}) \in R$ with $s_0 = s$. A (weighted) trace from s is a sequence $((\alpha_0, w_0), (\alpha_1, w_1), \ldots)$ of pairs $(\alpha_i, w_i) \in \Sigma \times \mathbb{R}_{\geq 0}$ for which there exists a path $((s_0, \alpha_0, s_1), (s_1\alpha_1, s_2), \ldots)$ from s for which $w_i = w(s_i, \alpha_i, s_{i+1})$.

 $^{^1{}m This}$ chapter is based on the journal paper [TFL10] published in the Journal of Logic and Algebraic Programming.

The set of traces from a state s is denoted Tr(s). Given a trace σ , we denote by $U(\sigma) \in \Sigma^{\omega}$ its label sequence (*i.e.*, the associated unweighted trace), and by σ_i its i'th label-weight pair.

2.2 Quantitative Analysis

In this section we introduce our quantitative analysis of WTS, both in a linear and in a branching setting. For ease of exposition we concentrate on *trace inclusion* and *simulation* here and defer treatment of both trace equivalence and bisimulation to other work. We shall introduce three different quantitative notions of trace inclusion and of simulation, all filling in the gap between the *unweighted* and the *weighted* relations, which we recall below:

- **2.3 Definition.** Let (S, s_0, Σ, R, w) be a WTS. A relation $R \subseteq S \times S$ is
 - an unweighted simulation provided that for all $(s,t) \in R$ and $s \xrightarrow{\alpha,c} s'$, also $t \xrightarrow{\alpha,d} t'$ for some $d \in \mathbb{R}_{\geq 0}$ and $(s',t') \in R$,
 - a (weighted) simulation provided that for all $(s,t) \in R$ and $s \xrightarrow{\alpha,c} s'$, also $t \xrightarrow{\alpha,c} t'$ for some $(s',t') \in R$.

We write

- $s \leq^u t$ if $(s,t) \in R$ for some unweighted simulation R,
- $s \leq t$ if $(s,t) \in R$ for some weighted simulation R.

Also, we write

- $s \leq^u t$ if $U(\mathsf{Tr}(s)) \subseteq U(\mathsf{Tr}(t))$,
- $s \le t$ if $\mathsf{Tr}(s) \subseteq \mathsf{Tr}(t)$.

We shall fill in the gap between unweighted and weighted relations using (asymmetric) distance functions $d: S \times S \to \mathbb{R}_{\geq 0} \cup \{\infty\}$. Any of the distances defined below will obey the properties given in the following definition.

- **2.4 Definition.** A hemimetric $d: S \times S \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ defined on the states of a WTS (S, s_0, Σ, R, w) is called
 - a linear distance if $s \le t$ implies d(s,t) = 0 and $s \not \le^u t$ implies $d(s,t) = \infty$,
 - a branching distance if $s \leq t$ implies d(s,t) = 0 and $s \nleq^u t$ implies $d(s,t) = \infty$.

As usual, we can generalize distances between states of a single WTS to distances between two different WTS by taking their disjoint union.

Our distance functions are essentially based on three different metrics on the set of sequences of real numbers. Throughout this work, these are referred to as *point-wise* (2.1), *accumulated* (2.2), and *maximum-lead* (2.3) distances, respectively. For sequences $a = (a_i)$, $b = (b_i)$ these are defined as follows:

$$d_{\bullet}(a,b) = \sup_{i} \left\{ |a_i - b_i| \right\} \tag{2.1}$$

$$d_{+}(a,b) = \sum_{i} |a_{i} - b_{i}| \tag{2.2}$$

$$d_{\pm}(a,b) = \sup_{i} \left\{ \left| \sum_{j=0}^{i} a_{j} - \sum_{j=0}^{i} b_{j} \right| \right\}$$
 (2.3)

The intuition behind these metrics is that d_{\bullet} measures the largest individual difference of sequence entries, d_{+} measures the accumulated sum of (the absolute values of) the entries' differences, and d_{\pm} measures the largest lead of one sequence over the other, i.e., the maximum difference in accumulated values. Hence the maximum-lead distance of two sequences is the same as the point-wise distance of their partial-sum sequences.

Besides the above three, other metrics on sequences of reals are also of interest, and we will see in Chapter 5 that linear and branching distances of WTS based on these other metrics can be developed similarly to the ones we introduce in this chapter.

In the following we will consider discounted distances, where the contribution of each step is decreased exponentially over time. To this end, we fix a discounting factor $\lambda \in [0,1]$; as extreme cases, $\lambda = 1$ means that the future is undiscounted, and $\lambda = 0$ means that only the present is considered.

Also, we fix a WTS $S = (S, s_0, \Sigma, R, w)$.

2.2.1 Linear distances

We will now introduce our quantitative trace distances.

2.5 Definition. For traces σ , τ , the point-wise, accumulating, and maximum-lead trace distances are given by $d_{\bullet}^{\mathsf{T}}(\sigma,\tau) = d_{+}^{\mathsf{T}}(\sigma,\tau) = d_{\pm}^{\mathsf{T}}(\sigma,\tau) = \infty$ if $U(\sigma) \neq U(\tau)$, and for $U(\sigma) = U(\tau)$,

$$d_{\bullet}^{\mathsf{T}}(\sigma, \tau) = \sup_{i} \left\{ \lambda^{i} \left| w(\sigma_{i}) - w(\tau_{i}) \right| \right\},$$

$$d_{+}^{\mathsf{T}}(\sigma, \tau) = \sum_{i} \lambda^{i} \left| w(\sigma_{i}) - w(\tau_{i}) \right|,$$

$$d_{\pm}^{\mathsf{T}}(\sigma, \tau) = \sup_{i} \left\{ \lambda^{i} \left| \sum_{j=0}^{i} w(\sigma_{j}) - \sum_{j=0}^{i} w(\tau_{j}) \right| \right\}.$$

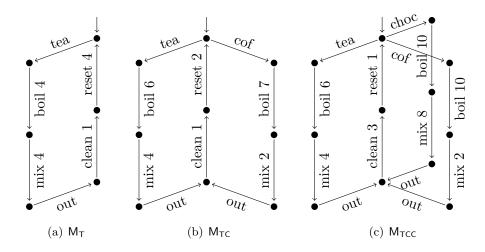


Figure 2.1: Three beverage machines

Observe that the above distances on traces are symmetric; they are indeed *metrics* on the set of traces. This is not the case when lifted to states:

2.6 Definition. For states $s, t \in S$, the point-wise, accumulating and maximum-lead *linear distances* are given as follows:

$$\begin{split} d_{\bullet}^{\mathsf{L}}(s,t) &= \sup_{\sigma \in \mathsf{Tr}(s)} \inf_{\tau \in \mathsf{Tr}(t)} d_{\bullet}^{\mathsf{T}}(\sigma,\tau) \\ d_{+}^{\mathsf{L}}(s,t) &= \sup_{\sigma \in \mathsf{Tr}(s)} \inf_{\tau \in \mathsf{Tr}(t)} d_{+}^{\mathsf{T}}(\sigma,\tau) \\ d_{\pm}^{\mathsf{L}}(s,t) &= \sup_{\sigma \in \mathsf{Tr}(s)} \inf_{\tau \in \mathsf{Tr}(t)} d_{\pm}^{\mathsf{T}}(\sigma,\tau) \end{split}$$

Note that this is precisely the Hausdorff-hemimetric construction, hence it can be generalized to other distances between traces. Also, it is quite natural, cf. Proposition 1.1. It can easily be shown that the distances defined above are indeed linear distances in the sense of Definition 2.4.

2.7 Example. To illustrate differences between the three linear distances introduced above, consider the three WTS models of beverage machines depicted in Figure 2.1; a Tea maker M_T , a Tea and Coffee maker M_{TC} and a Tea, Coffee and Chocolate maker M_{TCC} . In the figure, edges without specified weight have weight 0.

The production of a beverage consists of six operations: Selecting the drink, boiling the water, mixing the beverage, outputting the finished product, self cleaning, and resetting. Each operation consumes a certain amount of power depending on its implementation by electrical components. Weights thus model power consumption, and are given in such a way that in more

powerful machines, some operations, for example boiling, require more power, whereas some other, for example resetting, require less.

By design of the beverage machines, there are unweighted trace inclusions $M_T \leq^u M_{TC} \leq^u M_{TCC}$; any behavior of a "lesser" machine can be emulated qualitatively by a "better" one. What is less obvious is how they compare in power consumption.

Noting that any infinite behavior in the beverage machines consists of loops of width 6, we can introduce some ad-hoc notation to simplify calculations. Let $d_{\bullet}^{L}(M_{T}, M_{TC})^{6}$ denote point-wise distance from M_{T} to M_{TC} when only traces of length at most 6 are considered, and similarly for the other machines and distances. For a (realistic) discounting factor of $\lambda = .90$, the point-wise distances can be computed as follows:

$$\begin{split} d_{\bullet}^{\mathsf{L}}(\mathsf{M}_{\mathsf{T}},\mathsf{M}_{\mathsf{TC}}) &= \sup_{i} \big\{ d_{\bullet}^{\mathsf{L}}(\mathsf{M}_{\mathsf{T}},\mathsf{M}_{\mathsf{TC}})^{6} \cdot \lambda^{6i} \big\} = d_{\bullet}^{\mathsf{L}}(\mathsf{M}_{\mathsf{T}},\mathsf{M}_{\mathsf{TC}})^{6} &= 1.80 \\ d_{\bullet}^{\mathsf{L}}(\mathsf{M}_{\mathsf{T}},\mathsf{M}_{\mathsf{TCC}}) &= \sup_{i} \big\{ d_{\bullet}^{\mathsf{L}}(\mathsf{M}_{\mathsf{T}},\mathsf{M}_{\mathsf{TCC}})^{6} \cdot \lambda^{6i} \big\} = d_{\bullet}^{\mathsf{L}}(\mathsf{M}_{\mathsf{T}},\mathsf{M}_{\mathsf{TCC}})^{6} &= 1.80 \\ d_{\bullet}^{\mathsf{L}}(\mathsf{M}_{\mathsf{TC}},\mathsf{M}_{\mathsf{TCC}}) &= \sup_{i} \big\{ d_{\bullet}^{\mathsf{L}}(\mathsf{M}_{\mathsf{TC}},\mathsf{M}_{\mathsf{TCC}})^{6} \cdot \lambda^{6i} \big\} = d_{\bullet}^{\mathsf{L}}(\mathsf{M}_{\mathsf{TC}},\mathsf{M}_{\mathsf{TCC}})^{6} = 2.70 \end{split}$$

For the accumulating distances:

$$d_{+}^{L}(\mathsf{M}_{\mathsf{T}},\mathsf{M}_{\mathsf{TC}}) = \sum_{i} d_{\bullet}^{L}(\mathsf{M}_{\mathsf{T}},\mathsf{M}_{\mathsf{TC}})^{6} \cdot \lambda^{6i} = d_{\bullet}^{L}(\mathsf{M}_{\mathsf{T}},\mathsf{M}_{\mathsf{TC}})^{6} \frac{1}{1-\lambda^{6}} \approx 2.52$$

$$d_{+}^{L}(\mathsf{M}_{\mathsf{T}},\mathsf{M}_{\mathsf{TCC}}) = \sum_{i} d_{\bullet}^{L}(\mathsf{M}_{\mathsf{T}},\mathsf{M}_{\mathsf{TCC}})^{6} \cdot \lambda^{6i} = d_{\bullet}^{L}(\mathsf{M}_{\mathsf{T}},\mathsf{M}_{\mathsf{TCC}})^{6} \frac{1}{1-\lambda^{6}} \approx 8.80$$

$$d_{+}^{L}(\mathsf{M}_{\mathsf{TC}},\mathsf{M}_{\mathsf{TCC}}) = \sum_{i} d_{\bullet}^{L}(\mathsf{M}_{\mathsf{TC}},\mathsf{M}_{\mathsf{TCC}})^{6} \cdot \lambda^{6i} = d_{\bullet}^{L}(\mathsf{M}_{\mathsf{TC}},\mathsf{M}_{\mathsf{TCC}})^{6} \frac{1}{1-\lambda^{6}} \approx 7.41$$

Similarly, the maximum-lead distances can be computed as follows:

$$d_{\pm}^{\mathsf{L}}(\mathsf{M_T},\mathsf{M_{TC}}) \approx 1.62$$

 $d_{\pm}^{\mathsf{L}}(\mathsf{M_T},\mathsf{M_{TCC}}) \approx 2.62$
 $d_{\pm}^{\mathsf{L}}(\mathsf{M_{TC}},\mathsf{M_{TCC}}) \approx 3.34$

The following lemma provides recursive bounds on linear distances and will be useful as motivation for the definition of branching distance below. For the bound on the maximum-lead distance, we introduce a generalization of d_+^{L} by

$$\begin{split} d_{\pm}^{\mathsf{L}}(s,t)(\delta) &= \sup_{\sigma \in \mathsf{Tr}(s)} \inf_{\tau \in \mathsf{Tr}(t)} d_{\pm}^{\mathsf{L}}(\sigma,\tau)(\delta) \,, \\ d_{\pm}^{\mathsf{L}}(\sigma,\tau)(\delta) &= \sup_{i} \left\{ \lambda^{i} \left| \delta + \sum_{j=0}^{i} w(\sigma_{j}) - \sum_{j=0}^{i} w(\tau_{j}) \right| \right\}. \end{split}$$

Here $\delta \in \mathbb{R}$ is the *lead* which σ has already acquired over τ ; hence $d_{\pm}^{\mathsf{L}}(\sigma,\tau)(0) = d_{\pm}^{\mathsf{L}}(\sigma,\tau)$ and $d_{\pm}^{\mathsf{L}}(s,t)(0) = d_{\pm}^{\mathsf{L}}(s,t)$.

2.8 Lemma. For states $s, t \in S$,

$$\begin{split} d_{\bullet}^{\mathsf{L}}(s,t) &\leq \sup_{s \xrightarrow{\alpha,c} s'} \inf_{t \xrightarrow{\alpha,d} t'} \max \left(|c-d|, \lambda \, d_{\bullet}^{\mathsf{L}}(s',t') \right), \\ d_{+}^{\mathsf{L}}(s,t) &\leq \sup_{s \xrightarrow{\alpha,c} s'} \inf_{t \xrightarrow{\alpha,d} t'} |c-d| + \lambda \, d_{+}^{\mathsf{L}}(s',t'), \\ d_{\pm}^{\mathsf{L}}(s,t)(\delta) &\leq \sup_{s \xrightarrow{\alpha,c} s'} \inf_{t \xrightarrow{\alpha,d} t'} \max \left(|\delta|, \lambda \, d_{\pm}^{\mathsf{L}}(s',t') \big(\frac{\delta+c-d}{\lambda} \big) \right). \end{split}$$

Proof: We only show the proof for accumulated distance; the others are similar. If $\mathsf{Tr}(s) = \emptyset$, then $d_+^\mathsf{L}(s,t) = 0$ and we are done. Otherwise, let $\sigma \in \mathsf{Tr}(s)$; we need to show that

$$\inf_{\tau \in \mathsf{Tr}(t)} d^{\mathsf{L}}_{+}(\sigma,\tau) \leq \sup_{s \xrightarrow{\alpha,c} s'} \inf_{t \xrightarrow{\alpha,d} t'} |c-d| + \lambda \, d^{\mathsf{L}}_{+}(s',t') \,.$$

Let π be a path from s which realizes σ , write $\pi = s \xrightarrow{\alpha_1, c_1} s_1 \to \ldots$, and let σ_1 be the trace generated by the suffix of π starting in s_1 . If $t \xrightarrow{\alpha_1}$, then the infimum on the right hand side of the equation is ∞ , and we are done.

Assume that the infimum is finite and let $\varepsilon \in \mathbb{R}_+$. There exists $t \xrightarrow{\alpha_1, d_1} t_1$ for which

$$|c_1 - d_1| + \lambda d_+^{\mathsf{L}}(s_1, t_1) \le \inf_{t \xrightarrow{\alpha_1, d} t'} |c_1 - d| + \lambda d_+^{\mathsf{L}}(s_1, t') + \frac{\varepsilon}{2}.$$

Let $\tau_1 \in \text{Tr}(t_1)$ be such that $d_+^{\mathsf{L}}(\sigma_1, \tau_1) \leq d_+^{\mathsf{L}}(s_1, t_1) + \frac{\varepsilon}{2}$. Let $\tau = (\alpha_1, d_1).\tau_1$, the concatenation, then

$$d_{+}^{L}(\sigma,\tau) = |c_{1} - d_{1}| + \lambda d_{+}^{L}(\sigma_{1}, \tau_{1})$$

$$\leq |c_{1} - d_{1}| + \lambda d_{+}^{L}(s_{1}, t_{1}) + \frac{\varepsilon}{2}$$

$$\leq \inf_{t \xrightarrow{\alpha_{1}, d} t'} |c_{1} - d| + \lambda d_{+}^{L}(s_{1}, t') + \varepsilon.$$
(2.4)

We have shown that for all $\varepsilon \in \mathbb{R}_+$, there exists $\tau \in \text{Tr}(t)$ for which Equation (2.4) holds, hence

$$\inf_{\tau \in \mathsf{Tr}(t)} d^{\mathsf{L}}_+(\sigma,\tau) \leq \inf_{t \xrightarrow{\alpha_1,d} t'} |c_1 - d| + \lambda \, d^{\mathsf{L}}_+(s_1,t')$$

and the claim follows.

2.2.2 Simulation distances

In the following we use parametrized families $\{R_{\varepsilon} \subseteq S \times S\}$ and $\{R_{\varepsilon,\delta} \subseteq S \times S\}$, *i.e.*, functions $\mathbb{R}_{\geq 0} \to 2^{S \times S}$ and $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to 2^{S \times S}$, respectively; we shall show how these give rise to distances in Section 2.2.3.

2.9 Definition. A family of relations $\mathcal{R} = \{R_{\varepsilon} \subseteq S \times S \mid \varepsilon \geq 0\}$ is

- a point-wise simulation family provided that for all $(s,t) \in R_{\varepsilon} \in \mathcal{R}$ and $s \xrightarrow{\alpha,c} s'$, also $t \xrightarrow{\alpha,d} t'$ with $|c-d| \leq \varepsilon$ for some $d \in \mathbb{R}_{\geq 0}$ and $(s',t') \in R'_{\varepsilon} \in \mathcal{R}$ with $\varepsilon' \leq \frac{\varepsilon}{\lambda}$,
- an accumulating simulation family provided that for all $(s,t) \in R_{\varepsilon} \in \mathcal{R}$ and $s \xrightarrow{\alpha,c} s'$, also $t \xrightarrow{\alpha,d} t'$ with $|c-d| \leq \varepsilon$ for some $d \in \mathbb{R}_{\geq 0}$ and $(s',t') \in R'_{\varepsilon} \in \mathcal{R}$ with $\varepsilon' \leq \frac{\varepsilon |c-d|}{\lambda}$.

A family of relations $\mathcal{R} = \{R_{\varepsilon,\delta} \subseteq S \times S \mid \varepsilon \geq 0, -\varepsilon \leq \delta \leq \varepsilon\}$ is

• a maximum-lead simulation family provided that for all $(s,t) \in R_{\varepsilon,\delta} \in \mathcal{R}$ and $s \xrightarrow{\alpha,c} s'$, also $t \xrightarrow{\alpha,d} t'$ with $|\delta + c - d| \le \varepsilon$ for some $d \in \mathbb{R}_{\ge 0}$ and $(s',t') \in R_{\varepsilon',\delta'} \in \mathcal{R}$ with $\varepsilon' \le \frac{\varepsilon}{\lambda}$ and $\delta' \le \frac{\delta + c - d}{\lambda}$.

We write

- $s \preccurlyeq_{\varepsilon}^{\bullet} t$ if $(s,t) \in R_{\varepsilon} \in \mathcal{R}$ for some point-wise simulation family \mathcal{R} ,
- $s \leq_{\varepsilon}^{+} t$ if $(s,t) \in R_{\varepsilon} \in \mathcal{R}$ for some accumulating simulation family \mathcal{R} ,
- $s \preccurlyeq_{\varepsilon}^{\pm} t$ if $(s,t) \in R_{\varepsilon,0} \in \mathcal{R}$ for some maximum-lead simulation family \mathcal{R} .

Note that the relations defined in the last part above again can be collected into families $\preccurlyeq^{\bullet} = \{ \preccurlyeq^{\bullet}_{\varepsilon} | \varepsilon \geq 0 \}, \ \preccurlyeq^{+} = \{ \preccurlyeq^{+}_{\varepsilon} | \varepsilon \geq 0 \}, \ \text{and} \ \preccurlyeq^{\pm} = \{ \preccurlyeq^{\pm}_{\varepsilon,\delta} | \varepsilon, \delta \geq 0 \}.$

Some explanatory remarks regarding these definitions are in order. For point-wise simulation, $(s,t) \in R_{\varepsilon}$ means that any computation from s can be matched by one from t with the same labels and a point-wise weight difference of at most ε . Hence the requirement that $s \xrightarrow{\alpha,c} s'$ imply $t \xrightarrow{\alpha,d} t'$ with weight difference $|c-d| \le \varepsilon$, and that computations from the target states s', t' be matched with some (inversely) discounted point-wise distance $\varepsilon' \le \frac{\varepsilon}{\lambda}$.

For accumulated simulation, $(s,t) \in R_{\varepsilon}$ is interpreted so that any computation from s can be matched by one from t with the same labels and accumulated absolute-value weight difference at most ε . Hence we again require that $|c-d| \le \varepsilon$, but now computations from the target states have to be matched by what is left of ε after |c-d| has been used (and inverse discounting applied).

Maximum-lead simulation is slightly more complicated, because we need to keep track of the lead δ which one computation has accomplished over the other. Hence $(s,t) \in R_{\varepsilon,\delta}$ is to mean that any computation from s which starts with a lead of δ over t can be matched by a computation from t with accumulated weight difference at most ε . Thus we require that lead plus weight difference, $\delta + c - d$, be in-between $-\varepsilon$ and ε , and the new lead for computations from the target states is set to that value (again with inverse discounting applied).

For later use we collect the following easy facts about the above simulations:

2.10 Lemma.

- 1. The families \preccurlyeq^{\bullet} , \preccurlyeq^{+} and \preccurlyeq^{\pm} are the largest respective simulation families.
- 2. For $\varepsilon \leq \varepsilon'$ and $R_{\varepsilon}, R'_{\varepsilon} \in \mathcal{R}$ a point-wise or accumulating simulation family, $R_{\varepsilon} \subseteq R'_{\varepsilon}$. For $\varepsilon \leq \varepsilon'$, $-\varepsilon \leq \delta \leq \varepsilon$ and $R_{\varepsilon,\delta}, R_{\varepsilon',\delta} \in \mathcal{R}$ a maximum-lead simulation family, $R_{\varepsilon,\delta} \subseteq R_{\varepsilon',\delta}$.
- 3. For states $s, t \in S$ and $\varepsilon \leq \varepsilon'$, $s \preccurlyeq_{\varepsilon}^{\bullet} t$ implies $s \preccurlyeq_{\varepsilon'}^{\bullet} t$, $s \preccurlyeq_{\varepsilon}^{+} t$ implies $s \preccurlyeq_{\varepsilon'}^{+} t$, and $s \preccurlyeq_{\varepsilon}^{\pm} t$ implies $s \preccurlyeq_{\varepsilon'}^{\pm} t$.
- 4. For states $s, t \in S$, $s \preccurlyeq t$ implies $s \preccurlyeq_0^{\bullet} t$, $s \preccurlyeq_0^{+} t$, and $s \preccurlyeq_0^{\pm} t$.
- 5. For states $s, t \in S$, $s \not\preccurlyeq^u t$ implies $s \not\preccurlyeq^{\bullet}_{\varepsilon} t$, $s \not\preccurlyeq^{+}_{\varepsilon} t$, and $s \not\preccurlyeq^{\pm}_{\varepsilon} t$ for any ε .

2.2.3 Branching distances

We present an alternative characterization of the above simulation relations in form of recursive equations; note that these closely resemble the inequalities of Lemma 2.8:

2.11 Definition. For states $s, t \in S$, the *point-wise*, accumulated, and maximum-lead branching distances are the respective minimal fixed points to the following recursive equations:

$$\begin{split} d_{\bullet}^{\mathsf{B}}(s,t) &= \sup_{s \xrightarrow{\alpha,c} s'} \inf_{t \xrightarrow{\alpha,d} t'} \max \left(|c-d|, \lambda \, d_{\bullet}^{\mathsf{B}}(s',t') \right) \\ d_{+}^{\mathsf{B}}(s,t) &= \sup_{s \xrightarrow{\alpha,c} s'} \inf_{t \xrightarrow{\alpha,d} t'} |c-d| + \lambda \, d_{+}^{\mathsf{B}}(s',t') \\ d_{\pm}^{\mathsf{B}}(s,t) &= d_{\pm}^{\mathsf{B}}(s,t)(0) \\ &\quad \text{with } d_{\pm}^{\mathsf{B}}(s,t)(\delta) = \sup_{s \xrightarrow{\alpha,c} s'} \inf_{t \xrightarrow{\alpha,d} t'} \max \left(|\delta|, \lambda \, d_{\pm}^{\mathsf{B}}(s',t') \big(\frac{\delta+c-d}{\lambda} \big) \right) \end{split}$$

Again, some remarks regarding these definitions will be in order. First note that sup and inf are taken over the complete lattice $\mathbb{R}_{\geq 0} \cup \{\infty\}$ here, whence inf $\emptyset = \infty$ and $\sup \emptyset = 0$. Thus $d_{\bullet}^{\mathsf{B}}(s,t) = 0$ in case $s \not\to \text{and } d_{\bullet}^{\mathsf{B}}(s,t) = \infty$ in case $s \xrightarrow{\alpha,c} \text{but } t \not\to \text{for some } \alpha$, and similarly for the other distances.

The functionals defined by the first two equations above are endofunctions on the complete lattice of functions $S \times S \to \mathbb{R}_{\geq 0} \cup \{\infty\}$; they are easily shown to be monotone, hence the minimal fixed points exist. For the last equation, the functional is an endofunction on the complete lattice $\mathbb{R} \to (S \times S \to \mathbb{R}_{\geq 0} \cup \{\infty\})$, mapping each lead $\delta \in \mathbb{R}$ to a function $d_{\pm}^{\mathsf{B}}(\cdot,\cdot)(\delta)$. Also this

functional can be shown to be monotone and hence to have a minimal fixed point.

It is not difficult to see that the distances defined above are branching distances in the sense of Definition 2.4. Below we show that they are closely related to the simulations of Definition 2.9:

- **2.12 Proposition.** For states $s, t \in S$ and $\varepsilon \in \mathbb{R}_{>0}$, we have
 - $s \preccurlyeq_{\varepsilon}^{\bullet} t \text{ if and only if } d_{\bullet}^{\mathsf{B}}(s,t) \leq \varepsilon$,
 - $s \preccurlyeq_{\varepsilon}^+ t$ if and only if $d_+^{\mathsf{B}}(s,t) \leq \varepsilon$,
 - $s \preccurlyeq_{\varepsilon}^{\pm} t \text{ if and only if } d_{+}^{\mathsf{B}}(s,t) \leq \varepsilon.$

Proof: Each of the six implications involved can be shown using standard structural-induction arguments. \Box

2.3 Properties of distances

In this section we present a number of properties of the six distances introduced above.

2.3.1 Branching versus linear distance

For the qualitative relations, simulation implies trace inclusion, i.e., $s \leq^u t$ implies $s \leq^u t$, and $s \leq^t t$ implies $s \leq t$. Below we show a natural generalization of this to our quantitative setting, where implications translate to inequalities; note that an equivalent statement of the theorem is that for any ε , $d^{\mathsf{B}}(s,t) \leq \varepsilon$ implies $d^{\mathsf{L}}(s,t) \leq \varepsilon$ for all three distances considered.

2.13 Theorem. For all states $s, t \in S$, we have

$$d_{\bullet}^{\mathsf{L}}(s,t) \leq d_{\bullet}^{\mathsf{B}}(s,t)\,, \qquad d_{+}^{\mathsf{L}}(s,t) \leq d_{+}^{\mathsf{B}}(s,t)\,, \qquad d_{\pm}^{\mathsf{L}}(s,t) \leq d_{\pm}^{\mathsf{B}}(s,t)\,.$$

Proof: This follows from Lemma 2.8 by an easy structural-induction argument. \Box

Note that Example 1.2 shows that indeed, all distances in the equations above can be finite. Other, standard examples show however that WTS exist for which $s \not\leq t$ and yet $s \leq t$, hence $d^{\mathsf{B}}(s,t) = \infty$ and $d^{\mathsf{L}}(s,t) = 0$ for all three distances, showing the following theorem:

2.14 Theorem. The distances d_{\bullet}^{L} and d_{\bullet}^{B} are topologically inequivalent. Similarly, d_{+}^{L} and d_{+}^{B} , and also d_{\pm}^{L} and d_{\pm}^{B} , are topologically inequivalent.

2.3.2 Relationship between distances

The theorems below sum up the relationship between our three linear distances; note that the results depend heavily on whether or not discounting is applied. The following lemma is useful and easily shown:

2.15 Lemma. For states $s, t \in S$, we have

$$\begin{split} d_{\bullet}^{\mathsf{L}}(s,t) &\leq d_{+}^{\mathsf{L}}(s,t) \qquad d_{\pm}^{\mathsf{L}}(s,t) \leq d_{+}^{\mathsf{L}}(s,t) \qquad d_{\bullet}^{\mathsf{L}}(s,t) \leq 2 \, d_{\pm}^{\mathsf{L}}(s,t) \\ d_{\bullet}^{\mathsf{B}}(s,t) &\leq d_{+}^{\mathsf{B}}(s,t) \qquad d_{\pm}^{\mathsf{B}}(s,t) \leq d_{+}^{\mathsf{B}}(s,t) \qquad d_{\bullet}^{\mathsf{B}}(s,t) \leq 2 \, d_{\pm}^{\mathsf{B}}(s,t) \end{split}$$

The restrictions on traces mentioned below are understood to be applied to the sets Tr(s), Tr(t) in Definition 2.6.

- **2.16 Theorem.** Assume the discounting factor $\lambda = 1$.
 - 1. When restricted to traces of bounded length, the three linear distances d_{\bullet}^{L} , d_{+}^{L} and d_{\pm}^{L} are Lipschitz equivalent.
 - 2. For traces of unbounded length, the linear distances are mutually topologically inequivalent.

Proof: If the length of traces is bounded above by $N \in \mathbb{N}$, then $d_+^{\mathsf{L}}(s,t) \leq Nd_{\bullet}^{\mathsf{L}}(s,t)$ for all $s,t \in S$, and the result follows with Lemma 2.15.

For traces of unbounded length, topological inequivalence of d^{L}_{\bullet} and d^{L}_{\pm} , and of d^{L}_{\bullet} and d^{L}_{\pm} , can be shown by the following infinite WTS:

$$0 \begin{pmatrix} s & s_1 & s_2 & & s_n \\ 0 & \frac{1}{2} & & \frac{1}{4} & & & \frac{1}{2^n} & & & \cdots \end{pmatrix}$$

Here we have $d_{+}^{\mathsf{L}}(s, s_n) = d_{\pm}^{\mathsf{L}}(s, s_n) = \infty$ for all n, but for any $\delta \in \mathbb{R}_+$ there is an n for which $d_{\bullet}^{\mathsf{L}}(s, s_n) < \delta$. Similarly, topological inequivalence of d_{+}^{L} and d_{\pm}^{L} is shown by the infinite WTS below:

$$0 \left(\int_{s'}^{s} 1 - \frac{1}{2} \left(\int_{s'}^{s_1} 1 - \frac{1}{2} - \frac{1}{4} \left(\int_{s'}^{s_2} 1 - \frac{1}{4} - \cdots - \frac{1}{2^n} \left(\int_{s'}^{s_n} 1 - \frac{1}{2^n} - \cdots \right) \right) \right)$$

2.17 Theorem. For discounting factor $\lambda < 1$, the three linear distances d_{\bullet}^{L} , d_{+}^{L} and d_{+}^{L} are Lipschitz equivalent.

Proof: This is similar to the first claim of the previous theorem: For all states $s,t \in S$, we have $d_+^{\mathsf{L}}(s,t) \leq \frac{1}{1-\lambda} d_{\bullet}^{\mathsf{L}}(s,t)$, and the result follows with Lemma 2.15.

2.18 Theorem. For discounting factor $\lambda = 1$, the three branching distances d_{\bullet}^{B} , d_{+}^{B} and d_{\pm}^{B} are mutually topologically inequivalent. For $\lambda < 1$, they are Lipschitz equivalent.

Proof: The first claim can be shown using the same example WTS as for the second part of the proof of Theorem 2.16, and for the second claim we have $d_+^{\mathsf{B}}(s,t) \leq \frac{1}{1-\lambda} d_{\bullet}^{\mathsf{B}}(s,t)$ and can apply Lemma 2.15.

2.4 Conclusion

We have argued above that our proposed extension of the qualitative notion of trace inclusion and simulation to a quantitative setting is reasonable.

For the three types of distances considered in this chapter, we have seen that linear distances can easily be introduced, whereas definition of branching distances requires more work and involves fixed-point computations. Our Lemma 2.8 remedies some of these difficulties, and we expect this remedy to also be applicable for other interesting trace distances. We will show in Chapter 5 that a general procedure for obtaining branching distances from linear distances is available.

We have shown that all our three linear distances are topologically inequivalent to their corresponding branching distance, thus measure inherently different properties. Still, and analogously to the qualitative setting, the branching distance can be used as an over-approximation of the linear distance. Also, and perhaps more surprisingly, whether different linear or branching distances are mutually equivalent depends on the usage of discounting. We expect most of these results to also hold for other kinds of trace distances, see again Chapter 5.

We have mentioned earlier that in this work we concentrate on trace inclusion and simulation (asymmetric) distances, and of course similar treatment should be given to trace equivalence and bisimulation distances. Symmetric linear distances are easily defined as symmetrizations of the linear distances introduced here, but for the branching distances there are subtle differences between symmetrized simulation distances on the one hand and bisimulation distances on the other hand. The next chapter will be concerned with bisimulation distances.

3 A Quantitative Characterization of Weighted Kripke Structures in Temporal Logic¹

This chapter is concerned with weighted Kripke structures (WKS), which represent a straight-forward extension of Kripke structures with a weighted transition relation labeling each transition. It then proceeds to define a weighted version of Computation Tree Logic (WCTL) together with two different semantics which mirror the point-wise and accumulating distances of the previous chapter. It is then shown that WCTL is adequate and expressive for the corresponding bisimulation distances.

3.1 Preliminaries

As in Chapter 2, the results presented in this chapter are based on metrics on sequences of real numbers. Let $a = (a_i)$ and $b = (b_i)$ be such sequences, we then define for $\lambda \in]0,1[$ the following basic distances:

$$d_{+}(a,b) = \sum_{i} \lambda^{i} |a_{i} - b_{i}| \tag{3.1}$$

$$d_{\bullet}(a,b) = \sup_{i} \left\{ \lambda^{i} |a_{i} - b_{i}| \right\}$$
(3.2)

Throughout the chapter we will refer to (3.1) and (3.2), as well other distances based on these, as an accumulating distance and as a point-wise distance, respectively. For the remainder of this chapter we fix a discounting factor $\lambda \in]0,1[$; note that contrary to the previous chapter, we here assume $0 < \lambda < 1$.

We proceed to introduce WKS. A natural interpretation is to view the labellings as the cost of taking transitions in the structure. This extension is similar to the one presented in Chapter 2 for labeled transition systems, thus the results presented in the previous chapter are transferable to the current setting.

 $^{^1{\}rm This}$ chapter is based on the journal paper [FLT10] published in Computing and Informatics.

- **3.1 Definition.** For a finite set AP of atomic propositions, a weighted Kripke structure is a quadruple M = (S, T, L, w) where
 - S is a finite set of states,
 - $T \subseteq S \times S$ is a transition relation
 - $L: S \to 2^{AP}$ is the proposition labeling, and
 - $w: T \to \mathbb{R}_{\geq 0}$ assigns positive real-valued weights to transitions.

We write $s \to s'$ instead of $(s, s') \in T$ and $s \xrightarrow{w} s'$ to indicate w(s, s') = w.

A (weighted) path in a WKS M = (S, T, L, w) is a (possibly infinite) sequence $\sigma = ((s_0, w_0), (s_1, w_1), (s_2, w_2), \ldots)$ with $(s_i, w_i) \in S \times \mathbb{R}_{\geq 0}$ and such that $s_i \to s_{i+1}$ and $w_i = w(s_i, s_{i+1})$ for all i. We denote by Tr(s) the set of paths in M starting at state s, and by Tr(M) the set of all paths in M. Given a path σ , we write $\sigma(i) = (\sigma(i)_s, \sigma(i)_w)$ for its i'th state-weight pair, and σ^i for the suffix starting at $\sigma(i)$.

Notice that we have restricted ourselves to *finite* weighted Kripke structures here, *i.e.*, structures with a finite set of states and finitely many atomic propositions. Our characterization results in Section 3.4 only hold for such finite structures.

3.2 Example. Figure 3.1 gives a model of a simple printer as a WKS which we shall come back to again later. Resource usage is modeled as atomic propositions, and transition weights model the combined cost of the operations. Turning on the machine, it moves from the state **Off** to **Ready**, from where it can **Suspend** and wake up at a much lower cost. Input is processed in the **Receiving** state, and the chosen output form incurs different costs related to resource usage, clean-up and reset.

3.2 Weighted CTL

We now consider two interpretations of weighted CTL (WCTL), based on (3.1) and (3.2), which will encompass quantitative information by two means. First, as with TCTL and PCTL, a syntactic extension of path operators, by annotation with real weights, models requirements on path weights (the exact meaning of these are deferred to the choice of semantics). Second, satisfaction of a formula by a system is no longer interpreted in the Boolean domain $\{\top, \bot\}$, but rather assigns to a state a truth value in the domain $\mathbb{R}_{\geq 0} \cup \{\infty\}$. We will interpret 0 as an exact match, whereas ∞ indicates an incompatibility between the system and the specified atomic propositions of a formula. Any intermediate value is interpreted as a distance from an exact match. That is, a smaller distance means a closer (better) match of the specified weights in the

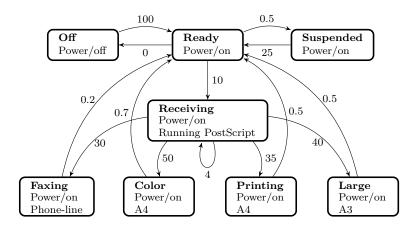


Figure 3.1: The behavior and cost and resource usage of a simple printer.

formula. We denote by $[\![\varphi]\!](s) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ the value obtained by evaluating formula φ at state s.

From here on, we fix a set AP of atomic propositions and a WKS (S, T, L, w). All definitions and results below will be given for the states of one single WKS, but we note that to relate states of different WKS, one can simply form the disjoint union.

3.3 Definition. For $p \in AP$, Φ generates the set of state formulae, and Ψ the set of path formulae, annotated by weights $c \in \mathbb{R}_{\geq 0}$, according to the following abstract syntax:

$$\Phi ::= p \mid \neg p \mid \Phi_1 \land \Phi_2 \mid \Phi_1 \lor \Phi_2 \mid \mathsf{E}\Psi \mid \mathsf{A}\Psi$$

$$\Psi ::= \mathsf{X}_c \Phi \mid \mathsf{G}_c \Phi \mid \mathsf{F}_c \Phi \mid [\Phi_1 \mathsf{U}_c \Phi_2]$$

The logic WCTL is the set of state formulae, written $\mathcal{L}_w(AP)$ or simply \mathcal{L}_w .

Before presenting the formal semantics, let us consider the usual meaning of the CTL modalities, as well as how these may be generalized to ensure adherence to bisimulation variants considered in the following section:

Given CTL propositions of the form $M, s \models \mathsf{E}\psi$ and $M, s \models \mathsf{A}\psi$, we may interpret these as infinite *existential*, respectively *universal*, quantifications over paths in M from s satisfying ψ . Similarly, $M, \sigma \models \mathsf{F}\varphi$ and $M, \sigma \models \mathsf{G}\varphi$ may be interpreted as an infinite *disjunction*, respectively *conjunction*, over propositions on the form $M, s_i \models \varphi$ for $i \geq 0$, where s_i is a state on σ .

This observation is in line with some arguments given in [KP04], and we expect that a generic approach to defining quantitative (or multi-valued) semantics for WCTL over the truth domain $\mathbb{R}_{\geq 0} \cup \{\infty\}$ is obtainable. To this end, the standard sup and inf operators are reasonable generalization of E, A, F and G (interpreted as disjunction and conjunction over the standard Boolean domain) to the complete lattice $\mathbb{R}_{\geq 0} \cup \{\infty\}$.

Furthermore, this approach requires only modification to the evaluation (i.e., semantics) of path formulae. Our semantics specializes to the usual one in two different ways: either by mapping to the designated set of truth values (i.e., to \top), all $\varepsilon < \infty$ and ∞ to \bot , or by mapping only 0 to \top and all $\varepsilon > 0$ to \bot .

3.2.1 Semantics

In the following we present two discounted semantics, derived from the distances d_+ from (3.1), and d_{\bullet} from (3.2) where weights of transition are accumulated or considered point-wise, respectively. Formally, the semantics of $\varphi \in \mathcal{L}_w$ defines a map from the set of states S to the set $\mathbb{R}_{\geq 0} \cup \{\infty\}$. The first definition gives a general weighted semantics to state formulae:

3.4 Definition (State semantics). The semantics of state formulae is defined inductively as follows:

In the last two formulae, $\llbracket \psi \rrbracket(\sigma)$ is the accumulating or point-wise semantics of σ with respect to ψ as appropriate, see below.

In the next definition, we give the two different weighted semantics to path formulae; an accumulated and a point-wise one. Note that the only difference between the two is an interchange of maximum and sum, which supports the findings in [KP04, LLM05] which advocate abstracting away from concrete operators and interpreting the semantics over general algebraic structures.

3.5 Definition (Path semantics). The *accumulating* semantics of path formulae is defined inductively as follows:

The *point-wise* semantics of path formulae is defined inductively as follows:

$$\llbracket \varphi \rrbracket_{\bullet}(\sigma) = \llbracket \varphi \rrbracket(\sigma(0)_{s})$$

$$\llbracket \mathsf{X}_{c}\varphi \rrbracket_{\bullet}(\sigma) = \max \left\{ |\sigma(0)_{w} - c|, \ \lambda \llbracket \varphi \rrbracket_{\bullet}(\sigma^{1}) \right\}$$

$$\llbracket \mathsf{F}_{c}\varphi \rrbracket_{\bullet}(\sigma) = \inf_{k} \left(\max \left\{ \max_{0 \leq j < k} \left\{ \lambda^{j} |\sigma(j)_{w} - c| \right\}, \ \lambda^{k} \llbracket \varphi \rrbracket_{\bullet}(\sigma^{k}) \right\} \right)$$

$$\llbracket \mathsf{G}_{c}\varphi \rrbracket_{\bullet}(\sigma) = \sup_{k} \left(\max \left\{ \max_{0 \leq j < k} \left\{ \lambda^{j} |\sigma(j)_{w} - c| \right\}, \ \lambda^{k} \llbracket \varphi \rrbracket_{\bullet}(\sigma^{k}) \right\} \right)$$

$$\llbracket \varphi_{1} \mathsf{U}_{c}\varphi_{2} \rrbracket_{\bullet}(\sigma) = \inf_{k} \left(\max \left\{ \max_{0 \leq j < k} \left\{ \lambda^{j} |\llbracket \varphi_{1} \rrbracket_{\bullet}(\sigma^{j}) - c| \right\}, \ \lambda^{k} \llbracket \varphi_{2} \rrbracket_{\bullet}(\sigma^{k}) \right\} \right)$$

Note that as usual, F_c can also be derived from U_c by $F_c\varphi \triangleq \mathbf{tt}U_c\varphi$ (where \mathbf{tt} is some tautology).

Compared to, for example, TCTL, the annotated operators specify an expected value, hence $X_c\varphi$ evaluated on σ means that c is expected of the first transition in σ . The difference is then added to (or the maximum is taken of it and) the value of φ over the remaining path σ^1 .

3.6 Example. In the context of the example from Figure 3.1 we consider a useful property of printers, that of having received a job, the printer cannot suspend before completing the job. The formula $\varphi = A(\neg Suspended \ U_{10} \ Ready)$ formalizes this qualitative property and also states that we expect to reach the **Ready** state using transitions with cost 10. With $\lambda = .9$, the point-wise interpretation $[\![\varphi]\!]_{\bullet}(Receiving) = 40$ is the cost (minus 10) of the transition in the computation tree which is furthest from 10. In the accumulating interpretation, $[\![\varphi]\!]_{+}(Receiving) = 48.37$ yields the sum of all such differences.

3.3 Bisimulation

We now consider extensions of *strong bisimulation* [Mil89] over WKS, based on (3.1) and (3.2). These are filling the gap between *unweighted* and *weighted* strong bisimulation as defined below; *cf.* also Def. 2.3.

- **3.7 Definition.** Let (S, T, L, w) be a WKS on a set AP of atomic propositions. A relation $R \subseteq S \times S$ is
 - an unweighted bisimulation provided that for all $(s,t) \in R$, L(s) = L(t) and
 - if $s \to s'$, then also $t \to t'$ and $(s', t') \in R$ for some $t' \in S'$,
 - if $t \to t'$, then also $s \to s'$ and $(s', t') \in R$ for some $s' \in S$;
 - a (weighted) bisimulation provided that for all $(s,t) \in R, L(s) = L(t)$ and

- if $s \xrightarrow{c} s'$, then also $t \xrightarrow{c} t'$ and $(s', t') \in R$ for some $t' \in S'$,
- if $t \xrightarrow{c} t'$, then also $s \xrightarrow{c} s'$ and $(s', t') \in R$ for some $s' \in S$.

We write $s \sim^u t$ if $(s,t) \in R$ for some unweighted bisimulation R, and $s \sim t$ if $(s,t) \in R$ for some weighted bisimulation R.

The motivation for the variants defined below is that, in order to relate structures, we do not always need perfect matching of transition weights; rather we would like to know how accurately weights are matched. As with the simulation distances of Chapter 2, we call a *bisimulation distance* any pseudometric on the states of a WKS which mediates between unweighted and weighted bisimilarity:

- **3.8 Definition.** A bisimulation distance on a WKS (S, T, L, w) is a function $d: S \times S \to \mathbb{R}_{>0} \cup \{\infty\}$ which satisfies the following for all $s_1, s_2, s_3 \in S$:
 - $d(s_1, s_1) = 0$,
 - $d(s_1, s_2) + d(s_2, s_3) \ge d(s_1, s_3)$,
 - $d(s_1, s_2) = d(s_2, s_1),$
 - $s_1 \sim s_2$ implies $d(s_1, s_2) = 0$ and
 - $d(s_1, s_2) \neq \infty$ implies $s_1 \sim^u s_2$

Our distances are based on distances of (infinite) sequences of real numbers, which is appropriate, as for (s,t) in \sim^u (or in \sim), any path $(s,a,s_1,a_1,s_2,\ldots) \in \mathsf{Tr}(s)$ must be matched by an equal-length path $(t,b,t_1,b_1,t_2,\ldots) \in \mathsf{Tr}(t)$ with (s_i,t_i) in \sim^u (respectively \sim).

By extending bisimulation with the d_+ and d_{\bullet} distances, we collect a family of relations $\{R_{\varepsilon} \subseteq S \times S\}$ (*i.e.*, a map $\mathbb{R}_{\geq 0} \to 2^{S \times S}$) since, due to discounting, for each step the distance between each successor pair may grow:

- **3.9 Definition.** A family of relations $\mathcal{R} = \{R_{\varepsilon} \subseteq S \times S \mid \varepsilon > 0\}$ is
 - an accumulating bisimulation family provided that for all $(s,t) \in R_{\varepsilon} \in \mathcal{R}$, L(s) = L(t) and
 - if $s \xrightarrow{c} s'$, then also $t \xrightarrow{d} t'$ with $|c d| \le \varepsilon$ for some $d \in \mathbb{R}_{\ge 0}$ and $(s', t') \in R_{\varepsilon'} \in \mathcal{R}$ with $\varepsilon' \lambda \le \varepsilon |c d|$, and
 - if $t \xrightarrow{c} t'$, then also $s \xrightarrow{d} s'$ with $|c d| \le \varepsilon$ for some $d \in \mathbb{R}_{\ge 0}$ and $(s', t') \in R_{\varepsilon'} \in \mathcal{R}$ with $\varepsilon' \lambda \le \varepsilon |c d|$.
 - a point-wise bisimulation family provided that for all $(s,t) \in R_{\varepsilon} \in \mathcal{R}$, L(s) = L(t) and

- if $s \stackrel{c}{\to} s'$, then also $t \stackrel{d}{\to} t'$ with $|c d| \le \varepsilon$ for some $d \in \mathbb{R}_{\ge 0}$ and $(s', t') \in R_{\varepsilon'} \in \mathcal{R}$ with $\varepsilon' \lambda \le \varepsilon$, and
- if $t \stackrel{c}{\to} t'$, then also $s \stackrel{d}{\to} s'$ with $|c d| \le \varepsilon$ for some $d \in \mathbb{R}_{\ge 0}$ and $(s', t') \in R_{\varepsilon'} \in \mathcal{R}$ with $\varepsilon' \lambda \le \varepsilon$.

We write $s \sim_{\varepsilon}^+ t$ and $s \sim_{\varepsilon}^{\bullet} t$, if $(s,t) \in R_{\varepsilon} \in \mathcal{R}$ for an accumulating, respectively point-wise, bisimulation family \mathcal{R} .

Both variants of bisimulation families give raise to a bisimulation distance in the sense of Definition 3.8 by $d_+(s,t) = \inf\{\varepsilon \mid s \sim_{\varepsilon}^+ t\}$ and $d_{\bullet}(s,t) = \inf\{\varepsilon \mid s \sim_{\varepsilon}^{\bullet} t\}$. Observe the following easy facts:

3.10 Lemma.

- 1. For $\varepsilon \leq \varepsilon'$ and members $R_{\varepsilon}, R_{\varepsilon'} \in \mathcal{R}$ of an accumulating or point-wise bisimulation family, $R_{\varepsilon} \subseteq R_{\varepsilon'}$.
- 2. Given $s \sim_{\varepsilon}^+ t$, then every path $\sigma = (s_0, w_0, s_1, w_1 s_2, \dots) \in \operatorname{Tr}(s)$ has a corresponding path $\sigma' = (t_0, w_0', t_1, w_1' t_2, \dots) \in \operatorname{Tr}(t)$ such that $\varepsilon = \varepsilon_0$ and $s_i \sim_{\varepsilon_i}^+ t_i$ for all i, where $\varepsilon_{i+1}\lambda = \varepsilon_i |w_i w_i'|$.
- 3. Given $s \sim_{\varepsilon}^{\bullet} t$, then every path $\sigma = (s_0, w_0, s_1, w_1 s_2, \dots) \in \operatorname{Tr}(s)$ has a corresponding path $\sigma' = (t_0, w'_0, t_1, w'_1 t_2, \dots) \in \operatorname{Tr}(t)$ such that $\varepsilon = \varepsilon_0$ and $s_i \sim_{\varepsilon_i}^{\bullet} t_i$ for all i, where $\varepsilon_{i+1} \lambda = \varepsilon_i$.

Note that as we only consider finite WKS, all R_{ε} relations are finite. Also, we shall speak of *corresponding paths* when referring to the second and third properties of the above lemma.

3.4 Characterization

In this section we show that the presented WCTL interpretations are adequate and expressive with respect to the appropriate bisimilarity variant.

3.4.1 Adequacy

The link between accumulating bisimilarity and our accumulating semantics for WCTL is as follows:

3.11 Theorem. For $s, t \in S$, $s \sim_{\varepsilon}^+ t$ iff $\forall \varphi \in \mathcal{L}_w : |\llbracket \varphi \rrbracket_+(s) - \llbracket \varphi \rrbracket_+(t)| \leq \varepsilon$.

The proof follows from Lemmas 3.15 and 3.16 below. Observe that this provides us with the following corollary which is precisely the standard notion of adequacy, see [HM85]:

3.12 Corollary. For $s, t \in S$, $s \sim_0^+ t$ iff $[\![\varphi]\!]_+(s) = [\![\varphi]\!]_+(t)$ for all $\varphi \in \mathcal{L}_w$.

We obtain an equivalent result for the point-wise semantics:

- **3.13 Theorem.** For $s, t \in S$, $s \sim_{\varepsilon}^{\bullet} t$ iff $\forall \varphi \in \mathcal{L}_w : |\llbracket \varphi \rrbracket_{\bullet}(s) \llbracket \varphi \rrbracket_{\bullet}(t)| \leq \varepsilon$.
- **3.14 Example.** We consider again the printer from Figure 3.1. When ignoring **Color** and **Printing** as atomic propositions, we have **Color** $\sim_{.2}^{+}$ **Printing**, as the two initial transition are the only difference. As a formula which realizes this bisimulation distance one can take $\varphi = \text{power/on} \land \text{A4} \land \text{AX}_{0.5} \text{Ready}$; then $\|\varphi\|_{+}(\text{Printing}) = 0$ and $\|\varphi\|_{+}(\text{Color}) = .2$.

The proofs of adequacy, and also of expressivity below, for the accumulating and point-wise cases are similar, hence we concentrate on the accumulating case. In the proof we will repeatedly make use of the lesser-known little brother of the triangle inequality

$$||x - y| - |x - z|| \le |y - z|$$
.

3.15 Lemma. Let $s, t \in S$ with $s \sim_{\varepsilon}^+ t$, and let $\sigma = (s, u, s_1, u_1, \ldots) \in \operatorname{Tr}(s)$, $\tau = (t, v, t_1, v_1, \ldots) \in \operatorname{Tr}(t)$ be corresponding paths. Then $|\llbracket \varphi \rrbracket_+(s) - \llbracket \varphi \rrbracket_+(t)| \leq \varepsilon$ for all state formulae φ , and $|\llbracket \varphi \rrbracket_+(\sigma) - \llbracket \varphi \rrbracket_+(\tau)| \leq \varepsilon$ for all path formulae φ .

Proof: We prove the lemma by structural induction in φ . The induction base is clear, as $s \sim_{\varepsilon}^+ t$ implies that $p \in L(s)$ if and only if $p \in L(t)$, hence $\llbracket \varphi \rrbracket_+(s) = \llbracket \varphi \rrbracket_+(t)$ for $\varphi = p$ or $\varphi = \neg p$. For the inductive step, we examine each syntactic construction in turn:

1. $\varphi = \varphi_1 \vee \varphi_2$

There are four cases to consider, corresponding to whether $[\![\varphi_1]\!]_+(s) \leq [\![\varphi_2]\!]_+(s)$ or $[\![\varphi_1]\!]_+(s) > [\![\varphi_2]\!]_+(s)$ and similarly for $[\![\varphi_1]\!]_+(t)$ and $[\![\varphi_2]\!]_+(t)$. We show the proof for one of the "mixed" cases; the other three are similar or easier:

Assume $[\![\varphi_1]\!]_+(s) \leq [\![\varphi_2]\!]_+(s)$ and $[\![\varphi_1]\!]_+(t) > [\![\varphi_2]\!]_+(t)$. Then $[\![\varphi_1]\!]_+(s) - [\![\varphi_1]\!]_+(s) - [\![\varphi_1]\!]_+(s) - [\![\varphi_2]\!]_+(t)$, and $[\![\varphi_1]\!]_+(s) - [\![\varphi_1]\!]_+(s) \leq [\![\varphi_1]\!]_+(s) - [\![\varphi_2]\!]_+(s) - [\![\varphi_2]\!]_+(t)$, and by induction hypothesis, $-\varepsilon \leq [\![\varphi_1]\!]_+(s) - [\![\varphi_1]\!]_+(t)$ and $[\![\varphi_2]\!]_+(s) - [\![\varphi_2]\!]_+(t) \leq \varepsilon$.

- 2. $\varphi = \varphi_1 \wedge \varphi_2$. This is similar to the previous case.
- 3. $\varphi = \mathsf{E}\varphi_1$

By definition of $\llbracket \mathsf{E}\varphi_1 \rrbracket_+$ there is a path $\sigma \in \mathsf{Tr}(s)$ for which $\llbracket \varphi_1 \rrbracket_+(\sigma) = \llbracket \varphi \rrbracket_+(s)$. By Lemma 3.10 there is a corresponding path $\tau \in \mathsf{Tr}(t)$, and from the induction hypothesis we know that $|\llbracket \varphi_1 \rrbracket_+(\sigma) - \llbracket \varphi_1 \rrbracket_+(\tau)| \leq \varepsilon$. Thus $|\llbracket \varphi \rrbracket_+(s) - \llbracket \varphi \rrbracket_+(t)| \leq \varepsilon$.

4. $\varphi = A\varphi_1$. This is similar to the previous case.

5. $\varphi = \mathsf{X}_c \varphi_1$

By definition, $\llbracket \varphi \rrbracket_+(\sigma) = \lambda \llbracket \varphi_1 \rrbracket_+(\sigma^1) + |c-u|$ and $\llbracket \varphi \rrbracket_+(\tau) = \lambda \llbracket \varphi_1 \rrbracket_+(\tau^1) + |c-v|$, where $\sigma = s \xrightarrow{u} \sigma^1$ and $\tau = t \xrightarrow{v} \tau^1$. Since $s \sim_{\varepsilon}^+ t$ and σ and τ correspond, we have $\sigma(1) \sim_{\varepsilon'}^+ \tau(1)$ with $\varepsilon' \lambda \leq \varepsilon - |u-v|$, and by induction hypothesis $|\llbracket \varphi_1 \rrbracket_+(\sigma^1) - \llbracket \varphi_1 \rrbracket_+(\tau^1)| \leq \varepsilon'$. Hence $|\llbracket \varphi \rrbracket_+(\sigma) - \llbracket \varphi \rrbracket_+(\tau)| \leq ||c-u| - |c-v|| + \lambda |\llbracket \varphi_1 \rrbracket_+(\sigma^1) - \llbracket \varphi_2 \rrbracket_+(\tau^1)| \leq |u-v| + \varepsilon - |u-v| = \varepsilon$.

6. $\varphi = \mathsf{F}_c \varphi_1$

Pick any $\delta > 0$, then there is $k \in \mathbb{N}$ for which $S_k = \sum_{j=0}^{k-1} \lambda^j |\sigma(j)_w - c| + \lambda^k \llbracket \phi \rrbracket_+(\sigma^k) \leq \llbracket \phi \rrbracket_+(\sigma) + \delta$. As the paths σ and τ correspond, we also have $T_k = \sum_{j=0}^{k-1} \lambda^j |\tau(j)_w - c| + \lambda^k \llbracket \phi \rrbracket_+(\tau^k) \leq \llbracket \phi \rrbracket_+(\tau) + \delta$. Repeated use of the definition of \sim_{ε}^+ yields $\sigma(k) \sim_{\varepsilon'}^+ \tau(k)$ with $\varepsilon' \lambda^k \leq \varepsilon - \sum_{j=0}^{k-1} \lambda^j |\sigma(j)_w - \tau(j)_w|$, hence by induction hypothesis, $|\llbracket \phi \rrbracket_+(\sigma^k) - \llbracket \phi \rrbracket_+(\tau^k)| \leq \varepsilon'$. Thus $|\llbracket \phi \rrbracket_+(\sigma) - \llbracket \phi \rrbracket_+(\tau)| \leq |S_k - T_k| + \delta \leq \sum_{j=0}^{k-1} \lambda^j ||\sigma(j)_w - c| - |\tau(j)_w - c|| + \lambda^k |\llbracket \phi \rrbracket_+(\sigma^k) - \llbracket \phi \rrbracket_+(\tau^k)| + \delta \leq \varepsilon + \delta$. As these considerations hold for any $\delta > 0$, we must have $|\llbracket \phi \rrbracket_+(\sigma) - \llbracket \phi \rrbracket_+(\tau)| \leq \varepsilon$.

- 7. $\varphi = \mathsf{G}_c \varphi_1$; $\varphi = \varphi_1 \mathsf{U}_c \varphi_2$. These are similar to the previous case.
- **3.16 Lemma.** Let $s, t \in S$ and assume that $|[\![\varphi]\!]_+(s) [\![\varphi]\!]_+(t)| \le \varepsilon$ for all state formulae $\varphi \in \mathcal{L}_w$. Then $s \sim_{\varepsilon}^+ t$.

Proof: This follows directly from Theorem 3.17 below, but one can also observe that the accumulating family $\mathcal{R} = \{R_{\varepsilon}\}$ defined by

$$R_{\varepsilon} = \{(s,t) \mid s,t \in S, \forall \varphi \in \mathcal{L}_w : |\llbracket \varphi \rrbracket_+(s) - \llbracket \varphi \rrbracket_+(t)| \le \varepsilon\}$$

is indeed an accumulating bisimulation in terms of Definition 3.9.

3.4.2 Expressivity

We show that WCTL with accumulating semantics is expressive with respect to accumulating bisimulation in the following sense:

3.17 Theorem. For each $s \in S$ and every $\gamma \in \mathbb{R}_+$, there exists a state formula $\varphi_{\gamma}^s \in \mathcal{L}_w$, interpreted over the accumulating semantics, which characterizes s up to accumulating bisimulation and up to γ , i.e., such that for all $s' \in S$, $s \sim_{\varepsilon}^+ s'$ if and only if $[\varphi_{\gamma}^s]_+(s') \in [\varepsilon - \gamma, \varepsilon + \gamma]$ for all γ .

Proof: We define characteristic formulae of unfoldings, as follows: For each $s \in S$ and $n \in \mathbb{N}$, denote $L(s) = \{p_1, \ldots, p_k\}$ and $AP \setminus L(s) = \{q_1, \ldots, q_\ell\}$ and let $\varphi(s, n)$ be the WCTL formula defined inductively as follows:

$$\varphi(s,0) = (p_1 \wedge \dots \wedge p_k) \wedge (\neg q_1 \wedge \dots \wedge \neg q_\ell)$$

$$\varphi(s,n+1) = \bigwedge_{s \xrightarrow{w} s'} \mathsf{EX}_w \varphi(s',n) \wedge \bigwedge_{w:s \xrightarrow{w} s'} \mathsf{AX}_w \Big(\bigvee_{s \xrightarrow{w} s'} \varphi(s',n)\Big) \wedge \varphi(s,0)$$

It is easy to see that $[\![\varphi(s,n)]\!]_+(s) = 0$ for all n.

To complete the proof, one observes that for each $\gamma > 0$, there is $n(\gamma) \in \mathbb{N}$ such that $\varphi(s, n(\gamma))$ can play the role of ϕ_{γ}^{s} in the theorem. Intuitively this is due to discounting: The further the unfolding in $\varphi(s, n)$, the higher are the weights discounted, hence from some $n(\gamma)$ on, maximum weight difference is below γ .

3.18 Theorem. For each $s \in S$ and every $\gamma \in \mathbb{R}_+$, there exists a state formula $\varphi_{\gamma}^s \in \mathcal{L}_w$, interpreted over the point-wise semantics, which characterizes s up to point-wise bisimulation and up to γ , i.e., such that for all $s' \in S$, $s \sim_{\varepsilon}^{\bullet} s'$ if and only if $[\![\varphi_{\gamma}^s]\!]_{\bullet}(s') \in [\varepsilon - \gamma, \varepsilon + \gamma]$ for all γ .

3.5 Conclusion

We have shown in this chapter that weighted CTL with an accumulating semantics is adequate and expressive for accumulating bisimulation for weighted Kripke structures. We have also seen that the same holds for the point-wise semantics for WCTL with respect to point-wise bisimulation.

We will see in Chapter 5 that these results can be lifted to a common abstract framework which also encompasses other weighted bisimulations such as the maximum-lead bisimulation of [HMP05] presented in the previous chapter.

4 Metrics for Weighted Transition Systems: Axiomatization¹

In this chapter we turn to axiomatizations of the point-wise and accumulating simulation distances. We first present axiomatizations for finite processes and then for regular processes. We then show that the axiomatizations for finite processes are sound and complete, whereas the ones for regular processes are sound and ε -complete.

4.1 Simulation distances

Throughout this chapter we fix a *finite* metric space \mathbb{K} of weights with a metric $d_{\mathbb{K}} : \mathbb{K} \times \mathbb{K} \to \mathbb{R}$. We also fix a discounting factor λ with $0 \le \lambda < 1$, which will be used in the definition of accumulating distance below.

4.1 Definition. A weighted transition system is a tuple (S, T), where S is a finite set of states and $T \subseteq S \times \mathbb{K} \times S$ is a set of (weighted) transitions.

Note that all transition systems in this chapter are indeed assumed finite, hence requiring finiteness of the metric space \mathbb{K} does not add extra restrictions.

We fix a weighted transition system (S,T) and introduce simulation distance between states in (S,T). We concentrate on two types here, accumulating and point-wise distance, but other kinds may indeed be defined.

4.1.1 Accumulating distance

4.2 Definition. For states $s, t \in S$, the accumulating simulation distance from s to t is defined to be the least fixed point to the set of equations

$$d_{+}(s,t) = \max_{s \xrightarrow{n} s'} \min_{t \xrightarrow{m} t'} \left(d_{\mathbb{K}}(n,m) + \lambda d_{+}(s',t') \right). \tag{4.1}$$

To justify this definition, we need to show that the equations (4.1) indeed have a least solution. To this end, write $S = \{s_1, \ldots, s_p\}$ and assume for

 $^{^1{\}rm This}$ chapter is based on the journal paper [LFT11] published in Theoretical Computer Science.

the moment that the transition system (S,T) is non-blocking such that every $s_i \in S$ has an outgoing transition $s_i \xrightarrow{n} s_k$ for some $s_k \in S$. Define a function $F: \mathbb{R}^{p \times p}_{\geq 0} \to \mathbb{R}^{p \times p}_{\geq 0}$ by

$$F(x)_{i,j} = \max_{s_i \xrightarrow{n} s_k} \min_{s_j \xrightarrow{m} s_\ell} \left(d_{\mathbb{K}}(n,m) + \lambda x_{k,\ell} \right).$$

Here we are using the standard linear-algebra notation $\mathbb{R}_{>0}^{p\times p}$ for $p\times p$ -matrices with entries in $\mathbb{R}_{>0}$ and $x_{k,\ell}$ for the entry in their k'th row and ℓ 'th column.

4.3 Lemma. With metric on $\mathbb{R}_{>0}^{p\times p}$ defined by $d(x,y)=\max_{i,j=1}^p|x_{i,j}-y_{i,j}|$, F is a contraction with Lipschitz constant λ .

Proof: (*Cf.* also the proof of [ZP96, Thm. 5.1].) We can partition $\mathbb{R}_{\geq 0}^{p \times p}$ into finitely many (indeed at most $2^{p^2q^2}$ with $q=|\mathbb{K}|$) closed polyhedral regions $R_{i,j}$ (some of which may be unbounded) such that for $x,y \in R_{i,j}$ in a common region, the p^2 max-min equations get resolved to the same transitions. In more precise terms, there are mappings $n, m, k, \ell : \{1, \dots, p\} \times \{1, \dots, p\} \rightarrow$ $\{1,\ldots,p\}$ such that $F(x)_{i,j}=d_{\mathbb{K}}(n(i,j),m(i,j))+\lambda x_{k(i,j),\ell(i,j)}$ for all $x\in R_{i,j}$.

Now if $x, y \in R_{i,j}$ are in a common region, then

$$\begin{split} d(F(x), F(y)) &\leq \lambda \max_{i,j} |x_{k(i,j),\ell(i,j)} - y_{k(i,j),\ell(i,j)}| \\ &\leq \lambda \max_{i,j} |x_{i,j} - y_{i,j}| = \lambda d(x,y) \,. \end{split}$$

If $x \in R_{i_1,j_1}, y \in R_{i_2,j_2}$ are in different regions, a bit more work is needed. The straight line segment between x and y admits finitely many intersection points with the regions $R_{i,j}$; denote these $x = z_0, \ldots, z_q = y$. We have

$$d(F(x), F(y)) \le d(F(z_0), F(z_1)) + \dots + d(F(z_{q-1}, z_q))$$

$$\le \lambda (d(z_0, z_1) + \dots + d(z_{q-1}, z_q)) = \lambda d(x, y).$$

Note that the last equality only holds because all z_i are on a straight line. \square

Using the Banach fixed-point theorem and completeness of $\mathbb{R}_{>0}^{p\times p}$ we can hence conclude that F has a unique fixed point. In the general case, where (S,T) may not be non-blocking, F is a function $[0,\infty] \to [0,\infty]$ with (extra) fixed point $[\infty, ..., \infty]$. Hence as a function $[0, \infty] \to [0, \infty]$, F has at most two fixed points. Now we can write the equation set from the definition as

$$\begin{bmatrix} d_{+}(s_{1}, s_{1}) & d_{+}(s_{1}, s_{2}) & \cdots & d_{+}(s_{1}, s_{p}) \\ d_{+}(s_{2}, s_{1}) & d_{+}(s_{2}, s_{2}) & \cdots & d_{+}(s_{2}, s_{p}) \\ \vdots & \vdots & \ddots & \vdots \\ d_{+}(s_{p}, s_{1}) & d_{+}(s_{p}, s_{2}) & \cdots & d_{+}(s_{p}, s_{p}) \end{bmatrix}$$

$$= F \begin{bmatrix} d_{+}(s_{1}, s_{1}) & d_{+}(s_{1}, s_{2}) & \cdots & d_{+}(s_{1}, s_{p}) \\ d_{+}(s_{2}, s_{1}) & d_{+}(s_{2}, s_{2}) & \cdots & d_{+}(s_{2}, s_{p}) \\ \vdots & \vdots & \ddots & \vdots \\ d_{+}(s_{p}, s_{1}) & d_{+}(s_{p}, s_{2}) & \cdots & d_{+}(s_{p}, s_{p}) \end{bmatrix},$$

hence (4.1) has indeed a unique least fixed point.

4.1.2 Point-wise distance

For point-wise simulation distance we follow a lattice-theoretic rather than a contraction approach.

4.4 Definition. For states $s, t \in S$, the *point-wise simulation distance* from s to t is defined to be the least fixed point to the set of equations

$$d_{\bullet}(s,t) = \max_{s \xrightarrow{n} s'} \min_{t \xrightarrow{m} t'} \max \left(d_{\mathbb{K}}(n,m), d_{\bullet}(s',t') \right).$$

Note that in this chapter, the point-wise distance is *undiscounted*. Let $G: [0, \infty]^{p \times p} \to [0, \infty]^{p \times p}$ be the function defined by

$$G(x)_{i,j} = \max_{s_i \xrightarrow{n} s_k} \min_{s_j \xrightarrow{m} s_\ell} \max (d_{\mathbb{K}}(n,m), x_{k,\ell}).$$

4.5 Lemma. With partial order on $[0,\infty]^{p\times p}$ defined by $x\leq y$ iff $x_{i,j}\leq y_{i,j}$ for all i,j,G is (weakly) increasing.

Now the Tarski fixed-point theorem allows us to conclude that G has a unique least fixed point, hence the above definition is justified.

4.1.3 Properties

4.6 Proposition. The functions d_+ and d_{\bullet} are hemimetrics on S.

Proof: To show that $d_+(s,s) = d_{\bullet}(s,s) = 0$ is trivial. The triangle inequalities can be shown inductively; we prove the one for d_+ : For $s, t, u \in S$, we have

$$\begin{aligned} d_{+}(s,t) + d_{+}(t,u) &= \max_{s \xrightarrow{n} s'} \min_{t \xrightarrow{m} t'} \left(d_{\mathbb{K}}(n,m) + \lambda d_{+}(s',t') \right) \\ &+ \max_{t \xrightarrow{m} t'} \min_{u \xrightarrow{z} u'} \left(d_{\mathbb{K}}(m,z) + \lambda d_{+}(t',u') \right) \\ &\geq \max_{s \xrightarrow{n} s'} \min_{t \xrightarrow{m} t'} \inf_{u \xrightarrow{z} u'} \left(d_{\mathbb{K}}(n,m) + d_{\mathbb{K}}(m,z) + \lambda \left(d_{+}(s',t') + d_{+}(t',u') \right) \right) \\ &\geq \max_{s \xrightarrow{n} s'} \min_{u \xrightarrow{z} u'} \left(d_{\mathbb{K}}(n,z) + \lambda d_{+}(s',u') \right) = d_{+}(s,u) \end{aligned}$$

assuming the triangle inequality has been proven for the triple (s', t', u'). \square

In the next proposition we take the standard liberty of comparing different (weighted) transition systems by considering their disjoint union.

4.7 Proposition. The weighted transition systems $\mathbf{0}$ and \mathbf{U} given as $\mathbf{0} = (\{s_1\}, \emptyset)$ and $\mathbf{U} = (\{s_1\}, \{(s_1, n, s_1) \mid n \in \mathbb{K}\})$ are respectively minimal and maximal elements with respect to both $d_+(\cdot, \cdot)$ and $d_{\bullet}(\cdot, \cdot)$, that is, $d_+(\mathbf{0}, A) = d_{\bullet}(\mathbf{0}, A) = d_+(A, \mathbf{U}) = d_{\bullet}(A, \mathbf{U}) = 0$ for any WTS A.

Proof: For $d_{+}(\mathbf{0}, A)$ and $d_{\bullet}(\mathbf{0}, A)$, the maximum $\max_{s_1} \xrightarrow{n} s'_1$ is taken over the empty set and hence is 0. For $d_{+}(A, \mathbf{U})$ and $d_{\bullet}(A, \mathbf{U})$, any transition $s \xrightarrow{n} s'$ in A can be matched by $s_1 \xrightarrow{n} s_1$ in \mathbf{U} , hence the distance is again 0.

4.2 Axiomatizations for Finite Weighted Processes

We now turn to a setting where our weighted transition systems are generated by finite or regular (weighted) process expressions. We construct a sound and complete axiomatization of simulation distance in a setting without recursion first and show afterwards how this may be extended to a setting with recursion.

Let \mathcal{P} be the set of process expressions generated by the following grammar:

$$E ::= \mathbf{0} \mid n.E \mid E + E \mid \qquad n \in \mathbb{K}$$

Here **0** is used to denote the *empty process*, *cf.* Proposition 4.7.

The semantics of finite process expressions is a weighted transition system generated by the following standard SOS rules:

$$\frac{E_1 \xrightarrow{n} E'_1}{n.E \xrightarrow{n} E} \qquad \frac{E_2 \xrightarrow{n} E'_2}{E_1 + E_2 \xrightarrow{n} E'_1} \qquad \frac{E_2 \xrightarrow{n} E'_2}{E_1 + E_2 \xrightarrow{n} E'_2}$$

$$(A1) \frac{}{\vdash [\mathbf{0}, E] \bowtie r} \quad 0 \bowtie r \qquad (A2) \frac{}{\vdash [n.E, \mathbf{0}] \bowtie r} \quad \infty \bowtie r$$

$$(R1) \frac{\vdash [E, F] \bowtie r_1}{\vdash [n.E, m.F] \bowtie r} \quad d_{\mathbb{K}}(n, m) + \lambda r_1 \bowtie r$$

$$(R2) \frac{\vdash [E_1, F] \bowtie r_1 \quad \vdash [E_2, F] \bowtie r_2}{\vdash [E_1 + E_2, F] \bowtie r} \quad \max(r_1, r_2) \bowtie r$$

$$(R3) \frac{\vdash [n.E, F_1] \bowtie r_1 \quad \vdash [n.E, F_2] \bowtie r_2}{\vdash [n.E, F_1 + F_2] \bowtie r} \quad \min(r_1, r_2) \bowtie r$$

Figure 4.1: The \mathfrak{F} proof system.

We can immediately get the following equalities

$$d_{+}(E + \mathbf{0}, E) = 0$$

$$d_{+}(n.E, m.F) = d_{\mathbb{K}}(n, m) + \lambda d_{+}(E, F)$$

$$d_{+}(E_{1} + E_{2}, F) = \max(d_{+}(E_{1}, F), d_{+}(E_{2}, F))$$

$$d_{+}(n.E, F_{1} + F_{2}) = \min(d_{+}(n.E, F_{1}), d_{+}(n.E, F_{2}))$$

$$(4.3)$$

For the point-wise distance, we again need only exchange (4.2) with

$$d_{\bullet}(n.E, m.F) = \max(d_{\mathbb{K}}(n, m), d_{\bullet}(E, F))$$

In order to show for example (4.3) we simply need to apply the definitions:

$$\begin{aligned} d_{+}(n.E, F_{1} + F_{2}) &= \inf_{F_{1} + F_{2} \xrightarrow{m} F'} d_{\mathbb{K}}(n, m) + \lambda \, d_{+}(E, F') \\ &= \min \begin{cases} \inf_{F_{1} \xrightarrow{m} F'} d_{\mathbb{K}}(n, m) + \lambda \, d_{+}(E, F') \\ \inf_{F_{2} \xrightarrow{m} F'} d_{\mathbb{K}}(n, m) + \lambda \, d_{+}(E, F') \end{cases} \\ &= \min \left(d_{+}(n.E, F_{1}), d_{+}(n.E, F_{2}) \right) \end{aligned}$$

For (4.2), the sup-inf expression ranges over singleton sets, hence the result is easy; the remaining equalities may shown in a similar way.

The inference system \mathfrak{F} as given in Figure 4.1 axiomatizes accumulating simulation distance for finite processes, as we shall prove below. Its sentences are inequalities of the form $[E,F]\bowtie r$ where $\bowtie\in\{=,\leq,\geq\}$ and $0\leq r\leq\infty$. Whenever $[E,F]\bowtie r$ may be concluded from \mathfrak{F} , we write $\vdash_{\mathfrak{F}} [E,F]\bowtie r$.

In addition to reflexivity and transitivity, we will need the following standard properties of \bowtie in latter proofs of soundness and completeness: Whenever $a \bowtie b$ then, for all $c: a + c \bowtie b + c$, $a \cdot c \bowtie b \cdot c$, and $\max\{a, c\} \bowtie \max\{b, c\}$.

We also remark that the left process indeed needs to be guarded in rule (R3) above, *i.e.*, the following proposed rule (R3') leads to an unsound inference system:

$$(R3') \frac{\vdash [E, F_1] \bowtie r_1 \qquad \vdash [E, F_2] \bowtie r_2}{\vdash [E, F_1 + F_2] \bowtie r} \quad \min(r_1, r_2) \bowtie r$$

Indeed, using this rule we can derive the following (incomplete) proof tree with a contradictory conclusion; the reason behind is that with $E = E_1 + E_2$ non-deterministic as below, both F_1 and F_2 may be needed to answer the challenge posed by E:

$$\frac{ \vdash [1.\mathbf{0}, 1.\mathbf{0}] \ge 0 \qquad \vdash [2.\mathbf{0}, 1.\mathbf{0}] \ge 1}{ \vdash [1.\mathbf{0} + 2.\mathbf{0}, 1.\mathbf{0}] \ge 1} \qquad \frac{ \vdash [1.\mathbf{0}, 2.\mathbf{0}] \ge 1 \qquad \vdash [2.\mathbf{0}, 2.\mathbf{0}] \ge 0}{ \vdash [1.\mathbf{0} + 2.\mathbf{0}, 1.\mathbf{0} + 2.\mathbf{0}] \ge 1}$$

4.8 Theorem (Soundness). If $\vdash_{\mathfrak{F}} [E, F] \bowtie r$, then $d_+(E, F) \bowtie r$.

Proof: By an easy induction in the proof tree for $\vdash_{\mathfrak{F}} [E, F] \bowtie r$, with a case analysis for the applied proof rule:

- (A1) follows from $d_{+}(\mathbf{0}, E) = 0$.
- (A2) follows from $d_{+}(n.E, \mathbf{0}) = \infty$ which is clear by the definition of d_{+} .
- (R1) By induction hypothesis we have $d_+(E, F) \bowtie r_1$, and as $d_+(n.E, m.F) = d_{\mathbb{K}}(n, m) + \lambda d_+(E, F)$, it follows that $d_+(n.E, m.F) = d_{\mathbb{K}}(n, m) + \lambda r_1$.
- (R2) By induction hypothesis, $d_{+}(E_{1}, F) \bowtie r_{1}$ and $d_{+}(E_{2}, F) \bowtie r_{2}$, hence $d_{+}(E_{1} + E_{2}, F) = \max(d_{+}(E_{1}, F), \max\{E_{2}, F\}) \bowtie \max(r_{1}, r_{2})$.
- (R3) By induction hypothesis, $d_+(n.E, F_1) \bowtie r_1$ and $d_+(n.E, F_2) \bowtie r_2$, hence $d_+(n.E, F_1 + F_2) = \min(d_+(n.E, F_1), d_+(n.E, F_2)) \bowtie \min(r_1, r_2)$.
- **4.9 Theorem (Completeness).** If $d_+(E,F) \bowtie r$, then $\vdash_{\mathfrak{F}} [E,F] \bowtie r$.

Proof: By an easy structural induction on E:

$$(E = \mathbf{0})$$
 We have $d_+(\mathbf{0}, F) = 0 \bowtie r$. By Axiom (A1), also $\vdash [\mathbf{0}, F] = 0$.

(E = n.E') We use an inner induction on F:

Case
$$F = 0$$
: Here $d_+(E, F) = d_+(n.E', 0) = \infty \bowtie r$. By Axiom (A2), also $\vdash [n.E', 0] = \infty$.

Case F = m.F': Here $d_{+}(E,F) = d_{+}(n.E',m.F') = d_{\mathbb{K}}(n,m) + \lambda d_{+}(E',F') \bowtie r$, hence with $r' = \lambda^{-1}(r - d_{\mathbb{K}}(n,m)), d_{+}(E',F') \bowtie r'$. By induction hypothesis it follows that $\vdash [E',F'] \bowtie r'$, and we can use Axiom (R1) to conclude that $\vdash [E,F] \bowtie r$.

Case $F = F_1 + F_2$: Using (4.3), we have $d_+(E, F) = d_+(n.E', F_1 + F_2) = \min(d_+(n.E', F_1), d_+(n.E', F_2))$. Let $d_+(n.E', F_1) \bowtie r_1$ and $d_+(n.E', F_2) \bowtie r_2$. By the previous case, we know $\vdash [n.E', F_1] \bowtie r_1$. As $\min\{r_1, r_2\} \bowtie r$ it follows using (R3) that $\vdash_{\mathfrak{F}} [n.E, F_1 + F_2] \bowtie r$.

 $(E = E_1 + E_2)$ By an argument similar to the one in the preceding subcase, we have $d_+(E,F) = \max(d_+(E_1,F),d_+(E_2,F))$. If $d_+(E_1,F) \bowtie r_1$ and $d_+(E_2,F) \bowtie r_2$ with $\max(r_1,r_2) \bowtie r$, we can use the induction hypothesis to conclude $\vdash [E_1,F] \bowtie r_1$ and $\vdash [E_1,F] \bowtie r_1$, whence $\vdash [E,F] \bowtie r$ by Axiom (R2).

4.2.1 Point-wise distance

We can devise a sound and complete inference system \mathfrak{F}^{\bullet} for point-wise distance (instead of accumulating) by replacing inference rule (R1) in System \mathfrak{F} by the rule

$$(\mathbf{R} \mathbf{1}^{\bullet}) \ \frac{\vdash [E, F] \bowtie r_1}{\vdash [n.E, m.F] \bowtie r} \quad \max(d_{\mathbb{K}}(n, m), \lambda r_1) \bowtie r$$

As before, we write $\vdash_{\mathfrak{F}^{\bullet}} [E, F] \bowtie r$ if $[E, F] \bowtie r$ can be proven by \mathfrak{F}^{\bullet} .

4.10 Theorem (Soundness & Completeness). $\vdash_{\mathfrak{F}^{\bullet}} [E,F] \bowtie r \text{ if and only if } d_{\bullet}(E,F) \bowtie r$

Proof: The proof is similar to the one for \mathfrak{F} .

4.2.2 Simulation distance zero

We show here that for distance zero, our inference system \mathfrak{F} specializes to a sound and complete inference system for simulation. The inference system \mathfrak{F}_0 is displayed in Figure 4.2.

4.11 Theorem (Soundness & Completeness). $\vdash_{\mathfrak{F}_0} E \preccurlyeq F$ if and only if $E \preccurlyeq F$.

Proof: Soundness follows immediately from the soundness of Proof system \mathfrak{F} , and for completeness we note that the arguments one uses in the inductive proof of Theorem 4.9 all specialize to distance zero.

$$(R1_0) \frac{\vdash \mathbf{0} \preccurlyeq E}{\vdash n.E \preccurlyeq F}$$

$$(R2_0) \frac{\vdash E \preccurlyeq F}{\vdash n.E \preccurlyeq n.F}$$

$$(R3_0') \frac{\vdash E_1 \preccurlyeq F \qquad \vdash E_2 \preccurlyeq F}{\vdash E_1 + E_2 \preccurlyeq F}$$

$$(R3_0') \frac{\vdash n.E \preccurlyeq F_1}{\vdash n.E \preccurlyeq F_1 + F_2} \qquad (R3_0) \frac{\vdash n.E \preccurlyeq F_2}{\vdash n.E \preccurlyeq F_1 + F_2}$$

Figure 4.2: The \mathfrak{F}_0 proof system

We remark that, contrary to the situation for general distance above, we may indeed replace the guarded process n.E in $(R3'_0)$ and $(R3_0)$ by a plain E without invalidating the rules. Note also that \mathfrak{F}_0 may similarly be obtained as a specialization \mathfrak{F}_0^{\bullet} of the axiomatization \mathfrak{F}_0^{\bullet} of point-wise distance above.

4.3 Axiomatizations for Regular Weighted Processes

Let $N = \max\{d_{\mathbb{K}}(n,m) \mid n,m \in \mathbb{K}\}$; by finiteness of \mathbb{K} , $N \in \mathbb{R}$. Let V be a fixed set of variables, then \mathcal{P}^R is the set of process expressions generated by the following grammar:

$$E ::= \mathbf{U} \mid X \mid n.E \mid E + E \mid \mu X.E \qquad n \in \mathbb{K}, X \in V$$

Here we use **U** to denote the *universal process* recursively offering any weight in \mathbb{K} , cf. Proposition 4.7. Note that we do not incorporate the empty process **0**. Semantically this will mean that all processes in \mathcal{P}^R are non-terminating, and that the accumulating distance between any pair of processes is finite. The reason for the exchange of **0** with **U** is precisely this last property; specifically, completeness of our axiomatization (Theorem 4.16) can only be shown if all accumulating distances are finite.

The semantics of processes in \mathcal{P}^R is given as weighted transition systems which are generated by the following standard SOS rules:

$$\begin{array}{ccc}
\hline
n.E \xrightarrow{n} E & \overline{\mathbf{U}} \xrightarrow{n} \mathbf{U} \\
\hline
E_1 \xrightarrow{n} E'_1 & E_2 \xrightarrow{n} E'_2 & E[\mu X.E/X] \xrightarrow{n} F \\
\hline
E_1 + E_2 \xrightarrow{n} E'_1 & E_1 + E_2 \xrightarrow{n} E'_2 & \mu X.E \xrightarrow{n} F
\end{array}$$

As usual we say that a variable X is guarded in an expression E if any occurrence of X in E is within a subexpression n.E'. Formally, we define the $guarding\ depth\ \mathsf{gd}(E,X)$ of variable X in expression E recursively by

$$\begin{split} \operatorname{gd}(\mathbf{U},X) &= \infty \\ \operatorname{gd}(X,X) &= 0 \\ \operatorname{gd}(n.E,X) &= 1 + \operatorname{gd}(E,X) \\ \operatorname{gd}(E_1 + E_2,X) &= \min\big(\operatorname{gd}(E_1,X),\operatorname{gd}(E_2,X)\big) \\ \operatorname{gd}(\mu X.E,Y) &= \begin{cases} \operatorname{gd}(E,X) & \text{if } X \neq Y \\ \infty & \text{if } X = Y \end{cases} \end{split}$$

and we say that X is guarded in E if $gd(E, X) \geq 1$.

Also as usual, we denote by E[F/X] the expression derived from E by substituting all free occurrences of variable X in E by F, and given tuples $\bar{F} = (F_1, \ldots, F_k)$, $\bar{X} = (X_1, \ldots, X_k)$, we write $E[\bar{F}/\bar{X}] = E[F_1/X_1, \ldots, F_k/X_k]$ for the simultaneous substitution.

Our inference system for regular processes consists of the set of rules \mathfrak{R} as shown in Figure 4.3; whenever $[E, F] \bowtie r$ may be concluded from \mathfrak{R} , we write $\vdash_{\mathfrak{R}} [E, F] \bowtie r$.

Compared to inference system \mathfrak{F} for finite processes, we note that we have to include the triangle inequality (R4) as an inference rule. Also, the precongruence property of simulation distance is expressed by rules (R1), (R5), and (R6). We will need all those extra rules in the proof of Lemma 4.13 which again is necessary for showing completeness.

4.12 Theorem (Soundness). For closed expressions $E, F \in \mathcal{P}^R$ we have that $\vdash_{\mathfrak{R}} [E, F] \bowtie r$ implies $d_+(E, F) \bowtie r$.

Proof: By an easy induction in the proof tree for $\vdash_{\mathfrak{R}} [E, F] \bowtie r$, using the definition of $d_+(\cdot, \cdot)$. In relation to Axiom (A3), we note that $N = \max\{d_{\mathbb{K}}(n,m) \mid n,m \in \mathbb{K}\}$ implies $d_+(E,F) \leq \sum_{i=0}^{\infty} \lambda^i N = \frac{N}{1-\lambda}$.

Our completeness result for regular processes will be based on the following lemmas; here we call an expression $E \in \mathcal{P}^R$ non-recursive if it does not contain any subexpressions $\mu X.E'$:

4.13 Lemma. For all $E \in \mathcal{P}^R$ and $k \in \mathbb{N}$ there exist a non-recursive expression F and tuples $\bar{E} = (E_1, \ldots, E_k)$, $\bar{X} = (X_1, \ldots, X_k)$ for which $\mathsf{gd}(F, X_i) \geq k$ for all i and

$$\vdash_{\mathfrak{R}} [E, F[\bar{E}/\bar{X}]] = 0\,, \qquad \vdash_{\mathfrak{R}} [F[\bar{E}/\bar{X}], E] = 0\,.$$

Proof: Repeated use of the unfolding axioms (A6) and (A7), the congruence rules (R1), (R5), and (R6) with r = 0 and of the triangle inequality (R4). \square

$$(A3) \frac{1}{\vdash [E, F] \leq r} \frac{N}{1-\lambda} \leq r$$

$$(A4) \frac{1}{\vdash [U, \sum_{n \in \mathbb{K}} n. \mathbf{U}] = 0}$$

$$(A5) \frac{1}{\vdash [\sum_{n \in \mathbb{K}} n. \mathbf{U}, \mathbf{U}] = 0} \frac{(A6)}{\vdash [\mu X. E, E[\mu X. E/X]] = 0}$$

$$(A7) \frac{1}{\vdash [E[\mu X. E/X], \mu X. E] = 0} \frac{(A8)}{\vdash [E, \mathbf{U}] = 0}$$

$$(R1) \frac{1}{\vdash [n. E, F] \bowtie r_1} \frac{1}{\vdash [n. E, m. F] \bowtie r} \frac{1}{\vdash [n. E, m. F] \bowtie r} \frac{1}{\vdash [E_1 + E_2, F] \bowtie r_2} \frac{1}{\vdash [E_1 + E_2, F] \bowtie r} \frac{1}{\vdash [n. E, F_1] \bowtie r} \frac{1}{\vdash [E, F] \leq r} \frac{1}$$

Figure 4.3: The \Re proof system

4.14 Lemma. Let F be a non-recursive expression and $\bar{E} = (E_1, \dots, E_k)$, $\bar{X} = (X_1, \dots, X_k)$ tuples for which $gd(F, X_i) \geq k$ for all i. Then

$$\vdash_{\mathfrak{R}} [F[\bar{E}/\bar{X}], F[\bar{\mathbf{U}}/\bar{X}]] = 0\,, \qquad \vdash_{\mathfrak{R}} [F[\bar{\mathbf{U}}/\bar{X}], F[\bar{E}/\bar{X}]] = \lambda^k \tfrac{N}{1-\lambda}\,.$$

Proof: Repeated use of Axioms (A3) and (A8) together with the congruence rules (R1), (R5), and (R6) with r = 0.

4.15 Lemma. For closed non-recursive expressions E, F, $d_+(E,F) \bowtie r$ implies $\vdash_{\mathfrak{R}} [E,F] \bowtie r$.

Proof: By structural induction similar to the proof of Theorem 4.9. \Box

We are now in a position to state our completeness result which enables arbitrary ε -close proofs in the sense below. The proof uses unfoldings of recursive expressions as in Lemma 4.13, and as these unfoldings are *finite* non-recursive processes, we cannot expect exact completeness.

4.16 Theorem (Completeness up to \varepsilon). Let E and F be closed expressions of \mathcal{P}^R and $\varepsilon > 0$. Then $d_+(E,F) = r$ implies $\vdash_{\mathfrak{R}} [E,F] \leq r + \varepsilon$ and $\vdash_{\mathfrak{R}} [E,F] \geq r - \varepsilon$.

Proof: Assume $d_+(E, F) = r$, and choose $k \in \mathbb{N}$ such that $2\lambda^k \frac{N}{1-\lambda} \leq \varepsilon$. By Lemma 4.13 we have non-recursive expressions E', F' and tuples \bar{E} , \bar{F} , \bar{X} , and \bar{Y} for which $gd(E', X_i) \geq k$ and $gd(F', Y_i) \geq k$ for all i, and such that

$$\vdash_{\mathfrak{R}} [E, E'[\bar{E}/\bar{X}]] = 0, \qquad \qquad \vdash_{\mathfrak{R}} [E'[\bar{E}/\bar{X}], E] = 0,$$

$$\vdash_{\mathfrak{R}} [F, F'[\bar{F}/\bar{Y}]] = 0, \qquad \qquad \vdash_{\mathfrak{R}} [F'[\bar{F}/\bar{Y}], F] = 0.$$

From Lemma 4.14 it follows that

$$\begin{split} & \vdash_{\mathfrak{R}} [E'[\bar{E}/\bar{X}], E'[\bar{\mathbf{U}}/\bar{X}]] = 0 \,, \\ & \vdash_{\mathfrak{R}} [E'[\bar{\mathbf{U}}/\bar{X}], E'[\bar{E}/\bar{X}]] \leq \lambda^k \frac{N}{1-\lambda} = \frac{\varepsilon}{2} \,, \\ & \vdash_{\mathfrak{R}} [F'[\bar{F}/\bar{Y}], F'[\bar{\mathbf{U}}/\bar{Y}]] = 0 \,, \\ & \vdash_{\mathfrak{R}} [F'[\bar{\mathbf{U}}/\bar{Y}], F'[\bar{F}/\bar{Y}]] \leq \lambda^k \frac{N}{1-\lambda} = \frac{\varepsilon}{2} \,. \end{split}$$

Using the triangle inequality and Theorem 4.12 we now have

$$\begin{aligned} d_{+}(E'[\bar{\mathbf{U}}/\bar{X}], F'[\bar{U}/\bar{X}]) &\leq d_{+}(E'[\bar{\mathbf{U}}/\bar{X}], E'[\bar{E}/\bar{X}]) + d_{+}(E'[\bar{E}/\bar{X}], E) \\ &+ d_{+}(E, F) + d_{+}(F, F'[\bar{F}/\bar{Y}]) \\ &+ d_{+}(F'[\bar{F}/\bar{Y}], F'[\bar{\mathbf{U}}/\bar{Y}]) \\ &\leq \frac{\varepsilon}{2} + 0 + r + 0 + 0 = r + \frac{\varepsilon}{2} \,. \end{aligned}$$

Only non-recursive expressions are involved here, so that we can invoke Lemma 4.15 to conclude

$$\vdash_{\mathfrak{R}} [E'[\bar{\mathbf{U}}/\bar{X}], F'[\bar{U}/\bar{X}]] \leq r + \frac{\varepsilon}{2}.$$

Now we can use the triangle inequality axiom (R4) together with the eight equations above to arrive at

$$\vdash_{\mathfrak{R}} [E, F] \leq r + \varepsilon$$
.

Similar arguments show that also $\vdash_{\Re} [E, F] \geq r - \varepsilon$,

4.3.1 Point-wise distance

Again we can easily convert our proof system \mathfrak{R} into one for point-wise (instead of accumulating) distance. In this case, we obtain \mathfrak{R}^{\bullet} by replacing inference rule (R1) by (R1 $^{\bullet}$) as we did for Proof system \mathfrak{F} , and (A3) needs to be replaced by

$$(A3^{\bullet})$$
 $\overline{\vdash [E, F] \leq N}$

With these replacements we have a sound and ε -complete axiomatization of point-wise simulation distance for recursive weighted processes:

4.17 Theorem (Soundness & Completeness up to ε). Let E and F be closed expressions of \mathcal{P}^R , then $\vdash_{\mathfrak{R}^{\bullet}} [E, F] \bowtie r$ implies $d_{\bullet}(E, F) \bowtie r$, and $d_{\bullet}(E, F) = r$ implies $\vdash_{\mathfrak{R}^{\bullet}} [E, F] \leq r + \varepsilon$ and $\vdash_{\mathfrak{R}^{\bullet}} [E, F] \geq r - \varepsilon$ for any $\varepsilon > 0$.

Proof: The proof is similar to that for accumulated distance. \Box

5 The Quantitative Linear-Time—Branching-Time Spectrum¹

This chapter generalizes the work presented so far in several ways and develops a general theory of linear and branching distances depending on a given, but unspecified, trace distance. It introduces quantitative Ehrenfeucht-Fraïssé games as a central tool for this generalization and then proceeds to define a spectrum of linear and branching distances which generalizes the one of van Glabbeek [vG01].

5.1 Traces, Trace Distances, and Transition Systems

For a finite non-empty sequence $a=(a_0,\ldots,a_n)$, we write $\operatorname{last}(a)=a_n$ and $\operatorname{len}(a)=n+1$ for the length of a; for an infinite sequence a we let $\operatorname{len}(a)=\infty$. Concatenation of finite sequences a and b is denoted $a \cdot b$. We denote by $a^k=(a_k,a_{k+1},\ldots)$ the k-shift and by a_i the (i+1)st element of a (finite or infinite) sequence, and by ε the empty sequence.

Throughout this chapter we fix a set \mathbb{K} of labels, and we let $\mathbb{K}^{\infty} = \mathbb{K}^* \cup \mathbb{K}^{\omega}$ denote the set of finite and infinite traces (*i.e.*, sequences) in \mathbb{K} . A hemimetric $d^{\mathsf{T}} : \mathbb{K}^{\infty} \times \mathbb{K}^{\infty} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is called a *trace distance* if $\mathsf{len}(\sigma) \neq \mathsf{len}(\tau)$ implies $d^{\mathsf{T}}(\sigma, \tau) = \infty$.

A labeled transition system (LTS) is a pair (S,T) consisting of states S and transitions $T \subseteq S \times \mathbb{K} \times S$. We often write $s \xrightarrow{x} t$ to signify that $(s,x,t) \in T$. Given $e = (s,x,t) \in T$, we write src(e) = s and tgt(e) = t for the source and target of e. A path in (S,T) is a finite or infinite sequence $\pi = ((s_0,x_0,t_0),(s_1,x_1,t_1),\ldots)$ of transitions $(s_j,x_j,t_j) \in T$ which satisfy $t_j = s_{j+1}$ for all j. We denote by $tr(\pi) = (x_0,x_1,\ldots)$ the trace induced by such a path π . For $s \in S$ we denote by $tr(\pi) = (s_0,s_1,\ldots)$ the set of (finite or infinite) paths from s and by $tr(s) = (tr(\pi) \mid \pi \in \mathsf{Pa}(s))$ the set of traces from s.

 $^{^1{\}rm This}$ chapter is based on the journal paper [FL14b] published in Theoretical Computer Science.

5.2 Examples of Trace Distances

We give a systematic treatment of trace distances with which our quantitative framework can be instantiated. Some of them have appeared in previous chapters; some others are new, but have been used elsewhere in the literature.

Most of the trace distances one finds in the literature are defined by giving a hemimetric d on \mathbb{K} and a method to combine the so-defined distances on individual symbols to a distance on traces. Three general methods are used for this combination:

- The point-wise trace distance: $PW_{\lambda}(d)(\sigma, \tau) = \sup_{i} \lambda^{j} d(\sigma_{i}, \tau_{i});$
- the accumulating trace distance: $AC_{\lambda}(d)(\sigma,\tau) = \sum_{j} \lambda^{j} d(\sigma_{j},\tau_{j});$
- The limit-average trace distance: $AV(d)(\sigma, \tau) = \liminf_{j \to 1} \sum_{i=0}^{j} d(\sigma_i, \tau_i)$.

Note that the trace distances are parametrized by the label distance $d : \mathbb{K} \times \mathbb{K} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$. Also, λ is a discounting factor with $0 < \lambda \leq 1$, and we assume that the involved traces have equal length; otherwise any trace distance has value ∞ . The point-wise distance thus measures the (discounted) greatest individual symbol distance in the traces, whereas accumulating and limit-average distance accumulate these individual distances along the traces.

If the distance on \mathbb{K} is the discrete distance given by $d_{\mathrm{disc}}(x,x)=0$ and $d_{\mathrm{disc}}(x,y)=\infty$ for $x\neq y$, then all trace distances above agree, for any λ . This defines the discrete trace distance $d_{\mathrm{disc}}^{\mathsf{T}}=\mathsf{PW}_{\lambda}(d_{\mathrm{disc}})=\mathsf{AC}_{\lambda}(d_{\mathrm{disc}})=\mathsf{AV}(d_{\mathrm{disc}})$ given by $d_{\mathrm{disc}}^{\mathsf{T}}(\sigma,\tau)=0$ if $\sigma=\tau$ and ∞ otherwise. We will show below that for the discrete trace distance, our quantitative linear-time-branching-time spectrum specializes to the qualitative one of [vG01].

If one lets d(x,x) = 0 and d(x,y) = 1 for $x \neq y$ instead, then $\mathsf{AC}_1(d)$ is $Hamming\ distance\ [Ham50]$ for finite traces, and $\mathsf{AC}_{\lambda}(d)$ with $\lambda < 1$ and $\mathsf{AV}(d)$ are two sensible ways to define Hamming distance also for infinite traces. $\mathsf{PW}_1(d)$ is topologically equivalent to the discrete distance; indeed, $\mathsf{PW}_1(d)(\sigma,\tau) = 1$ iff $d^\mathsf{T}_{\mathrm{disc}}(\sigma,\tau) = \infty$.

A generalization of the above distances may be obtained by equipping \mathbb{K} with a preorder $\preceq \subseteq \mathbb{K} \times \mathbb{K}$ indicating that a label $x \in \mathbb{K}$ may be replaced by any $y \in \mathbb{K}$ with $x \preceq y$, as for example in [Tho87]. If we define d(x,y) = 0 if $x \preceq y$ and $d(x,y) = \infty$ otherwise (note that this is a hemimetric which is not necessarily symmetric), then again $\mathsf{PW}_{\lambda}(d) = \mathsf{AC}_{\lambda}(d) = \mathsf{AV}(d)$ for any λ .

Point-wise and accumulating distances have been studied in a number of papers [dAFS09, ČHR12, CDH10, vB05] and in previous chapters. $PW_1(d)$ is the point-wise distance from [dAFS09, DLT08], and $PW_{\lambda}(d)$ for $\lambda < 1$ is the discounted distance from [dAFS09, dAHM03]. Accumulating distance $AC_{\lambda}(d)$ has been studied in [dAFS09], and AV(d) in [CDH10, ČHR12]. Both $AC_{\lambda}(d)$ and AV(d) are well-known from the theory of discounted and mean-payoff games [EM79, ZP96].

All distances above were obtained from distances on individual symbols in \mathbb{K} . A trace distance for which this is *not* the case is the *maximum-lead* distance from [HMP05] defined for $\mathbb{K} \subseteq \Sigma \times \mathbb{R}$, where Σ is an alphabet. Writing $x \in \mathbb{K}$ as $x = (x^{\ell}, x^{w})$, it is given by

$$d_{\pm}^{\mathsf{T}}(\sigma,\tau) = \begin{cases} \sup_{j} \left| \sum_{i=0}^{j} \sigma_{i}^{w} - \sum_{i=0}^{j} \tau_{i}^{w} \right| & \text{if } \sigma_{j}^{\ell} = \tau_{j}^{\ell} \text{ for all } j, \\ \infty & \text{otherwise.} \end{cases}$$

As this measures differences of accumulated labels along runs, it is especially useful for real-time systems, *cf.* [HMP05, FL12].

As a last example of a trace distance we mention the *Cantor* distance given by $d_{\rm C}^{\rm T}(\sigma,\tau)=(1+\inf\{j\mid\sigma_j\neq\tau_j\})^{-1}$. Cantor distance hence measures the (inverse of the) length of the common prefix of the sequences and has been used for verification e.g., in [DHLN10]. Both Hamming and Cantor distance have applications in information theory and pattern matching.

We will return to our example trace distances in Section 5.7 to show how our framework may be applied to yield concrete formulations of distances in the linear-time-branching-time spectrum relative to these.

5.3 Quantitative Ehrenfeucht-Fraïssé Games

To lift the linear-time—branching-time spectrum to the quantitative setting, we define below a quantitative Ehrenfeucht-Fraïssé game [Ehr61, Fra54] which is similar to the well-known bisimulation game of [Sti95].

Let (S,T) be a LTS and $d^{\mathsf{T}}: \mathbb{K}^{\infty} \times \mathbb{K}^{\infty} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ a trace distance.

The intuition of the game is as follows: The two players, with Player 1 starting the game, alternate to choose transitions, or moves, in T, starting with transitions from given start states s and t and continuing their choices from the targets of the transitions chosen in the previous step. At each of his turns, Player 1 also makes a choice whether to choose a transition from the target of his own previous choice, or from the target of his opponent's previous choice (to "switch paths"). We use a $switch\ counter$ to keep track of how often Player 1 has chosen to switch paths. Player 2 has then to respond with a transition from the remaining target. This game is played for an infinite number of rounds, or until one player runs out of choices, thus building two finite or infinite paths. The value of the game is then the trace distance of the traces of these two paths.

We proceed to formalize the above intuition. A Player-1 configuration of the game is a tuple $(\pi, \rho, m) \in T^n \times T^n \times \mathbb{N}$, for $n \in \mathbb{N}$, such that for all $i \in \{0, \ldots, n-2\}$, either $src(\pi_{i+1}) = tgt(\pi_i)$ and $src(\rho_{i+1}) = tgt(\rho_i)$, or $src(\pi_{i+1}) = tgt(\rho_i)$ and $src(\rho_{i+1}) = tgt(\pi_i)$. Similarly, a Player-2 configuration is a tuple $(\pi, \rho, m) \in T^{n+1} \times T^n \times \mathbb{N}$ such that for all $i \in \{0, \ldots, n-2\}$, either $src(\pi_{i+1}) = tgt(\pi_i)$ and $src(\rho_{i+1}) = tgt(\rho_i)$, or $src(\pi_{i+1}) = tgt(\rho_i)$ and

 $src(\rho_{i+1}) = tgt(\pi_i)$; and $src(\pi_n) = tgt(\pi_{n-1})$ or $src(\pi_n) = tgt(\rho_{n-1})$. The set of all Player-*i* configurations is denoted Conf_i.

Intuitively, the configuration (π, ρ, m) keeps track of the history of the game; π stores the choices of Player 1, ρ the choices of Player 2, and m is the switch counter. Hence π and ρ are sequences of transitions in T which can be arranged by suitable swapping to form two paths $(\bar{\pi}, \bar{\rho})$. How exactly these sequences are constructed is determined by a pair of *strategies* which specify for each player which edge to play from any configuration.

A Player-1 strategy is hence a partial mapping $\theta_1 : \mathsf{Conf}_1 \to T \times \mathbb{N}$ such that for all $(\pi, \rho, m) \in \mathsf{Conf}_1$ for which $\theta_1(\pi, \rho, m) = (e', m')$ is defined,

- $src(e') = tgt(last(\pi))$ and m' = m or m' = m + 1, or
- $src(e') = tgt(last(\rho))$ and m' = m + 1.

A Player-2 strategy is a partial mapping θ_2 : $\mathsf{Conf}_2 \to T \times \mathbb{N}$ such that for all $(\pi \cdot e, \rho, m) \in \mathsf{Conf}_2$ for which $\theta_2(\pi \cdot e, \rho, m) = (e', m')$ is defined, m' = m, and $src(e') = tgt(\mathsf{last}(\rho))$ if $src(e) = tgt(\mathsf{last}(\pi))$, $src(e') = tgt(\mathsf{last}(\pi))$ if $src(e) = tgt(\mathsf{last}(\rho))$. The sets of Player-1 and Player-2 strategies are denoted Θ_1 and Θ_2 .

Note that if Player 1 chooses a transition from the end of the previous choice of Player 2 (case $src(e') = tgt(last(\rho))$ above), then the switch counter is increased; but Player 1 may also choose to increase the switch counter without switching paths. Player 2 does not touch the switch counter.

We can now define what it means to update a configuration according to a strategy: For $\theta_1 \in \Theta_1$ and $(\pi, \rho, m) \in \mathsf{Conf}_1$, $\mathsf{upd}_{\theta_1}(\pi, \rho, m)$ is defined if $\theta_1(\pi, \rho, m) = (e', m')$ is defined, and then $\mathsf{upd}_{\theta_1}(\pi, \rho, m) = (\pi \cdot e', \rho, m')$. Similarly, for $\theta_2 \in \Theta_2$ and $(\pi \cdot e, \rho, m) \in \mathsf{Conf}_2$, $\mathsf{upd}_{\theta_2}(\pi \cdot e, \rho, m)$ is defined if $\theta_2(\pi \cdot e, \rho, m) = (e', m')$ is defined, and then $\mathsf{upd}_{\theta_2}(\pi \cdot e, \rho, m) = (\pi \cdot e, \rho \cdot e', m')$.

For any pair of states $(s,t) \in S \times S$, a pair of strategies $(\theta_1, \theta_2) \in \Theta_1 \times \Theta_2$ inductively determines a sequence (π^j, ρ^j, m^j) of configurations, by

$$\begin{split} (\pi^0,\rho^0,m^0) &= (s,t,0); \\ (\pi^{2j+1},\rho^{2j+1},m^{2j+1}) &= \begin{cases} \text{undef.} & \text{if } \mathsf{upd}_{\theta_1}(\pi^{2j},\rho^{2j},m^{2j}) \text{ is undefined,} \\ \mathsf{upd}_{\theta_1}(\pi^{2j},\rho^{2j},m^{2j}) & \text{otherwise;} \end{cases} \\ (\pi^{2j},\rho^{2j},m^{2j}) &= \begin{cases} \text{undef.} & \text{if } \mathsf{upd}_{\theta_2}(\pi^{2j-1},\rho^{2j-1},m^{2j-1}) \\ & \text{is undefined,} \\ \mathsf{upd}_{\theta_2}(\pi^{2j-1},\rho^{2j-1},m^{2j-1}) & \text{otherwise.} \end{cases} \end{split}$$

Note that indeed, we are updating configurations by alternating between the two strategies θ_1 , θ_2 .

The configurations in this sequence satisfy $\pi^j \leq_p \pi^{j+1}$ and $\rho^j \leq_p \rho^{j+1}$ for all j, where \leq_p denotes prefix ordering, hence the direct limits $\pi = \varinjlim \pi^j$, $\rho =$

 $\varinjlim \rho^j$ exist (as finite or infinite paths). By our conditions on configurations, the pair (π, ρ) in turn determines a pair $(\bar{\pi}, \bar{\rho})$ of paths in S, as follows:

$$(\bar{\pi}_{1}, \bar{\rho}_{1}) = \begin{cases} (\pi_{1}, \rho_{1}) & \text{if } src(\pi_{1}) = s \\ (\rho_{1}, \pi_{1}) & \text{if } src(\pi_{1}) = t \end{cases}$$
$$(\bar{\pi}_{j}, \bar{\rho}_{j}) = \begin{cases} (\pi_{j}, \rho_{j}) & \text{if } src(\pi_{j}) = tgt(\bar{\pi}_{j-1}) \\ (\rho_{j}, \pi_{j}) & \text{if } src(\pi_{j}) = tgt(\bar{\rho}_{j-1}) \end{cases}$$

The *outcome* of the game when played from (s,t) according to a strategy pair (θ_1, θ_2) is defined to be $\operatorname{out}(\theta_1, \theta_2)(s,t) = (\bar{\pi}, \bar{\rho})$, and its *utility* is defined by $\operatorname{util}(\theta_1, \theta_2)(s,t) = d^{\mathsf{T}}(\operatorname{tr}(\operatorname{out}(\theta_1, \theta_2)(s,t))) = d^{\mathsf{T}}(\operatorname{tr}(\bar{\pi}), \operatorname{tr}(\bar{\rho}))$.

Recall that d^{T} is given as a parameter to the game; if we want to make explicit the parametrization on the trace distance d^{T} on which utility depends, we write $\mathsf{util}_{d^{\mathsf{T}}}(\theta_1, \theta_2)(s, t)$.

Note that $\operatorname{util}(\theta_1,\theta_2)(s,t)$ is defined both in case the paths $\bar{\pi}$ and $\bar{\rho}$ are finite and in case they are infinite (the case where one is finite and the other is infinite cannot occur). Also, if the paths are finite because $\theta_1(\pi^j,\rho^j,m^j)$ was undefined for some configuration (π^j,ρ^j,m^j) in the sequence, then $\operatorname{len}(\bar{\pi}) = \operatorname{len}(\bar{\rho})$; if on the other hand the reason is that $\theta_2(\pi^j,\rho^j,m^j)$ was undefined, then $\operatorname{len}(\bar{\pi}) = \operatorname{len}(\bar{\rho}) \pm 1$, and $\operatorname{util}(\theta_1,\theta_2)(s,t) = \infty$. Hence if the game reaches a configuration in which Player 2 has no moves available, the utility is ∞ .

The objective of Player 1 in the game is to maximize utility, whereas Player 2 wants to minimize it. Hence we define the *value* of the game from (s,t) to be

$$v(s,t) = \sup_{\theta_1 \in \Theta_1} \inf_{\theta_2 \in \Theta_2} \operatorname{util}(\theta_1,\theta_2)(s,t) \,.$$

For a given subset $\Theta'_1 \subseteq \Theta_1$ we will write

$$v(\Theta_1')(s,t) = \sup_{\theta_1 \in \Theta_1'} \inf_{\theta_2 \in \Theta_2} \mathsf{util}(\theta_1,\theta_2)(s,t) \,,$$

and if we need to emphasize dependency of the value on the given trace distance, we write $v(d^{\mathsf{T}}, \Theta_1')$. The following lemma states the immediate fact that if Player 1 has fewer strategies available, the game value decreases.

5.1 Lemma. For all
$$\Theta_1' \subseteq \Theta_1'' \subseteq \Theta_1$$
 and all $s, t \in S$, $v(\Theta_1')(s, t) \leq v(\Theta_1'')(s, t)$.

The above definition of strategies is slightly too general in that whether or not a strategy is defined in a given configuration should only depend on the actual part of the configuration on which the strategy has an effect. We hence define a notion of *uniformity* which we will assume from now:

5.2 Definition. A strategy $\theta_1 \in \Theta_1$ is *uniform* if it holds for all configurations $(\pi, \rho, m), (\pi, \tilde{\rho}, m), (\tilde{\pi}, \rho, m) \in \mathsf{Conf}_1$ that whenever $\theta_1(\pi, \rho, m) = (e', m')$ is defined,

- if $src(e') = tgt(last(\pi))$, then also $\theta_1(\pi, \tilde{\rho}, m)$ is defined, and
- if $src(e') = tgt(last(\rho))$, then also $\theta_1(\tilde{\pi}, \rho, m)$ is defined.

A strategy $\theta_2 \in \Theta_2$ is uniform if it holds for all configurations $(\pi \cdot e, \rho, m), (\tilde{\pi} \cdot \tilde{e}, \rho, m), (\pi \cdot e, \tilde{\rho}, m) \in \mathsf{Conf}_2$ that whenever $\theta_2(\pi \cdot e, \rho, m) = (e', m')$ is defined,

- if $src(e') = tgt(last(\rho))$, then also $\theta_2(\tilde{\pi} \cdot \tilde{e}, \rho, m)$ is defined, and
- if $src(e') = tgt(last(\pi))$, then also $\theta_2(\pi \cdot e, \tilde{\rho}, m)$ is defined.

A subset $\Theta'_1 \subseteq \Theta_1$ is uniform if all strategies in Θ'_1 are uniform. Uniformity of strategies is used to combine paths built from different starting states in the proof of Proposition 5.5 below, and it allows us to show a minimax theorem for our setting.

The concrete strategy subsets we will consider in later sections will all be uniform, so from now on we only consider the subsets of Θ_1 and Θ_2 consisting of uniform strategies. Abusing notation, we will also denote these by Θ_1 and Θ_2 .

5.3 Lemma. For any uniform $\Theta'_1 \subseteq \Theta_1$ and all $s, t \in S$,

$$\sup_{\theta_1 \in \Theta_1'} \inf_{\theta_2 \in \Theta_2} \operatorname{util}(\theta_1, \theta_2)(s, t) = \inf_{\theta_2 \in \Theta_2} \sup_{\theta_1 \in \Theta_1'} \operatorname{util}(\theta_1, \theta_2)(s, t) \,.$$

Proof: By uniformity, neither of the two players has any possibility to influence the configurations reachable by the other's strategies. Hence it is immaterial which player gets to choose strategy first.

5.4 General Properties

We show here that under the uniformity condition, the game value is indeed a distance, and that results concerning inequalities in the qualitative dimension can be transfered to topological inequivalences in the quantitative setting. Say that a Player-1 strategy $\theta_1 \in \Theta_1$ is non-switching if it holds for all (π, ρ, m) for which $\theta_1(\pi, \rho, m) = (e', m')$ is defined that m = m', and let Θ_1^0 be the set of non-switching Player-1 strategies. We first show a lemma which shows that any pair of traces can be generated by a non-switching strategy:

5.4 Lemma. For all $s, t \in S$ and all $\sigma \in \mathsf{Tr}(s)$, $\tau \in \mathsf{Tr}(t)$ there exist $\theta_1 \in \Theta_1^0$ and $\theta_2 \in \Theta_2$ for which $\mathsf{util}(\theta_1, \theta_2)(s, t) = d^\mathsf{T}(\sigma, \tau)$.

Proof: Let $(\pi, \rho, 0) \in \mathsf{Conf}_1$ for finite paths π , ρ with $\mathsf{len}(\pi) = \mathsf{len}(\rho) = k \ge 0$ and $\mathsf{tr}(\pi) = \sigma_0 \dots \sigma_{k-1}$, $\mathsf{tr}(\rho) = \tau_0 \dots \tau_{k-1}$. If $\mathsf{len}(\sigma) \ge k$, then there is $e = (\mathsf{last}(\pi), \sigma_k, s') \in T$, and we define $\theta_1(\pi, \rho, 0) = (e, 0)$. If also $\mathsf{len}(\tau) \ge k$, then there is $e' = (\mathsf{last}(\rho), \tau_k, t') \in T$, and we let $\theta_2(\pi \cdot e, \rho, 0) = (e', 0)$.

Let $(\bar{\pi}, \bar{\rho}) = \operatorname{out}(\theta_1, \theta_2)(s, t)$. If both σ and τ are infinite traces, then $\operatorname{tr}(\bar{\pi}) = \sigma$ and $\operatorname{tr}(\bar{\rho}) = \tau$; otherwise, $\operatorname{tr}(\bar{\pi})$ and $\operatorname{tr}(\bar{\rho})$ will be finite prefixes of σ and τ for which $d^{\mathsf{T}}(\operatorname{tr}(\bar{\pi}), \operatorname{tr}(\bar{\rho})) = d^{\mathsf{T}}(\sigma, \tau)$.

The following proposition shows that the distance defined by our quantitative game is a hemimetric. Note that the proof of the triangle inequality uses uniformity.

5.5 Proposition. For all $\Theta'_1 \subseteq \Theta_1$ with $\Theta^0_1 \subseteq \Theta'_1$, $v(\Theta'_1)$ is a hemimetric on S.

Proof: We write $v = v(\Theta'_1)$ during this proof. It is clear that v(s,s) = 0 for all $s \in S$: if the players are making their choices from the same state, Player 2 can always answer by choosing exactly the same transition as Player 1. For proving the triangle inequality $v(s,u) \leq v(s,t) + v(t,u)$, let $\varepsilon > 0$ and use Lemma 5.3 to choose Player-2 strategies $\theta_2^{s,t}, \theta_2^{t,u} \in \Theta_2$ for which

$$\sup_{\theta_1 \in \Theta_1'} \operatorname{util}(\theta_1, \theta_2^{s,t})(s, t) < v(s, t) + \frac{\varepsilon}{2},$$

$$\sup_{\theta_1 \in \Theta_1'} \operatorname{util}(\theta_1, \theta_2^{t,u})(t, u) < v(t, u) + \frac{\varepsilon}{2}.$$
(5.1)

We define a strategy $\theta_2^{s,u} \in \Theta_2$ which uses three paths and two configurations in S as extra memory. This is only for convenience, as these can be reconstructed by Player 2 at any time; hence we do not extend the capabilities of Player 2:

$$\theta_{2}^{s,u}(\pi \cdot e, \chi, m; \bar{\pi}, \bar{\rho}', \bar{\chi}, \pi', \rho'_{1}, \rho'_{2}, \chi') = \begin{cases} \left(\theta_{2}^{t,u}(\rho'_{2} \cdot \theta_{2,1}^{s,t}(\pi' \cdot e, \rho'_{1}, m), \chi', m); \\ \bar{\pi} \cdot e, \\ \bar{\rho}' \cdot \theta_{2,1}^{s,t}(\pi' \cdot e, \rho'_{1}, m), \\ \bar{\chi} \cdot \theta_{2,1}^{t,u}(\rho'_{2} \cdot \theta_{2,1}^{s,t}(\pi' \cdot e, \rho'_{1}, m)), \\ \pi' \cdot e, \\ \rho'_{1} \cdot \theta'_{2,1}(\pi' \cdot e, \rho'_{1}, m), \\ \rho'_{2} \cdot \theta'_{2,1}(\pi' \cdot e, \rho'_{1}, m), \\ \chi' \cdot \theta'_{2,1}(\rho'_{2} \cdot \theta'_{2,1}(\pi' \cdot e, \rho'_{1}, m)) \right) & \text{if } src(e) = tgt(\text{last}(\bar{\pi})), \end{cases} \\ \left\{ \begin{pmatrix} \theta_{2}^{s,t}(\pi' \cdot \theta'_{2,1}(\rho'_{2} \cdot e, \chi', m), \rho'_{1}, m); \\ \bar{\pi} \cdot \theta'_{2,1}(\pi' \cdot \theta'_{2,1}(\rho'_{2} \cdot e, \chi', m), \rho'_{1}, m); \\ \bar{\pi} \cdot \theta'_{2,1}(\pi' \cdot \theta'_{2,1}(\rho'_{2} \cdot e, \chi', m), \\ \bar{\rho}' \cdot \theta'_{2,1}(\rho'_{2} \cdot e, \chi', m), \\ \bar{\chi} \cdot e, \\ \pi' \cdot \theta'_{2,1}(\rho'_{2} \cdot e, \chi', m), \\ \bar{\rho}'_{2} \cdot \theta'_{2,1}(\pi' \cdot \theta'_{2,1}(\rho'_{2} \cdot e, \chi', m)) \\ \rho'_{2} \cdot e, \\ \chi' \cdot \theta'_{2,1}(\rho'_{2} \cdot e, \chi', m) \right) & \text{if } src(e) = tgt(\text{last}(\bar{\chi})). \end{cases}$$

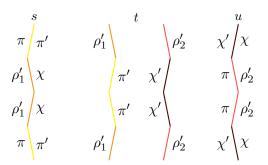


Figure 5.1: Configuration update in the game used for showing the triangle inequality

In the beginning of the game, all memory paths are initialized to be empty.

In the expression above, the strategy $\theta_2^{s,u}$ is constructed from the strategies $\theta_2^{s,t}$ and $\theta_2^{t,u}$ by using the answer to the move of Player 1 in one of the games as an emulated Player-1 move in the other. The paths $\bar{\pi}$, $\bar{\chi}$ are constructed from the configuration (π,χ) of the (s,u)-game and are only kept in memory so that we can see whether Player 1 is playing an edge prolonging $\bar{\pi}$ or $\bar{\chi}$. The pair (π', ρ'_1) is the configuration in the (s,t)-game we are emulating, and (ρ'_2, χ') is the (t,u)-configuration. The path $\bar{\rho}' = \bar{\rho}'_1 = \bar{\rho}'_2$ is common for the paths $(\bar{\pi}', \bar{\rho}'_1)$, $(\bar{\rho}'_2, \bar{\chi}')$ constructed from (π', ρ'_1) and (ρ'_2, χ') .

If Player 1 has played an edge e prolonging $\bar{\pi}$ (first case above), we compute an answer move $(e',m)=\theta_2^{s,t}(\pi'\cdot e,\rho_1',m)$ to this in the (s,t)-game. This answer is then used to emulate a Player-1 move in the (t,u)-game, and the answer $\theta_2^{t,u}(\rho_2'\cdot e',\chi',m)$ to this is what Player 2 plays in the (s,u)-game. The memory is updated accordingly. If on the other hand, Player 1 has played an edge e prolonging $\bar{\chi}$, we play in the (t,u)-game first and use the answer $(e',m)=\theta_2^{t,u}(\rho_2'\cdot e,\chi',m)$ in the (s,t)-game to compute $\theta_2^{s,t}(\pi'\cdot e',\rho_1',m)$. Figure 5.1 gives an illustration of how the configurations are updated during the game; note that uniformity of Θ_1' is necessary for being able to emulate Player-1 moves from one game in another.

Take now any $\theta_1^{s,u} \in \Theta_1'$, let $(\bar{\pi}, \bar{\chi}) = \operatorname{out}(\theta_1^{s,u}, \theta_2^{s,u})(s,u)$, and let $\bar{\rho}'$ be the corresponding memory path. By Lemma 5.4 there exist $\theta_1^{s,t}, \theta_1^{t,u} \in \Theta_1'$ for which $d^{\mathsf{T}}(\operatorname{tr}(\bar{\rho}'), \operatorname{tr}(\bar{\rho}')) = \operatorname{util}(\theta_1^{s,t}, \theta_2^{s,t})(s,t)$ and $d^{\mathsf{T}}(\operatorname{tr}(\bar{\rho}'), \operatorname{tr}(\bar{\chi})) = \operatorname{util}(\theta_1^{t,u}, \theta_2^{t,u})(t,u)$. Using Equation (5.1) we have

$$\begin{aligned} \operatorname{util}(\theta_1^{s,u},\theta_2^{s,u})(s,u) &= d^{\mathsf{T}}(\operatorname{tr}(\bar{\pi}),\operatorname{tr}(\bar{\chi})) \\ &\leq d^{\mathsf{T}}(\operatorname{tr}(\bar{\pi}),\operatorname{tr}(\bar{\rho})) + d^{\mathsf{T}}(\operatorname{tr}(\bar{\rho}),\operatorname{tr}(\bar{\chi})) \\ &< v(s,t) + v(t,u) + \varepsilon \end{aligned}$$

and hence also $\inf_{\theta_2 \in \Theta_2} \mathsf{util}(\theta_1^{s,u}, \theta_2)(s,u) < v(s,t) + v(t,u) + \varepsilon$. As the choice

of $\theta_1^{s,u}$ was arbitrary, this implies

$$\sup_{\theta_1 \in \Theta_1'} \inf_{\theta_2 \in \Theta_2} \operatorname{util}(\theta_1, \theta_2)(s, u) \le v(s, t) + v(t, u) + \varepsilon,$$

and as also ε was chosen arbitrarily, we have $v(s,u) \leq v(s,t) + v(t,u)$.

Next we show a transfer principle which allows us to generalize counterexamples regarding the equivalences in the qualitative linear-time—branching-time spectrum [vG01] to the qualitative setting. We will make use of this principle later to show that all distances we introduce are topologically inequivalent.

5.6 Lemma. Let $\Theta'_1, \Theta''_1 \subseteq \Theta_1$, and assume d^{T} to be separating. If there exist states $s, t \in S$ for which $v(d^{\mathsf{T}}_{\mathrm{disc}}, \Theta'_1)(s, t) = 0$ and $v(d^{\mathsf{T}}_{\mathrm{disc}}, \Theta''_1)(s, t) = \infty$, then $v(d^{\mathsf{T}}, \Theta'_1)$ and $v(d^{\mathsf{T}}, \Theta''_1)$ are topologically inequivalent.

Proof: By $v(d_{\mathrm{disc}}^\mathsf{T}, \Theta_1')(s,t) = 0$, we know that for any $\theta_1 \in \Theta_1'$ there exists $\theta_2 \in \Theta_2$ for which $(\bar{\pi}, \bar{\rho}) = \mathsf{out}(\theta_1, \theta_2)(s,t)$ satisfy $\mathsf{tr}(\bar{\pi}) = \mathsf{tr}(\bar{\rho})$, hence also $v(d^\mathsf{T}, \Theta_1')(s,t) = 0$. Conversely, as d^T is separating, $v(d^\mathsf{T}, \Theta_1'')(s,t) = 0$ would imply that also $v(d_{\mathrm{disc}}^\mathsf{T}, \Theta_1'')(s,t) = 0$, hence we must have $v(d^\mathsf{T}, \Theta_1'')(s,t) \neq 0$, entailing topological inequivalence.

5.5 The Distance Spectrum

In this section we introduce the distances depicted in Figure 5.2 and show their relationship. Note again that the results obtained here are independent of the particular trace distance considered. Throughout this section, we fix a LTS (S,T) and a trace distance $d^T: \mathbb{K}^{\infty} \times \mathbb{K}^{\infty} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$.

5.5.1 Branching Distances

If the switching counter in the game introduced in Section 5.3 is unbounded, Player 1 can choose at any move whether to prolong the previous choice or to switch paths, hence this resembles the bisimulation game [Sti95].

- **5.7 Definition.** The bisimulation distance between s and t is $d^{\text{bisim}}(s,t) = v(s,t)$.
- **5.8 Theorem.** For $d^{\mathsf{T}} = d^{\mathsf{T}}_{\mathrm{disc}}$ the discrete trace distance, $d^{\mathrm{bisim}}_{\mathrm{disc}}(s,t) = 0$ iff s and t are bisimilar.

Proof: By discreteness of $d_{\mathrm{disc}}^{\mathsf{T}}$, we have $d_{\mathrm{disc}}^{\mathrm{bisim}}(s,t)=0$ iff it holds that for all $\theta_1 \in \Theta_1$ there exists $\theta_2 \in \Theta_2$ for which $\mathrm{util}(\theta_1,\theta_2)(s,t)=0$. Hence for each reachable Player-1 configuration (π,ρ,m) with $\theta_1(\pi,\rho,m)=(e',m')$, we have $\theta_2(\pi\cdot e',\rho,m')=(e'',m')$ with $\mathrm{tr}(e')=\mathrm{tr}(e'')$, i.e., Player 2 matches the labels chosen by Player 1 precisely, implying that s and t are bisimilar. The proof of the other direction is trivial.

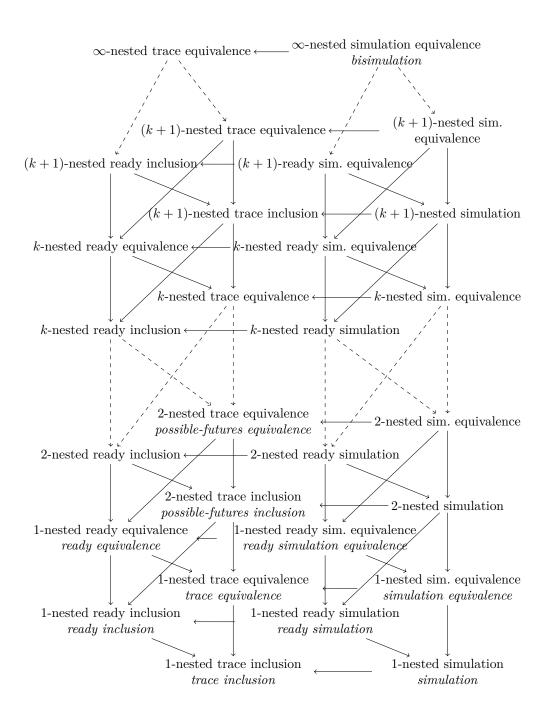


Figure 5.2: The quantitative linear-time-branching-time spectrum. The nodes are the different system distances introduced in this chapter, and an edge $d_1 \longrightarrow d_2$ or $d_1 \dashrightarrow d_2$ indicates that $d_1(s,t) \ge d_2(s,t)$ for all states s, t, and that d_1 and d_2 in general are topologically inequivalent.

We can restrict the strategies available to Player 1 by allowing only a pre-defined finite number of switches:

$$\Theta_1^{k\text{-sim}} = \{\theta_1 \in \Theta_1 \mid \text{if } \theta_1(\pi, \rho, m) = (e', m') \text{ is defined, then } m' \leq k - 1\}$$

In the so-defined k-nested simulation game, Player 1 is only allowed to switch paths k-1 times during the game. Note that $\Theta_1^{1-\text{sim}} = \Theta_1^0$ is the set of non-switching strategies.

- **5.9 Definition.** The k-nested simulation distance from s to t, for $k \in \mathbb{N}_+$, is $d^{k-\text{sim}}(s,t) = v(\Theta_1^{k-\text{sim}})(s,t)$. The k-nested simulation equivalence distance between s and t is $d^{k-\text{sim}-\text{eq}}(s,t) = \max(v(\Theta_1^{k-\text{sim}})(s,t),v(\Theta_1^{k-\text{sim}})(t,s))$.
- **5.10 Theorem.** For $d^{\mathsf{T}} = d^{\mathsf{T}}_{\mathrm{disc}}$ the discrete trace distance,
 - $d_{\mathrm{disc}}^{k\mathrm{-sim}}(s,t)=0$ iff there is a k-nested simulation from s to t,
 - $d_{\text{disc}}^{k\text{-sim-eq}}(s,t) = 0$ iff there is a k-nested simulation equivalence between s and t.

Especially, $d_{\rm disc}^{1-{\rm sim}}$ corresponds to the usual *simulation* preorder, and $d_{\rm disc}^{2-{\rm sim}}$ to *two-nested simulation*. Similarly, $d_{\rm disc}^{1-{\rm sim-eq}}$ is *similarity*, and $d_{\rm disc}^{2-{\rm sim-eq}}$ is *two-nested simulation equivalence*. We refer to [GV92, HM85] for definitions and discussion of two-nested and *k*-nested simulation.

Proof: This is similar to the proof of Theorem 5.8: If $d_{\text{disc}}^{k-\text{sim}}(s,t) = 0$, then any $\theta_1 \in \Theta_1^{k-\text{sim}}$ has a counter-strategy $\theta_2 \in \Theta_2$ which matches the labels chosen by Player 1 precisely, implying k-nested simulation from s to t. The other direction is again trivial.

5.11 Theorem. For all $k, \ell \in \mathbb{N}_+$ with $k < \ell$ and all $s, t \in S$,

$$d^{k\text{-sim-eq}}(s,t) \le d^{\ell\text{-sim}}(s,t) \le d^{\ell\text{-sim-eq}}(s,t) \le d^{\text{bisim}}(s,t).$$

If the trace distance d^{T} is separating, then all distances above are topologically inequivalent.

Proof: The first part of the theorem follows from the inclusions $\Theta_1^{k\text{-sim-eq}} \subseteq \Theta_1^{\ell\text{-sim}} \subseteq \Theta_1^{\ell\text{-sim-eq}} \subseteq \Theta_1$ and Lemma 5.1. Topological inequivalence follows from Lemma 5.6 and the fact that for the discrete relations corresponding to the distances above (obtained by letting $d^{\mathsf{T}} = d_{\mathrm{disc}}^{\mathsf{T}}$), the inequalities are strict [vG01].

As a variation of k-nested simulation, we can consider strategies which allow Player 1 to switch paths k times during the game, but at the last switch,

he may only pose *one* transition as a challenge, to which Player 2 must answer, and then the game finishes:

$$\Theta_1^{k\text{-rsim}} = \{\theta_1 \in \Theta_1 \mid \text{if } \theta_1(\pi, \rho, m) \text{ is defined, then } m \leq k-1\}$$

Hence after his k'th switch, Player 1 has no more moves available, and the game finishes after the answer move of Player 2. Again, we allow Player 1 to increase the switch counter without actually switching paths.

5.12 Definition. The k-nested ready simulation distance from s to t, for $k \in \mathbb{N}_+$, is $d^{k\text{-rsim}}(s,t) = v(\Theta_1^{k\text{-rsim}})(s,t)$. The k-nested ready simulation equivalence distance between s and t is $d^{k\text{-rsim-eq}}(s,t) = \max(v(\Theta_1^{k\text{-rsim}})(s,t), v(\Theta_1^{k\text{-rsim}})(t,s))$.

For the discrete case, it seems only k=1 has been considered; the proof is similar to the one of Theorem 5.8.

- **5.13 Theorem.** For $d^{\mathsf{T}} = d_{\mathsf{disc}}^{\mathsf{T}}$ the discrete trace distance,
 - $d_{\mathrm{disc}}^{1-\mathrm{rsim}}(s,t)=0$ iff there is a ready simulation from s to t,
 - $d_{\mathrm{disc}}^{1-\mathrm{rsim-eq}}(s,t)=0$ iff s and t are ready simulation equivalent.

The next theorem finishes our work on the right half of Figure 5.2.

5.14 Theorem. For all $k, \ell \in \mathbb{N}_+$ with $k < \ell$ and all $s, t \in S$,

$$d^{k\text{-sim}}(s,t) \le d^{k\text{-rsim}}(s,t) \le d^{\ell\text{-sim}}(s,t) \,,$$

$$d^{k\text{-sim-eq}}(s,t) \le d^{k\text{-rsim-eq}}(s,t) \le d^{\ell\text{-sim-eq}}(s,t) \,.$$

Additionally, $d^{k\text{-rsim}}$ and $d^{k\text{-sim-eq}}$ are incomparable, and also $d^{k\text{-rsim-eq}}$ and $d^{(k+1)\text{-sim}}$ are incomparable. If the trace distance d^{T} is separating, then all distances above are topologically inequivalent.

Proof: Like in the proof of Theorem 5.11, the inequalities follow from strategy set inclusions and topological inequivalence from Lemma 5.6. The incomparability results follow from the corresponding results for $d_{\rm disc}^{\sf T}$ and Lemma 5.6.

5.5.2 Linear Distances

Above we have introduced the distances in the right half of the quantitative linear-time—branching-time spectrum in Figure 5.2 and shown the relations claimed in the diagram. To develop the left half, we need the notion of *blind* strategies. For any subset $\Theta'_1 \subseteq \Theta_1$ we define the set of blind Θ'_1 -strategies by

$$\ddot{\Theta}_1' = \{ \theta_1 \in \Theta_1' \mid \forall \pi, \rho, \rho', m : \theta_1(\pi, \rho, m) = \theta_1(\pi, \rho', m), \\ \text{or } \theta_1(\pi, \rho, m) = (e, m + 1) \text{ and } tgt(\text{last}(\rho)) \neq tgt(\text{last}(\rho')) \}.$$

Hence in such a blind strategy, either the edge chosen by Player 1 does not depend on the choices of Player 2, or the switch counter is increased, in which case the Player-1 choice only depends on the target of the last choice of Player 2 (note that this dependency is necessary if Player 1 wants to switch paths).

Now we can define, for $s, t \in S$ and $k \in \mathbb{N}_+$,

- the ∞ -nested trace equivalence distance: $d^{\infty\text{-trace-eq}}(s,t) = v(\ddot{\Theta}_1)(s,t)$,
- the k-nested trace distance: $d^{k\text{-trace}}(s,t) = v(\ddot{\Theta}_1^{k\text{-sim}})(s,t)$,
- the k-nested trace equivalence distance: $d^{k\text{-trace-eq}}(s,t) = \max(v(\ddot{\Theta}_1^{k\text{-sim}})(s,t), v(\ddot{\Theta}_1^{k\text{-sim}})(t,s)),$
- the k-nested ready distance: $d^{k\text{-ready}}(s,t) = v(\ddot{\Theta}_1^{k\text{-rsim}})(s,t)$, and
- the k-nested ready equivalence distance: $d^{k\text{-ready-eq}}(s,t) = \max(v(\ddot{\Theta}_1^{k\text{-rsim}})(s,t), v(\ddot{\Theta}_1^{k\text{-rsim}})(t,s)).$

Our approach is justified by the following lemma which shows that the (1-nested) trace distance from s to t is precisely the Hausdorff distance between the sets of traces available from s and t, respectively.

5.15 Lemma. For
$$s, t \in S$$
, $d^{1-\text{trace}}(s, t) = \sup_{\sigma \in \text{Tr}(s)} \inf_{\tau \in \text{Tr}(t)} d^{\mathsf{T}}(\sigma, \tau)$.

Proof: We have $d^{1-\operatorname{trace}}(s,t) = v(\ddot{\Theta}_1^0)(s,t)$, with $\ddot{\Theta}_1^0 = \{\theta_1 \in \Theta_1^0 \mid \forall \pi, \rho, \rho', m : \theta_1(\pi,\rho,m) = \theta_1(\pi,\rho',m)\}$. Hence, and as strategies in Θ_1^0 are non-switching, every strategy $\theta_1 \in \ddot{\Theta}_1^0$ gives rise to precisely one trace $\sigma = \sigma(\theta_1) \in \operatorname{Tr}(s)$ independently of Player-2 strategy $\theta_2 \in \Theta_2$. Conversely, by Lemma 5.4 (noticing that indeed, we have constructed a blind Player-1 strategy in the proof of that lemma), every trace $\sigma \in \operatorname{Tr}(s)$ is generated by a strategy $\theta_1 \in \ddot{\Theta}_1^0$ with $\sigma = \sigma(\theta_1)$.

We can finish the proof by showing that for all $\theta_1 \in \ddot{\Theta}_1^0$,

$$\inf_{\theta_2 \in \Theta_2} d^{\mathsf{T}}(\sigma(\theta_1), \operatorname{tr}(\bar{\rho}(\theta_1, \theta_2))) = \inf_{\tau \in \mathsf{Tr}(t)} d^{\mathsf{T}}(\sigma(\theta_1), \tau).$$

But again using Lemma 5.4, we see that any $\tau \in \mathsf{Tr}(t)$ is generated by a strategy $\theta_2 \in \Theta_2$, hence this is clear.

Using the discrete trace distance, we recover the following standard relations [vG01]. The theorem follows by Lemma 5.15 and arguments similar to the ones used in the proofs of the corresponding theorems in the preceding section. We refer to [RB81, HM85] for definitions and discussion of possible-futures inclusion and equivalence.

- **5.16 Theorem.** For $d^{\mathsf{T}} = d^{\mathsf{T}}_{\mathrm{disc}}$ the discrete trace distance and $s, t \in S$ we have
 - $d_{\mathrm{disc}}^{1-\mathrm{trace}}(s,t)=0$ iff there is a trace inclusion from s to t,

- $d_{\mathrm{disc}}^{1-\mathrm{trace-eq}}(s,t) = 0$ iff s and t are trace equivalent,
- $d_{\mathrm{disc}}^{2-\mathrm{trace}}(s,t)=0$ iff there is a possible-futures inclusion from s to t,
- $d_{\mathrm{disc}}^{2\mathrm{-trace-eq}}(s,t)=0$ iff s and t are possible-futures equivalent,
- $d_{\text{disc}}^{1-\text{ready}}(s,t) = 0$ iff there is a readiness inclusion from s to t,
- $d_{\mathrm{disc}}^{1-\mathrm{ready-eq}}(s,t)=0$ iff s and t are ready equivalent.

The following theorem entails all relations in the left side of Figure 5.2; the right-to-left arrows follow from the strategy set inclusions $\Theta'_1 \subseteq \Theta'_1$ for any $\Theta'_1 \subseteq \Theta_1$ and Lemma 5.1. As with Theorems 5.11 and 5.14, the theorem follows by strategy set inclusion, Lemma 5.6, and corresponding results for the discrete relations.

5.17 Theorem. For all $k, \ell \in \mathbb{N}_+$ with $k < \ell$ and $s, t \in S$,

$$\begin{split} & d^{k\text{-trace-eq}}(s,t) \leq d^{\ell\text{-trace}}(s,t) \leq d^{\ell\text{-trace-eq}}(s,t) \leq d^{\infty\text{-trace-eq}}(s,t), \\ & d^{k\text{-trace}}(s,t) \leq d^{k\text{-ready}}(s,t) \leq d^{\ell\text{-trace}}(s,t), \\ & d^{k\text{-trace-eq}}(s,t) \leq d^{k\text{-ready-eq}}(s,t) \leq d^{\ell\text{-trace-eq}}(s,t). \end{split}$$

Additionally, $d^{k\text{-ready}}$ and $d^{k\text{-trace-eq}}$ are incomparable, and also $d^{k\text{-ready-eq}}$ and $d^{(k+1)\text{-trace}}$ are incomparable. If the trace distance d^{T} is separating, then all distances above are topologically inequivalent.

5.6 Recursive Characterizations

We now turn our attention to an important special case in which the given trace distance has a specific recursive characterization; we show that in this case, all distances in the spectrum can be characterized as least fixed points. We will see in Section 5.7 that this can be applied to all examples of trace distances mentioned in Section 5.2.

Note that all theorems require the LTS in question to be finitely branching; this is a standard assumption which goes back to [Sti95]. In most cases it may be relaxed to *compact branching* in the sense of [vB96], but to keep things simple, we do not do this here.

5.6.1 Fixed-Point Characterizations

Let L be a complete lattice with order \sqsubseteq and bottom and top elements \bot , \top . Let $f: \mathbb{K}^{\infty} \times \mathbb{K}^{\infty} \to L$, $g: L \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ and $F: \mathbb{K} \times \mathbb{K} \times L \to L$ such that $d^{\mathsf{T}} = g \circ f$, g is monotone, $F(x, y, \cdot) : L \to L$ is monotone for all $x, y \in \mathbb{K}$, and

$$f(\sigma,\tau) = \begin{cases} F(\sigma_0, \tau_0, f(\sigma^1, \tau^1)) & \text{if } \sigma, \tau \neq \varepsilon, \\ \top & \text{if } \sigma = \varepsilon, \tau \neq \varepsilon \text{ or } \sigma \neq \varepsilon, \tau = \varepsilon, \\ \bot & \text{if } \sigma = \tau = \varepsilon \end{cases}$$
 (5.2)

for all $\sigma, \tau \in \mathbb{K}^{\infty}$.

We hence assume that d^{T} has a recursive characterization (using F) on top of a complete (but otherwise arbitrary) lattice L which we introduce between \mathbb{K}^{∞} and $\mathbb{R}_{>0} \cup \{\infty\}$ to serve as a *memory*. Below we will work with different endofunctions I on the set of mappings $(\mathbb{N}_+ \cup \{\infty\}) \times \{1,2\} \to L^{S \times S}$ which are parametrized by the number m of switches in $\mathbb{N}_+ \cup \{\infty\}$ which Player 1 has left, and a value $p \in \{1, 2\}$ which keeps track of whether Player 1 currently is building the left or the right path.

5.18 Theorem. The endofunction I on $(\mathbb{N}_+ \cup \{\infty\}) \times \{1,2\} \to L^{S \times S}$ defined by

$$I(h_{m,p})(s,t) = \begin{cases} \sup_{s \to s'} \inf_{t \to t'} F(x,y,h_{m,1}(s',t')) \\ \sup_{s \to s'} \inf_{t \to t'} F(x,y,h_{m-1,2}(s',t')) \\ \sup_{t \to t'} \inf_{s \to s'} F(x,y,h_{m-1,2}(s',t')) \end{cases} \quad \text{if } m \geq 2, p = 1$$

$$I(h_{m,p})(s,t) = \begin{cases} \sup_{s \to s'} \inf_{t \to t'} F(x,y,h_{m,1}(s',t')) \\ \sup_{s \to s'} \inf_{t \to t'} F(x,y,h_{m,2}(s',t')) \\ \sup_{s \to s'} \inf_{t \to t'} F(x,y,h_{m-1,1}(s',t')) \end{cases} \quad \text{if } m \geq 2, p = 2$$

$$\sup_{s \to s'} \inf_{t \to t'} F(x,y,h_{m,2}(s',t')) \quad \text{if } m \geq 1, p = 2$$

$$\sup_{t \to t'} \inf_{s \to s'} F(x,y,h_{m,2}(s',t')) \quad \text{if } m = 1, p = 2$$

has a least fixed point $h^*: (\mathbb{N}_+ \cup \{\infty\}) \times \{1,2\} \to L^{S \times S}$, and if the LTS (S,T) is finitely branching, then $d^{k\text{-sim}} = g \circ h_{k,1}^*$, $d^{k\text{-sim-eq}} = g \circ \max(h_{k,1}^*, h_{k,2}^*)$ for all $k \in \mathbb{N}_+ \cup \{\infty\}$.

Hence I iterates the function h over the branching structure of (S, T), computing all nested branching distances at the same time. Note the specialization of this to simulation and bisimulation distance, where we have the following fixed-point equations, using $h_{1,1}^* = h^{1-\text{sim}}$ and $h_{\infty,1}^* = h^{\text{bisim}}$:

$$\begin{split} h^{\text{1-sim}}(s,t) &= \sup_{s \xrightarrow{x} s'} \inf_{t \xrightarrow{y} t'} F(x,y,h^{\text{1-sim}}(s',t')) \\ h^{\text{bisim}}(s,t) &= \max \begin{cases} \sup_{s \xrightarrow{x} s'} \inf_{t \xrightarrow{y} t'} F(x,y,h^{\text{bisim}}(s',t')) \\ \sup_{t \xrightarrow{y} t'} \sup_{s \xrightarrow{x} s'} F(x,y,h^{\text{bisim}}(s',t')) \end{cases} \end{split}$$

Proof: The lattice of mappings $(\mathbb{N}_+ \cup \{\infty\}) \times \{1,2\} \to L^{S \times S}$ with the pointwise partial order is complete, and I is monotone because F is, so by Tarski's fixed-point theorem, I has indeed a least fixed point h^* . To show that $d^{k\text{-sim}} = g \circ h_{k,1}^*$ for all k, we pull back $d^{k\text{-sim}}$ along g: Define $w : (\mathbb{N}_+ \cup \{\infty\}) \times \{1,2\} \to L^{S \times S}$ by

$$\begin{split} w_{k,1}(s,t) &= \sup_{\theta_1 \in \Theta_1^{k\text{-sim}}} \inf_{\theta_2 \in \Theta_2} f(\operatorname{tr}(\operatorname{out}(\theta_1,\theta_2)(s,t))) \\ w_{k,2}(s,t) &= \sup_{\theta_1 \in \Theta_1^{k\text{-sim}}} \inf_{\theta_2 \in \Theta_2} f(\operatorname{tr}(\operatorname{out}(\theta_1,\theta_2)(t,s))) \end{split}$$

then $d^{k\text{-sim}} = g \circ f(k,1)$ for all k by monotonicity of g. We will be done once we can show that $w = h^*$.

We first show that w is a fixed point for I. Let $s, t \in S$, then (assuming $k \geq 2$)

$$I(w_{k,1})(s,t) = \max \begin{cases} \sup \inf_{s \xrightarrow{s} s'} F(x,y,w_{k,1}(s',t')) \\ \sup \inf_{t \xrightarrow{y} t'} F(x,y,w_{k-1,2}(s',t')) \\ \sup \inf_{t \xrightarrow{y} t'} F(x,y,\sup_{\theta_1 \in \Theta_1^{k-\sin \theta_2} \in \Theta_2} \inf_{\theta_2 \in \Theta_2} F(\operatorname{tr}(\operatorname{out}(\theta_1,\theta_2)(s',t')))) \\ \sup \inf_{t \xrightarrow{y} t'} F(x,y,\sup_{\theta_1 \in \Theta_1^{k-\sin \theta_2} \in \Theta_2} \inf_{\theta_2 \in \Theta_2} F(\operatorname{tr}(\operatorname{out}(\theta_1,\theta_2)(t',s')))) \\ = \max \begin{cases} \sup \inf_{t \xrightarrow{y} t'} F(x,y,\sup_{\theta_1 \in \Theta_1^{(k-1)-\sin \theta_2} \in \Theta_2} \inf_{\theta_2 \in \Theta_2} F(x,y,f(\operatorname{tr}(\operatorname{out}(\theta_1,\theta_2)(t',s')))) \\ \sup \inf_{t \xrightarrow{y} t'} \sup_{s \xrightarrow{y} s'} \inf_{\theta_1 \in \Theta_1^{(k-1)-\sin \theta_2} \in \Theta_2} F(x,y,f(\operatorname{tr}(\operatorname{out}(\theta_1,\theta_2)(t',s')))) \\ \sup \inf_{t \xrightarrow{y} t'} \sup_{s \xrightarrow{y} s'} \inf_{\theta_1 \in \Theta_1^{(k-1)-\sin \theta_2} \in \Theta_2} \inf_{t \xrightarrow{x} t'} \sup_{\theta_1 \in \Theta_1^{(k-1)-\sin \theta_2} \in \Theta_2} \inf_{t \xrightarrow{y} t'} \inf_{\theta_1 \in \Theta_1^{(k-1)-\sin \theta_2} \in \Theta_2} \inf_{t \xrightarrow{y} t'} \sup_{s \xrightarrow{x} s'} \inf_{\theta_1 \in \Theta_1^{(k-1)-\sin \theta_2} \in \Theta_2} \int_{t} F(x,t(\operatorname{out}(\theta_1,\theta_2)(t',s')),y \cdot \operatorname{tr}(\operatorname{out}_2(\theta_1,\theta_2)(t',s'))),y \cdot \operatorname{tr}(\operatorname{out}_2(\theta_1,\theta_2)(t',s')),y \cdot \operatorname{tr}(\operatorname{out}_2(\theta_1,\theta_2)(t',s'))),y \cdot \operatorname{tr}(\operatorname{out}_2(\theta_1,\theta_2)(t',s')),y \cdot \operatorname{tr}(\operatorname{out}_2(\theta_1,\theta_2)(t',s'))),y \cdot \operatorname{tr}(\operatorname{out}_2(\theta_1,\theta_2)(t',s')),y \cdot \operatorname{tr}(\operatorname{out}_2(\theta_1,\theta_2)(t',s')),y \cdot \operatorname{tr}(\operatorname{out}_2(\theta_1,\theta_2)(t',s')),y \cdot \operatorname{tr}(\operatorname{out}_2(\theta_1,\theta_2)(t',s')),y \cdot \operatorname{tr}(\operatorname{out}_2(\theta_1,\theta_2)(t',s')),y \cdot \operatorname{tr}(\operatorname{out}_2(\theta_1,\theta_2)(t',s')),y \cdot$$

the next-to-last step by monotonicity of F. By uniformity, the choices of $t \xrightarrow{y} t'$ and $\theta_1 \in \Theta_1^{k-\text{sim}}$ do not depend on each other, so the corresponding inf

and sup can be exchanged, whence

$$\begin{split} I(w_{k,1})(s,t) &= \max \begin{cases} \sup \sup \inf \inf \\ s \xrightarrow{x} s' \; \theta_1 \in \Theta_1^{k\text{-}\mathrm{sim}} \; t \xrightarrow{y} t' \; \theta_2 \in \Theta_2 \\ f(x \cdot \operatorname{tr}(\operatorname{out}_1(\theta_1,\theta_2)(s',t')), y \cdot \operatorname{tr}(\operatorname{out}_2(\theta_1,\theta_2)(s',t'))) \\ \sup \sup \sup \inf \inf \\ t \xrightarrow{y} t' \; \theta_1 \in \Theta_1^{(k-1)\text{-}\mathrm{sim}} \; s \xrightarrow{x} s' \; \theta_2 \in \Theta_2 \\ f(x \cdot \operatorname{tr}(\operatorname{out}_1(\theta_1,\theta_2)(t',s')), y \cdot \operatorname{tr}(\operatorname{out}_2(\theta_1,\theta_2)(t',s'))) \\ &= \max \begin{cases} \sup \inf f(\operatorname{tr}(\operatorname{out}(\theta_1,\theta_2)(s,t))) \\ \theta_1 \in \Theta_{1,\operatorname{ns}}^{k\text{-}\mathrm{sim}} \; \theta_2 \in \Theta_2 \\ \sup \inf f(\operatorname{tr}(\operatorname{out}(\theta_1,\theta_2)(s,t))) \\ \theta_1 \in \Theta_{1,\operatorname{s}}^{k\text{-}\mathrm{sim}} \; \theta_2 \in \Theta_2 \end{cases} \\ &= w_{k,1}(s,t). \end{split}$$

In the last max expression, $\Theta_{1,\mathrm{ns}}^{k\text{-sim}} \subseteq \Theta_{1}^{k\text{-sim}}$ is the subset of Player-1 strategies θ_{1} which do not switch from the configuration (s,t,0), *i.e.*, for which $src(\theta_{1,1}(s,t,0)) = s$, and $\Theta_{1,\mathrm{s}}^{k\text{-sim}} = \Theta_{1}^{k\text{-sim}} \setminus \Theta_{1,\mathrm{ns}}^{k\text{-sim}}$ consists of the strategies which do switch from (s,t,0). The other cases in the definition of $I-I(w_{1,1})$, $I(w_{1,2})$, and $I(w_{k,2})$ for $k \geq 2$ —can be shown similarly, and we can conclude that $I(w_{k,p}) = w_{k,p}$ for all $k \in \mathbb{N}_+ \cup \{\infty\}$, $p \in \{1,2\}$.

To show that w is the least fixed point for I, let $\bar{h}: (\mathbb{N}_+ \cup \{\infty\}) \times \{1, 2\} \to L^{S \times S}$ be such that $I(\bar{h}) = \bar{h}$. We prove that $w \leq \bar{h}$, and again we show only the case $w_{k,1} \leq \bar{h}_{k,1}$ for $k \geq 2$. Note first that as the LTS (S,T) is finitely branching, we can use the equation for $I(\bar{h}_{k,1})(s,t)$ to conclude that for all $s,t \in S$,

for any
$$s \xrightarrow{x} s'$$
 there is $t \xrightarrow{y} t'$ such that $F(x, y, \bar{h}_{k,1}(s', t')) \le I(\bar{h}_{k,1})(s, t),$ (5.3)
for any $t \xrightarrow{y} t'$ there is $s \xrightarrow{x} s'$ such that $F(x, y, \bar{h}_{k-1,2}(s', t')) \le I(\bar{h}_{k,1})(s, t).$ (5.4)

Now let $\theta_1 \in \Theta_1^{k\text{-sim}}$; the proof will be finished once we can find $\theta_2 \in \Theta_2$ for which $f(\operatorname{tr}(\operatorname{out}(\theta_1,\theta_2)(s,t))) \leq \bar{h}_{k,1}(s,t)$. Let $(\pi \cdot e,\rho,m) \in \operatorname{Conf}_2$ and write $s = tgt(\operatorname{last}(\pi)), \ t = tgt(\operatorname{last}(\rho))$. Assume first that e = (s,x,s'), let $t = tgt(\operatorname{last}(\rho))$ and e = (t,y,t') an edge which satisfies the inequality of (5.3), and define $\theta_2(\pi \cdot e,\rho,m) = (e',m)$. For the so-defined Player-2 strategy θ_2 we have $f(\operatorname{tr}(\operatorname{out}(\theta_1,\theta_2)(s,t))) \leq \sup_{s \xrightarrow{x} s'} \inf_{t \xrightarrow{y} t'} F(x,y,\bar{h}_{k,1}(s',t')) \leq I(\bar{h}_{k,1})(s,t) = \bar{h}_{k,1}(s,t)$ for all $s,t \in S$. The case e = (t,y,t') is shown similarly, using (5.4) instead.

The fixed-point characterization for the ready simulation distances is similar (and so is its proof, which we hence omit):

5.19 Theorem. The endofunction I on $(\mathbb{N}_+ \cup \{\infty\}) \times \{1,2\} \to L^{S \times S}$ defined by

$$I(h_{m,p})(s,t) = \begin{cases} \sup \inf_{s \xrightarrow{x} s'} F(x,y,h_{m,1}(s',t')) \\ \sup \inf_{t \xrightarrow{y} t'} F(x,y,h_{m-1,2}(s',t')) \\ \sup \inf_{t \xrightarrow{y} t'} F(x,y,h_{m,1}(s',t')) \end{cases} \quad \text{if } m \ge 2, p = 1$$

$$I(h_{m,p})(s,t) = \begin{cases} \sup \inf_{t \xrightarrow{y} t'} F(x,y,h_{m,1}(s',t')) \\ \sup \inf_{t \xrightarrow{y} t'} F(x,y,h_{m,2}(s',t')) \\ \sup \inf_{t \xrightarrow{y} t'} F(x,y,h_{m,2}(s',t')) \\ \sup \inf_{t \xrightarrow{y} t'} F(x,y,h_{m,2}(s',t')) \end{cases} \quad \text{if } m = 1, p = 1$$

$$\max \begin{cases} \sup \inf_{t \xrightarrow{y} t'} F(x,y,h_{m,2}(s',t')) \\ \sup \inf_{t \xrightarrow{y} t'} F(x,y,h_{m,2}(s',t')) \\ \sup \inf_{t \xrightarrow{y} t'} F(x,y,h_{m,2}(s',t')) \end{cases} \quad \text{if } m = 1, p = 2$$

has a least fixed point $h^*: (\mathbb{N}_+ \cup \{\infty\}) \times \{1,2\} \to L^{S \times S}$, and if the LTS (S,T) is finitely branching, then $d^{k\text{-rsim}} = g \circ h_{k,1}^*$, $d^{k\text{-rsim-eq}} = g \circ \max(h_{k,1}^*, h_{k,2}^*)$ for all $k \in \mathbb{N}_+ \cup \{\infty\}$.

For the *linear* distances, we extend F to a function $\mathbb{K}^n \times \mathbb{K}^n \times L \to L$, for $n \in \mathbb{N}$, by

$$F(\varepsilon, \varepsilon, \alpha) = \alpha, \qquad F(x \cdot \sigma, y \cdot \tau, \alpha) = F(x, y, F(\sigma, \tau, \alpha)).$$

We also extend the \xrightarrow{x} relation to finite traces so we can write $s \xrightarrow{\sigma} s'$ below, by letting $s \xrightarrow{\varepsilon} s$ for all $s \in S$ and $s \xrightarrow{x \cdot \sigma} s'$ iff $s \xrightarrow{x} s'' \xrightarrow{\sigma} s'$ for some $s'' \in S$. We write $s \xrightarrow{\sigma}$ if there is a (finite or infinite) trace σ from s. The proofs of the below theorems are similar to the one of Theorem 5.18.

5.20 Theorem. The endofunction I on $(\mathbb{N}_+ \cup \{\infty\}) \times \{1,2\} \to L^{S \times S}$ defined by

$$I(h_{m,p})(s,t) = \begin{cases} \sup \inf_{s \xrightarrow{\sigma} t \xrightarrow{\tau} \\ \sup \inf_{s \xrightarrow{\sigma} s' t \xrightarrow{\tau} t'} F(\sigma,\tau,h_{m-1,1}(s',t')) & \text{if } m \geq 2, p = 1 \\ \sup \inf_{s \xrightarrow{\sigma} s' t \xrightarrow{\tau} t'} F(\sigma,\tau,h_{m-1,2}(s',t')) & \text{if } m = 1, p = 1 \end{cases}$$

$$I(h_{m,p})(s,t) = \begin{cases} \sup \inf_{s \xrightarrow{\sigma} t \xrightarrow{\tau} t'} F(\sigma,\tau,h_{m-1,2}(s',t')) & \text{if } m = 1, p = 1 \\ \sup \inf_{t \xrightarrow{\tau} t' s \xrightarrow{\sigma} s'} F(\sigma,\tau,h_{m-1,2}(s',t')) & \text{if } m \geq 2, p = 2 \\ \sup \inf_{t \xrightarrow{\tau} t' s \xrightarrow{\sigma} s'} F(\sigma,\tau,h_{m-1,1}(s',t')) & \text{if } m = 1, p = 2 \end{cases}$$

$$\sup \inf_{t \xrightarrow{\tau} t' s \xrightarrow{\sigma} s'} F(\sigma,\tau,h_{m-1,1}(s',t')) & \text{if } m = 1, p = 2 \end{cases}$$

has a least fixed point $h^*: (\mathbb{N}_+ \cup \{\infty\}) \times \{1,2\} \to L^{S \times S}$, and if the LTS (S,T) is finitely branching, then $d^{k\text{-trace}} = g \circ h_{k,1}^*$, $d^{k\text{-trace-eq}} = g \circ \max(h_{k,1}^*, h_{k,2}^*)$ for all $k \in \mathbb{N}_+ \cup \{\infty\}$.

5.21 Theorem. The endofunction I on $(\mathbb{N}_+ \cup \{\infty\}) \times \{1,2\} \to L^{S \times S}$ defined by

$$I(h_{m,p})(s,t) = \begin{cases} \sup_{s \to t^{-}} \inf_{t \to t^{-}} F(\sigma,\tau,h_{m-1,1}(s',t')) & \text{if } m \geq 2, p = 1 \\ \sup_{s \to s'} \inf_{t \to t'} F(\sigma,\tau,h_{m-1,2}(s',t')) & \text{if } m \geq 1, p = 1 \\ \sup_{s \to s'} \inf_{t \to t'} F(\sigma,\tau,h_{m-1,2}(s',t')) & \text{if } m = 1, p = 1 \end{cases}$$

$$I(h_{m,p})(s,t) = \begin{cases} \sup_{s \to t^{-}} \inf_{t \to t'} \sup_{s \to t^{-}} \inf_{t \to t'} F(\sigma,\tau,h_{m-1,2}(s',t')) & \text{if } m = 1, p = 1 \\ \sup_{s \to s'} \inf_{t \to t'} \inf_{t \to t'} \lim_{s \to s'} \lim_{t \to t'} F(\sigma,\tau,t') & \text{if } m \geq 2, p = 2 \end{cases}$$

$$\max_{s \to s'} \begin{cases} \sup_{t \to t'} \inf_{t \to t'} F(\sigma,\tau,h_{m-1,2}(s',t')) & \text{if } m \geq 2, p = 2 \\ \sup_{t \to t'} \inf_{s \to s'} F(\sigma,\tau,h_{m-1,1}(s',t')) & \text{if } m \geq 2, p = 2 \end{cases}$$

$$\max_{t \to t'} \begin{cases} \sup_{t \to t'} \inf_{s \to s'} F(\sigma,\tau,h_{m-1,1}(s',t')) & \text{if } m \geq 1, p = 2 \\ \sup_{t \to t'} \inf_{s \to s'} \lim_{t \to t'} \sup_{s \to t'} \inf_{t \to t'} F(\sigma,\tau,t') & \text{if } m = 1, p = 2 \end{cases}$$

$$\max_{t \to t'} \begin{cases} \sup_{t \to t'} \inf_{s \to t'} F(\sigma,\tau,t') & \text{if } m = 1, p = 2 \\ \sup_{t \to t'} \inf_{s \to t'} F(\sigma,\tau,t') & \text{if } m = 1, p = 2 \end{cases}$$

$$\max_{t \to t'} \begin{cases} \sup_{t \to t'} \inf_{s \to t'} F(\sigma,\tau,t') & \text{if } m = 1, p = 2 \\ \sup_{t \to t'} \inf_{s \to t'} F(\sigma,\tau,t') & \text{if } m = 1, p = 2 \end{cases}$$

has a least fixed point $h^*: (\mathbb{N}_+ \cup \{\infty\}) \times \{1,2\} \to L^{S \times S}$, and if the LTS (S,T) is finitely branching, then $d^{k\text{-ready}} = g \circ h_{k,1}^*$, $d^{k\text{-ready-eq}} = g \circ \max(h_{k,1}^*, h_{k,2}^*)$ for all $k \in \mathbb{N}_+ \cup \{\infty\}$.

The fixed-point characterizations above immediately lead to iterative semi-algorithms for computing the respective distances: to compute for example simulation distance, we can initialize $h^{1-\text{sim}}(s,t)=0$ for all states $s,t\in S$ and then iteratively apply the above equality. This assumes the LTS (S,T) to be finitely branching and uses Kleene's fixed-point theorem and continuity of F. However, this computation is only guaranteed to converge to simulation distance in finitely many steps in case the lattice $L^{S\times S}$ is finite; otherwise, the procedure might not terminate.

5.6.2 Relation Families

Below we show that both simulation and bisimulation distance admit a relational characterization akin to the one of the standard Boolean notions. Using switching counters like we did in the previous section, this can easily be generalized to give relational characterizations to all distances in this chapter.

- **5.22 Theorem.** If the LTS (S,T) is finitely branching, then $d^{1-\text{sim}}(s,t) \leq \varepsilon$ iff there exists a relation family $\mathcal{R} = \{R_{\alpha} \subseteq S \times S \mid \alpha \in L\}$ for which $(s,t) \in R_{\beta} \in R$ for some β with $g(\beta) \leq \varepsilon$, and such that for any $\alpha \in L$ and for all $(s',t') \in R_{\alpha} \in \mathcal{R}$,
 - for all $s' \xrightarrow{x} s''$, there exists $t' \xrightarrow{y} t''$ such that $(s'', t'') \in R_{\alpha'} \in \mathcal{R}$ for some $\alpha' \in L$ with $F(x, y, \alpha') \sqsubseteq \alpha$.

Similarly, $d^{\text{bisim}}(s,t) \leq \varepsilon$ iff there exists a relation family $\mathcal{R} = \{R_{\alpha} \subseteq S \times S \mid \alpha \in L\}$ for which $(s,t) \in R_{\beta} \in \mathcal{R}$ for some β with $g(\beta) \leq \varepsilon$, and such that for any $\alpha \in L$ and for all $(s',t') \in R_{\alpha} \in \mathcal{R}$,

- for all $s' \xrightarrow{x} s''$, there exists $t' \xrightarrow{y} t''$ such that $(s'', t'') \in R_{\alpha'} \in \mathcal{R}$ for some $\alpha' \in L$ with $F(x, y, \alpha') \sqsubseteq \alpha$:
- for all $t' \xrightarrow{y} t''$, there exists $s' \xrightarrow{x} s''$ such that $(s'', t'') \in R_{\alpha'} \in \mathcal{R}$ for some $\alpha' \in L$ with $F(x, y, \alpha') \sqsubseteq \alpha$.

Proof: We only show the proof for simulation distance; for bisimulation distance it is analogous. Assume first that $d^{1-\text{sim}}(s,t) \leq \varepsilon$, then we have $h: S \times S \to L$ for which $g(h(s,t)) \leq \varepsilon$ and

$$h(s',t') = \sup_{s' \xrightarrow{x} s''} \inf_{t' \xrightarrow{y} t''} F(x,y,h(s'',t''))$$

for all $s',t' \in S$. Let $\beta = h(s,t)$, and define a relation family $\mathcal{R} = \{R_{\alpha} \mid \alpha \in L\}$ by $R_{\alpha} = \{(s',t') \mid h(s',t') \sqsubseteq \alpha\}$. Let $\alpha \in L$ and $(s',t') \in R_{\alpha}$, then $\sup_{s'\xrightarrow{x}s''}\inf_{t'\xrightarrow{y}t''}F(x,y,h(s'',t'')) = h(s',t')\sqsubseteq \alpha$, and as (S,T) is finitely branching, this implies that for all $s'\xrightarrow{x}s''$ there is $t'\xrightarrow{y}t''$ and $\alpha'=h(s'',t'')$ such that $(s'',t'')\in R_{\alpha'}$ and $F(x,y,\alpha')\sqsubseteq \alpha$.

For the other direction, assume a relation family as in the theorem and define $h: S \times S \to L$ by $h(s',t') = \inf\{\alpha \mid (s',t' \in R_{\alpha}\})$. Then $(s,t) \in R_{\beta}$ implies that $h(s,t) \sqsubseteq \beta$ and hence $g(h(s,t)) \le \varepsilon$. Let $s',t' \in S$, then $(s',t') \in R_{h(s',t')}$, hence for all $s' \xrightarrow{x} s''$ there is $t' \xrightarrow{y} t''$ and $\alpha' \in L$ for which $F(x,y,\alpha') \sqsubseteq h(s',t')$ and $(s'',t'') \in R_{\alpha'}$, implying $h(s'',t'') \sqsubseteq \alpha'$ and hence $F(x,y,h(s'',t'')) \sqsubseteq h(s',t')$. Collecting the pieces, we get $I(h)(s',t') = \sup_{s'\xrightarrow{x} s''} \inf_{t'\xrightarrow{y} t''} F(x,y,h(s'',t'')) \sqsubseteq h(s',t')$, hence h is a pre-fixed point for I. But then $h^* \sqsubseteq h$, hence $d^{1-\text{sim}}(s,t) = g(h^*(s,t)) \le g(h(s,t)) \le \varepsilon$.

5.7 Recursive Characterizations for Example Distances

We show that the considerations in Section 5.6 apply to all the example distances we have introduced in Section 5.2. We apply Theorem 5.18 to derive fixed-point formulae for corresponding simulation distances, but of course all

other distances in the quantitative linear-time-branching-time spectrum have similar characterizations.

Let d be a hemimetric on \mathbb{K} , then for all $\sigma, \tau \in \mathbb{K}^{\infty}$ and $0 < \lambda \leq 1$,

$$\mathsf{PW}_{\lambda}(d)(\sigma,\tau) = \begin{cases} \max(d(\sigma_0,\tau_0),\lambda\,\mathsf{PW}_{\lambda}(d)(\sigma^1,\tau^1)) & \text{if } \sigma,\tau\neq\varepsilon,\\ \infty & \text{if } \sigma=\varepsilon,\tau\neq\varepsilon \text{ or } \sigma\neq\varepsilon,\tau=\varepsilon,\\ 0 & \text{if } \sigma=\tau=\varepsilon, \end{cases}$$

$$\mathsf{AC}_{\lambda}(d)(\sigma,\tau) = \begin{cases} d(\sigma_0,\tau_0) + \lambda\,\mathsf{AC}_{\lambda}(d)(\sigma^1,\tau^1) & \text{if } \sigma,\tau\neq\varepsilon,\\ \infty & \text{if } \sigma=\varepsilon,\tau\neq\varepsilon \text{ or } \sigma\neq\varepsilon,\tau=\varepsilon,\\ 0 & \text{if } \sigma=\tau=\varepsilon, \end{cases}$$

hence we can apply the iteration theorems with lattice $L = \mathbb{R}_{\geq 0} \cup \{\infty\}$, $g = \mathsf{id}$ the identity function, and the recursion function F given like the formulae above. Using Theorem 5.18 we can derive the following fixed-point expressions for simulation distance:

$$\begin{aligned} \mathsf{PW}_{\lambda}(d)^{\text{1-sim}}(s,t) &= \sup_{s \xrightarrow{x} s'} \inf_{t \xrightarrow{y} t'} \max(d(x,y), \lambda \, \mathsf{PW}_{\lambda}(d)^{\text{1-sim}}(s',t')) \\ \mathsf{AC}_{\lambda}(d)^{\text{1-sim}}(s,t) &= \sup_{s \xrightarrow{x} s'} \inf_{t \xrightarrow{y} t'} (d(x,y) + \lambda \, \mathsf{AC}_{\lambda}(d)^{\text{1-sim}}(s',t')) \end{aligned}$$

Incidentally, these are exactly the expressions introduced in [dAFS09] and in previous chapters.

Also note that if S is finite with |S| = n, then undiscounted pointwise distance $\mathsf{PW}_1(d)$ can only take on the finitely many values $\{d(x,y) \mid (s,x,s'), (t,y,t') \in T\}$, hence the fixed-point algorithm given by Kleene's theorem converges in at most n^2 steps. This algorithm is used in [dAFS09, DLT08, LFT11]. For undiscounted accumulating distance $\mathsf{AC}_1(d)$, it can be shown [LFT11] that with $D = \max\{d(x,y) \mid (s,x,s'), (t,y,t') \in T\}$, distance is either infinite or bounded above by $2n^2D$, hence the $\mathsf{AC}_1(d)$ algorithm either converges in at most $2n^2D$ steps or diverges.

For the limit-average distance AV(d), we let $L = (\mathbb{R}_{\geq 0} \cup \{\infty\})^{\mathbb{N}}$, $g(h) = \lim\inf_j h(j)$, and $f(\sigma,\tau)(j) = \frac{1}{j+1}\sum_{i=0}^j d(\sigma_i,\tau_i)$ the j'th average. The intuition is that L is used for "remembering" how long in the traces we have progressed with the computation. With F given by $F(x,y,h)(n) = \frac{1}{n+1}d(x,y) + \frac{n}{n+1}h(n-1)$ it can be shown that (5.2) holds, giving the following fixed-point expression for limit-average simulation distance (which to the best of our knowledge is new):

$$h_n^{1-\text{sim}}(s,t) = \sup_{s \xrightarrow{x} s'} \inf_{t \xrightarrow{y} t'} \left(\frac{1}{n+1} d(x,y) + \frac{n}{n+1} h_{n-1}^{1-\text{sim}}(s',t') \right)$$

For the maximum-lead distance, we let $L = (\mathbb{R}_{\geq 0} \cup \{\infty\})^{\mathbb{R}}$, the lattice of mappings from leads to maximum leads. Using the notation from Section 5.2,

we let g(h) = h(0) and $f(\sigma, \tau)(\delta) = \max(|\delta|, \sup_j |\delta + \sum_{i=0}^j \sigma_i^w - \sum_{i=0}^j \tau_j^w|)$ the maximum-lead distance between σ and τ assuming that σ already has a lead of δ over τ . With $F(x, y, h)(\delta) = \max(|\delta|, h(\delta + x - y))$ it can be shown that (5.2) holds, and then the fixed-point expression for maximum-lead simulation distance becomes the one given in Chapter 2:

$$h^{1-\text{sim}}(\delta)(s,t) = \sup_{s \xrightarrow{x} s' t} \inf_{t \to t'} \max(|\delta|, h^{1-\text{sim}}(s', t')(\delta + x - y))$$

Again it can be shown [HMP05] that for S finite with |S| = n and $D = \max\{d(x,y) \mid (s,x,s'), (t,y,t') \in T\}$, the iterative algorithm for computing maximum-lead distance either converges in at most $2n^2D$ steps or diverges.

Regarding Cantor distance, a useful recursive formulation is

$$f(\sigma,\tau)(n) = \begin{cases} f(\sigma^1,\tau^1)(n+1) & \text{if } \sigma_0 = \tau_0, \\ n & \text{otherwise,} \end{cases}$$

which iteratively counts the number of matching symbols in σ and τ . Here we use $L = (\mathbb{R}_{\geq 0} \cup \{\infty\})^{\mathbb{N}}$ and $g(h) = \frac{1}{h(0)}$; note that the order on L has to be reversed for g to be monotone. The fixed-point expression for Cantor simulation distance becomes

$$h_n^{\text{1-sim}}(s,t) = \max(n, \sup_{s \xrightarrow{x} s'} \inf_{t \xrightarrow{x} t'} h_{n+1}^{\text{1-sim}}(s',t'))$$

but as the order on L is reversed, the sup now means that Player 1 is trying to *minimize* this expression, and Player 2 tries to maximize it. Hence Player 2 tries to find maximal matching *subtrees*; the corresponding Cantor simulation equivalence distance between s and t hence is the inverse of the maximum depth of matching subtrees under s and t. The Cantor bisimulation distance in turn is the same as the inverse of *bisimulation depth* [HM85].

6 Weighted Modal Transition Systems¹

In this chapter we lift the accumulating distance to *modal specifications*, a specification formalism which permits incremental and compositional design. To this end, we replace the *refinement* relation of standard modal specifications by a refinement *distance*. We then show that our quantitative generalization does *not* admit any notions of determinization or conjunction, but that structural composition and quotient do satisfy the expected quantitative properties.

6.1 Weighted Modal Transition Systems

In this section we present the formalism we use for implementations and specifications. As implementations we choose the model of weighted transition systems, i.e., labeled transition systems with integer weights at transitions. Specifications both have a modal dimension, specifying discrete behavior which must be implemented and behavior which may be present in implementations, and a quantitative dimension, specifying intervals of weights on each transition within are permissible for an implementation.

Let $\mathbb{I} = \{[x,y] \mid x \in \mathbb{Z} \cup \{-\infty\}, y \in \mathbb{Z} \cup \{\infty\}, x \leq y\}$ be the set of closed extended-integer intervals and let Σ be a finite set of actions. Our set of specification labels is $\mathsf{Spec} = \Sigma \times \mathbb{I}$, pairs of actions and intervals. The set of implementation labels is defined as $\mathsf{Imp} = \Sigma \times \{[x,x] \mid x \in \mathbb{Z}\} \approx \Sigma \times \mathbb{Z}$. Hence a specification imposes labels and integer intervals which constrain the possible weights of an implementation.

We define a partial order on \mathbb{I} (representing inclusion of intervals) by $[x,y] \preceq [x',y']$ if $x' \leq x$ and $y \leq y'$, and we extend this order to specification labels by $(a,I) \preceq (a',I')$ if a=a' and $I \preceq I'$. The partial order on Spec is hence a refinement order; if $k_1 \preceq k_2$ for $k_1,k_2 \in \text{Spec}$, then no more implementation labels are contained in k_1 than in k_2 .

Specifications and implementations are defined as follows:

6.1 Definition. A weighted modal transition system (WMTS) is a quadruple $(S, s^0, -- \rightarrow, \longrightarrow)$ consisting of a set of states S with an initial state $s^0 \in S$

 $^{^{1}}$ This chapter is based on the journal paper [BFJ $^{+}$ 13] published in Formal Methods in System Design.

and $must\ (\longrightarrow)$ and $may\ (\dashrightarrow)$ transition relations $\longrightarrow, \dashrightarrow \subseteq S \times \operatorname{Spec} \times S$ such that for every $(s, k, s') \in \longrightarrow$ there is $(s, \ell, s') \in \dashrightarrow$ where $k \leq \ell$. A WMTS is an implementation if $\longrightarrow = \dashrightarrow \subseteq S \times \operatorname{Imp} \times S$.

Note the natural requirement that any required (must) behavior is also allowed (may) above, and that implementations correspond to standard integer-weighted transition systems, where all optional behavior and positioning in the intervals has been decided on.

A WMTS is *finite* if S and \dashrightarrow (and hence also \longrightarrow) are finite sets, and it is deterministic if it holds that for all $s \in S$, $a \in \Sigma$, $(s, (a, I_1), t_1), (s, (a, I_2), t_2) \in \longrightarrow$ imply $I_1 = I_2$ and $t_1 = t_2$. Hence a deterministic specification allows at most one transition under each discrete action from every state. In the rest of the paper we will write $s \xrightarrow{k} s'$ for $(s, k, s') \in \longrightarrow$ and similarly for \longrightarrow , and we will always write $S = (S, s^0, \dashrightarrow)$ or $S_i = (S_i, s_i^0, \dashrightarrow)$ for WMTS and $I = (I, i^0, \longrightarrow)$ for implementations. Note that an implementation is just a usual integer-weighted transition system.

Our theory will work with infinite WMTS, though we will require them to be *compactly branching*. This is a natural generalization of the standard requirement on systems to be *finitely branching* which was first used in [vB96]; see Def. 6.11 below.

The implementation semantics of a specification is given through modal refinement, as follows:

- **6.2 Definition.** A modal refinement of WMTS S_1 , S_2 is a relation $R \subseteq S_1 \times S_2$ such that for any $(s_1, s_2) \in R$
 - whenever $s_1 \xrightarrow{k_1} t_1$ for some $k_1 \in \operatorname{Spec}$, $t_1 \in S_1$, then there exists $s_2 \xrightarrow{k_2} t_2$ for some $k_2 \in \operatorname{Spec}$, $t_2 \in S_2$, such that $k_1 \preceq k_2$ and $(t_1, t_2) \in R$,
 - whenever $s_2 \xrightarrow{k_2} t_2$ for some $k_2 \in \operatorname{Spec}$, $t_2 \in S_2$, then there exists $s_1 \xrightarrow{k_1} t_1$ for some $k_1 \in \operatorname{Spec}$, $t_1 \in S_1$, such that $k_1 \leq k_2$ and $(t_1, t_2) \in R$.

We write $S_1 \leq_{\mathsf{m}} S_2$ if there is a modal refinement relation R for which $(s_1^0, s_2^0) \in R$.

Hence in such a modal refinement, behavior which is required in S_2 is also required in S_1 , no more behavior is allowed in S_1 than in S_2 , and the quantitative requirements in S_1 are refinements of the ones in S_2 . The implementation semantics of a specification can then be defined as the set of all implementations which are also refinements:

6.3 Definition. The implementation semantics of a WMTS S is the set $[S] = \{I \mid I \leq_{\mathsf{m}} S \text{ and } I \text{ is an implementation}\}.$

This conforms with the intuition developed above: if $I \in \llbracket S \rrbracket$, then any (reachable) behavior $i \xrightarrow{a,x} j$ in I must be allowed by a matching transition $s \xrightarrow{a,[l,r]} t$ in S with $l \leq x \leq r$; correspondingly, any (reachable) required behavior $s \xrightarrow{a,[l,r]} t$ in S must be implemented by a matching transition $i \xrightarrow{a,x} j$ in I with $l \leq x \leq r$.

6.2 Thorough and Modal Refinement Distances

For the quantitative specification formalism we have introduced in the last section, the standard Boolean notions of satisfaction and refinement are too fragile. To be able to reason not only whether a given quantitative implementation satisfies a given quantitative specification, but also to what extent, we introduce a notion of *distance* between both implementations and specifications.

We first define the distance between *implementations*; for this we introduce a distance on implementation labels by

$$d_{\mathsf{Imp}}((a_1, x_1), (a_2, x_2)) = \begin{cases} \infty & \text{if } a_1 \neq a_2, \\ |x_1 - x_2| & \text{if } a_1 = a_2. \end{cases}$$
 (6.1)

In the rest of the chapter, let $\lambda \in \mathbb{R}$ with $0 < \lambda < 1$ be a discounting factor.

6.4 Definition. The implementation distance $d: I_1 \times I_2 \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ between the states of implementations I_1 and I_2 is the least fixed point of the equations

$$d(i_1, i_2) = \max \begin{cases} \sup_{\substack{i_1 \xrightarrow{k_1} \\ i_1 \xrightarrow{k_1} j_1}} \inf_{\substack{i_2 \xrightarrow{k_2} \\ 2j_2}} d_{\mathsf{Imp}}(k_1, k_2) + \lambda d(j_1, j_2), \\ \sup_{\substack{i_2 \xrightarrow{k_2} \\ 2j_2}} \inf_{\substack{i_1 \xrightarrow{k_1} \\ i_1 \xrightarrow{k_1} \\ 1 \neq i}} d_{\mathsf{Imp}}(k_1, k_2) + \lambda d(j_1, j_2). \end{cases}$$

We define $d(I_1, I_2) = d(i_1^0, i_2^0)$.

6.5 Lemma. The implementation distance is well-defined, and is a pseudometric.

Proof: This is precisely the accumulating bisimulation distance from Chapter 5, so the statement follows from Proposition 5.5. See also the proof of Lemma 4.3.

We remark that besides this accumulating distance, other interesting system distances may be defined depending on the application at hand, *cf.* Chapter 5. We concentrate here on this distance and leave a generalization to other distances for the next chapter.

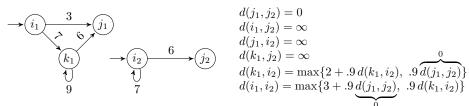


Figure 6.1: Two weighted transition systems with branching distance $d(I_1, I_2) = 18$.

6.6 Example. Consider the two implementations I_1 and I_2 in Figure 6.1 with a single action (elided for simplicity) and with discounting factor $\lambda = .9$. The equations in the illustration have already been simplified by removing all expressions that evaluate to ∞ . What remains to be done is to compute the least fixed point of the equation $d(k_1, i_2) = \max\{2 + .9 d(k_1, i_2), 0\}$. Clearly 0 is not a fixed point, and solving the equation $d(k_1, i_2) = 2 + .9 d(k_1, i_2)$ gives $d(k_1, i_2) = 20$. Hence $d(i_1, i_2) = \max\{3, .9 \cdot 20\} = 18$.

Note that the interpretation of the distance between two implementations depends entirely on the application one has in mind; but it can easily be shown that the distance between two implementations is zero iff they are weighted bisimilar. The intuition is then that the smaller the distance, the closer the implementations are to being bisimilar.

To lift the implementation distance to specifications, we need first to consider the distance between *sets* of implementations. Given implementation sets $\mathcal{I}_1, \mathcal{I}_2 \subseteq \mathsf{Imp}$, we define

$$d(\mathcal{I}_1, \mathcal{I}_2) = \sup_{I_1 \in \mathcal{I}_1} \inf_{I_2 \in \mathcal{I}_2} d(I_1, I_2)$$

Note that in case \mathcal{I}_2 is finite, we have that for all $\varepsilon \geq 0$, $d(\mathcal{I}_1, \mathcal{I}_2) \leq \varepsilon$ if and only if for each implementation $I_1 \in \mathcal{I}_1$ there exists $I_2 \in \mathcal{I}_2$ for which $d(I_1, I_2) \leq \varepsilon$, hence this is quite a natural notion of distance. Especially, $d(\mathcal{I}_1, \mathcal{I}_2) = 0$ if \mathcal{I}_1 is a subset of \mathcal{I}_2 up to bisimilarity. For infinite \mathcal{I}_2 , we have the slightly more complicated property that $d(\mathcal{I}_1, \mathcal{I}_2) \leq \varepsilon$ iff for all $\delta > 0$ and any $I_1 \in \mathcal{I}_1$, there is $I_2 \in \mathcal{I}_2$ for which $d(I_1, I_2) \leq \varepsilon + \delta$.

We lift this distance to specifications as follows:

- **6.7 Definition.** The thorough refinement distance between WMTS S_1 and S_2 is defined as $d_{\mathsf{th}}(S_1, S_2) = d(\llbracket S_1 \rrbracket, \llbracket S_2 \rrbracket)$. We write $S_1 \leq_{\mathsf{th}}^{\varepsilon} S_2$ if $d_{\mathsf{th}}(S_1, S_2) \leq \varepsilon$.
- **6.8 Lemma.** The thorough refinement distance is a hemimetric.

Proof: To show that $d_{\mathsf{th}}(S,S) = 0$ is trivial, and the triangle inequality $d_{\mathsf{th}}(S_1,S_2) + d_{\mathsf{th}}(S_2,S_3) \geq d_{\mathsf{th}}(S_1,S_3)$ follows like in the proof of [AB07, Lemma 3.72].

Indeed this permits us to measure incompatibility of specifications; intuitively, if two specifications have thorough distance ε , then any implementation of the first specification can be matched by an implementation of the second up to ε . Also observe the special case where $S_1 = I_1$ is an implementation: then $d_{\mathsf{th}}(I_1, S_2) = \inf_{I_2 \in \llbracket S_2 \rrbracket} d(I_1, I_2)$, which measures how close I_1 is to satisfy the specification S_2 .

We now proceed to introduce modal refinement distance as an overapproximation of thorough refinement distance.

First we generalize the distance on implementation labels from Equation (6.1) to specification labels, again using a Hausdorff-type construction. For $k, \ell \in \mathsf{Spec}$ we define

$$d_{\mathsf{Spec}}(k,\ell) = \sup_{k' \preccurlyeq k, k' \in \mathsf{Imp}} \inf_{\ell' \preccurlyeq \ell, \ell' \in \mathsf{Imp}} d_{\mathsf{Imp}}(k',\ell').$$

Note that d_{Spec} is asymmetric, and that $d_{\mathsf{Spec}}(k,\ell) = 0$ if and only if $k \leq \ell$. Also, $d_{\mathsf{Spec}}(k,\ell) = d_{\mathsf{Imp}}(k,\ell)$ for all $k,\ell \in \mathsf{Imp}$. In more elementary terms, we can express d_{Spec} as follows:

$$\begin{aligned} d_{\mathsf{Spec}}\big((a_1,I_1),(a_2,I_2)\big) &= \infty \quad \text{if } a_1 \neq a_2 \\ d_{\mathsf{Spec}}\big((a,[x_1,y_1]),(a,[x_2,y_2])\big) &= \max(x_2-x_1,y_1-y_2,0) \end{aligned}$$

6.9 Definition. Let S_1 , S_2 be WMTS. The modal refinement distance $d_{\mathsf{m}}: S_1 \times S_2 \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ from states of S_1 to states of S_2 is the least fixed point of the equations

$$d_{\mathsf{m}}(s_{1}, s_{2}) = \max \begin{cases} \sup_{\substack{s_{1} \\ s_{1} - \rightarrow 1} t_{1}} \inf_{s_{2} - \rightarrow 2} d_{\mathsf{Spec}}(k_{1}, k_{2}) + \lambda d_{\mathsf{m}}(t_{1}, t_{2}), \\ \sup_{\substack{s_{2} \text{ } k_{2} \\ s_{2} \neq 2}} \inf_{s_{1} \text{ } k_{1} + \lambda t_{1}} d_{\mathsf{Spec}}(k_{1}, k_{2}) + \lambda d_{\mathsf{m}}(t_{1}, t_{2}). \end{cases}$$

We define $d_{\mathsf{m}}(S_1,S_2)=d_{\mathsf{m}}(s_1^0,s_2^0)$, and we write $S_1\leq_{\mathsf{m}}^{\varepsilon}S_2$ if $d_{\mathsf{m}}(S_1,S_2)\leq\varepsilon$.

6.10 Lemma. The modal refinement distance is well-defined, and is a hemimetric.

Proof: Like in the proof of Lemma 6.5, the argument for existence of a unique least fixed point to the defining equations is that they define a contraction. The triangle inequality can again be shown inductively, and the property $d_{\mathsf{m}}(s,s)=0$ is clear.

We can now give a precise definition of compact branching:

6.11 Definition. A WMTS S is said to be *compactly branching* if the sets $\{(s',k) \mid s \xrightarrow{k} s'\}, \{(s',k) \mid s \xrightarrow{k} s'\} \subseteq S \times \text{Spec}$ are compact under the symmetrized product distance $\bar{d}_{m} \times \bar{d}_{\text{Spec}}$ for every $s \in S$.

The notion of compact branching was first introduced, for a formalism of metric transition systems, in [vB96]. It is a natural generalization of the standard requirement on transition systems to be finitely branching to a distance setting; we will need it for the property that continuous functions defined on the sets $\{(s',k) \mid s \xrightarrow{k} s'\}, \{(s',k) \mid s \xrightarrow{k} s'\} \subseteq S \times \text{Spec}$, for some $s \in S$, attain their infimum and supremum, see Lemma 6.13 and its proof below.

Thus, we shall henceforth assume all our WMTS to be compactly branching. The following lemma sets up some sufficient conditions for this to be the case.

6.12 Lemma. Let S be a WMTS and define the sets $L_i(s, a)$, $U_i(s, a)$ for all $s \in S$, $a \in \Sigma$ and $i \in \{1, 2\}$ by

$$L_{1}(s,a) = \{l \mid s \xrightarrow{a,[l,r]} s'\}, \qquad L_{2}(s,a) = \{l \mid s \xrightarrow{a,[l,r]} s'\},$$

$$U_{1}(s,a) = \{r \mid s \xrightarrow{a,[l,r]} s'\}, \qquad U_{2}(s,a) = \{r \mid s \xrightarrow{a,[l,r]} s'\}.$$

Then S is compactly branching if

- for all $s \in S$, any Cauchy sequence $(s'_n)_{n \in \mathbb{N}}$ in $\{s' \mid s \dashrightarrow s'\}$ (with pseudometric \bar{d}_m) has $\lim_{n \to \infty} s_n \in \{s' \mid s \dashrightarrow s'\}$, and likewise, any Cauchy sequence $(s'_n)_{n \in \mathbb{N}}$ in $\{s' \mid s \longrightarrow s'\}$ has $\lim_{n \to \infty} s_n \in \{s' \mid s \longrightarrow s'\}$, and
- for all $s \in S$, $a \in \Sigma$ and $i \in \{1, 2\}$, $L_i(s, a)$ is finite or $-\infty \in L_i(s, a)$, and $U_i(s, a)$ is finite or $\infty \in U_i(s, a)$.

Note that the first property mimicks (and generalizes) standard properties of finite branching and saturation, cf. [San09, Sect. 3.3]. The intuition is that if s has (either may or must) transitions to a converging sequence of states, then it also has a transition to the limit.

Proof: The first condition implies that the sets $\{s' \in S \mid s \dashrightarrow s'\}$ and $\{s' \in S \mid s \longrightarrow s'\}$ are compact in the pseudometric \bar{d}_m for all $s \in S$. By Tychonoff's theorem, products of compact sets are compact, so we need only show that the second condition implies that the sets $\{k \in \text{Spec} \mid s \xrightarrow{k} s'\}$ and $\{k \in \text{Spec} \mid s \xrightarrow{k} s'\}$ are compact in the pseudometric \bar{d}_{Spec} for every $s \in S$.

Let $s \in S$. By definition of d_{Spec} , the sets $\{k \mid s \xrightarrow{k} s'\}$, $\{k \mid s \xrightarrow{k} s'\}$ fall into connected components $\{I \mid s \xrightarrow{a,I} s'\}$, $\{I \mid s \xrightarrow{a,I} s'\}$ for all $a \in \Sigma$, hence the former are compact iff all the latter are. These in turn are compact iff the four sets L_i , U_i in the lemma, collecting lower and upper bounds of intervals, are compact. Now interval bounds are extended integers, so a sequence in L_i or U_i converges iff it is eventually stable or goes towards $-\infty$ or ∞ . If the sets are finite, eventual stability is the only option; if they are infinite, they

need to include the limit points $-\infty$ (for the lower interval bounds in L_i) or ∞ (for the upper interval bounds in U_i).

We extend the notion of relation families from revious chapters to modal refinement distance. We define a modal refinement family as an $\mathbb{R}_{\geq 0}$ -indexed family of relations $\mathcal{R} = \{R_{\varepsilon} \subseteq S_1 \times S_2 \mid \varepsilon \geq 0\}$ such that for any ε and any $(s_1, s_2) \in R_{\varepsilon} \in \mathcal{R}$,

- whenever $s_1 \xrightarrow{k_1} t_1$ for some $k_1 \in \operatorname{Spec}$, $t_1 \in S_1$, then there exists $s_2 \xrightarrow{k_2} t_2$ for some $k_2 \in \operatorname{Spec}$, $t_2 \in S_2$, such that $d_{\operatorname{Spec}}(k_1, k_2) \leq \varepsilon$ and $(t_1, t_2) \in R_{\varepsilon'} \in \mathcal{R}$ for some $\varepsilon' \leq \lambda^{-1} (\varepsilon d_{\operatorname{Spec}}(k_1, k_2))$,
- whenever $s_2 \xrightarrow{k_2} t_2$ for some $k_2 \in \operatorname{Spec}$, $t_2 \in S_2$, then there exists $s_1 \xrightarrow{k_1} t_1$ for some $k_1 \in \operatorname{Spec}$, $t_1 \in S_1$, such that $d_{\operatorname{Spec}}(k_1, k_2) \leq \varepsilon$ and $(t_1, t_2) \in R_{\varepsilon'} \in \mathcal{R}$ for some $\varepsilon' \leq \lambda^{-1} (\varepsilon d_{\operatorname{Spec}}(k_1, k_2))$.

Note that modal refinement families are

- upward closed in the sense that $(s_1, s_2) \in R_{\varepsilon}$ implies that $(s_1, s_2) \in R_{\varepsilon'}$ for all $\varepsilon' \geq \varepsilon$, and
- downward compact in the sense that for any set $E \subseteq \mathbb{R}_{\geq 0}$, if $(s_1, s_2) \in R_{\varepsilon}$ for all $\varepsilon \in E$, then also $(s_1, s_2) \in R_{\inf E}$. This property follows from the assumption that our WMTS are compactly branching.

Following the proof strategy developed in previous chapters for implementations, we can show the following characterization of modal refinement distance by modal refinement families:

6.13 Lemma. $S_1 \leq_{\mathsf{m}}^{\varepsilon} S_2$ iff there is a modal refinement family \mathcal{R} with $(s_1^0, s_2^0) \in R_{\varepsilon} \in \mathcal{R}$.

Proof: First, assume that $S_1 \leq_{\mathsf{m}}^{\varepsilon} S_2$, *i.e.*, $d_{\mathsf{m}}(s_1^0, s_2^0) \leq \varepsilon$, and define a relation family $\mathcal{R} = \{R_{\delta} \mid \delta \geq 0\}$ by $R_{\delta} = \{(s_1, s_2) \in S_1 \times S_2 \mid d_{\mathsf{m}}(s_1, s_2) \leq \delta\}$ for all $\delta \geq 0$, then $(s_1^0, s_2^0) \in R_{\varepsilon}$ holds by assumption. We show that \mathcal{R} is a modal refinement family. Let $(s_1, s_2) \in R_{\delta}$ for some $\delta \geq 0$, then by definition we know that $d_{\mathsf{m}}(s_1, s_2) \leq \delta$. Assume $s_1 \xrightarrow{k_1} t_1$. From $d_{\mathsf{m}}(s_1, s_2) \leq \delta$ we can infer that

$$\inf_{\substack{k_2\\s_2 \to \to_2 t_2}} d_{\mathsf{Spec}}(k_1, k_2) + \lambda d_{\mathsf{m}}(t_1, t_2) \leq \delta.$$

Hence, because S_2 is compactly branching, there exists a may-transition $s_2 \xrightarrow{k_2} t_2$ such that $d_{\mathsf{Spec}}(k_1, k_2) \leq \delta$ and $d_{\mathsf{m}}(t_1, t_2) \leq \lambda^{-1}(\delta - d_{\mathsf{Spec}}(k_1, k_2))$. The latter implies that $(t_1, t_2) \in R_{\delta'}$ for some $\delta' \leq \lambda^{-1}(\delta - d_{\mathsf{Spec}}(k_1, k_2))$ which was to be shown. The argument for the other assertion for must-transitions is

symmetric. This proves that there is a modal refinement family \mathcal{R} such that $(s_1^0, s_2^0) \in R_{\varepsilon} \in \mathcal{R}$.

For the reverse direction, assume that $(s_1^0, s_2^0) \in R_{\varepsilon} \in \mathcal{R}$ for some modal refinement family $\mathcal{R} = \{R_{\varepsilon} \mid \varepsilon \geq 0\}$. We prove that $(s_1, s_2) \in R_{\delta}$, for some $\delta \geq 0$, implies $d_{\mathsf{m}}(s_1, s_2) \leq \delta$. The claim $S_1 \leq_{\mathsf{m}}^{\varepsilon} S_2$ then follows from the assumption $(s_1^0, s_2^0) \in R_{\varepsilon}$.

To this end, observe that the space of functions $\Delta = [S_1 \times S_2 \to \mathbb{R}_{\geq 0} \cup \{\infty\}]$ forms a complete lattice, when the partial order \leq_{Δ} is defined such that for $f, f' \in \Delta, f \leq_{\Delta} f'$ iff $f(s_1, s_2) \leq f'(s_1, s_2)$ for all $s_1 \in S_1, s_2 \in S_2$. Moreover, since max, sup, inf and + are monotone, the function D defined for all $f \in \Delta$ by

$$D(f) = \max \begin{cases} \sup_{\substack{k_1 \\ s_1 - \cdots >_1 t_1}} \inf_{\substack{k_2 \\ s_2 - \cdots >_2 t_2}} d_{\mathsf{Spec}}(k_1, k_2) + \lambda f(t_1, t_2), \\ \sup_{\substack{k_2 \\ s_2 + \cdots >_2 t_2}} \inf_{\substack{k_1 \\ s_1 - \cdots >_1 t_1}} d_{\mathsf{Spec}}(k_1, k_2) + \lambda f(t_1, t_2), \end{cases}$$

is a monotone endofunction on Δ , hence by Tarski's fixed point theorem, D has a least fixed point. Now define $h(s_1, s_2) = \inf\{\delta \mid (s_1, s_2) \in R_\delta \in \mathcal{R}\}$; since R_δ is downward compact, we have $(s_1, s_2) \in R_{h(s_1, s_2)}$. By showing that h is a pre-fixed point of D, *i.e.*, that $D(h) \leq_{\Delta} h$, we get that $(s_1, s_2) \in R_\delta$ implies that $d_{\mathbf{m}}(s_1, s_2) \leq \delta$, since $h(s_1, s_2) \leq \delta$ and $d_{\mathbf{m}}(s_1, s_2) \leq h(s_1, s_2)$.

Since $(s_1, s_2) \in R_{h(s_1, s_2)}$, every $s_1 \xrightarrow{k_1} s_1'$ can be matched by some $s_2 \xrightarrow{k_2} s_2'$ such that $d_{\mathsf{Spec}}(k_1, k_2) + \lambda \delta' \leq h(s_1, s_2)$ for some δ' where $(s_1', s_2') \in R_{\delta'}$, implying $h(s_1', s_2') \leq \delta'$, but then also $d_{\mathsf{Spec}}(k_1, k_2) + \lambda h(s_1', s_2') \leq h(s_1, s_2)$. Similarly, every $s_2 \xrightarrow{k_2} s_2'$ has a match $s_1 \xrightarrow{k_1} s_1'$ such that $d_{\mathsf{Spec}}(k_1, k_2) + \lambda h(s_1', s_2') \leq h(s_1, s_2)$. Hence we have $D(h) \leq_{\Delta} h$ which was to be shown. \square

The next theorems show that modal refinement distance indeed overapproximates thorough refinement distance, and that it is exact for deterministic WMTS. Note that nothing general can be said about the precision of the overapproximation in the nondeterministic case; the standard counterexample given for the Boolean case in [BKLS09] shows that there exist WMTS for which $d_{\mathsf{th}}(S_1, S_2) = 0$ but $d_{\mathsf{m}}(S_1, S_2) = \infty$.

6.14 Theorem. For WMTS S_1 , S_2 we have $d_{\mathsf{th}}(S_1, S_2) \leq d_{\mathsf{m}}(S_1, S_2)$.

Proof: If $d_{\mathsf{m}}(S_1, S_2) = \infty$, we have nothing to prove. Otherwise, let $\mathcal{R} = \{R_{\varepsilon} \subseteq S_1 \times S_2 \mid \varepsilon \geq 0\}$ be a modal refinement family which witnesses $d_{\mathsf{m}}(S_1, S_2)$, *i.e.*, such that $(s_1^0, s_2^0) \in R_{d_{\mathsf{m}}(S_1, S_2)}$, and let $I_1 \in \llbracket S_1 \rrbracket$. We have to expose $I_2 \in \llbracket S_2 \rrbracket$ for which $d(I_1, I_2) \leq d_{\mathsf{m}}(S_1, S_2)$.

Let $R \subseteq I_1 \times S_1$ be a witness for $I_1 \leq_{\mathsf{m}} S_1$, define $R'_{\varepsilon} = R \circ R_{\varepsilon} \subseteq I_1 \times S_2$ for all $\varepsilon \geq 0$, and let $\mathcal{R}' = \{R'_{\varepsilon} \mid \varepsilon \geq 0\}$. The states of $I_2 = (I_2, i_2^0, \mathsf{Imp}, \to_{I_2})$ are $I_2 = S_2$ with $i_2^0 = s_2^0$, and the transitions we define as follows:

For any $i_1 \xrightarrow{k'_1} j_1$ and any $s_2 \in S_2$ for which $(i_1, s_2) \in R'_{\varepsilon} \in \mathcal{R}'$ for some ε , we have $s_2 \xrightarrow{k_2} t_2$ in S_2 with $d_{\mathsf{Spec}}(k'_1, k_2) \leq \varepsilon$ and $(j_1, t_2) \in R'_{\varepsilon'} \in \mathcal{R}'$ for some $\varepsilon' \leq \lambda^{-1} (\varepsilon - d_{\mathsf{Spec}}(k'_1, k_2))$. Write $k'_1 = (a'_1, x'_1)$ and $k_2 = (a_2, [x_2, y_2])$, then we must have $a'_1 = a_2$. Let

$$x_2' = \begin{cases} x_2 & \text{if } x_1' < x_2, \\ x_1' & \text{if } x_2 \le x_1' \le y_2, \\ y_2 & \text{if } x_1' > y_2 \end{cases}$$
 (6.2)

and $k_2' = (a_2, x_2')$, and put $s_2 \xrightarrow{k_2'} I_2$ in I_2 . Note that

$$d_{\mathsf{Spec}}(k_1', k_2') = d_{\mathsf{Spec}}(k_1', k_2). \tag{6.3}$$

Similarly, for any $s_2 \xrightarrow{k_2} t_2$ in S_2 and any $i_1 \in I_1$ with $(i_1, s_2) \in R'_{\varepsilon} \in \mathcal{R}'$ for some ε , we have $i_1 \xrightarrow{k'_1} j_1$ with $d_{\mathsf{Spec}}(k'_1, k_2) \leq \varepsilon$ and $(j_1, t_2) \in R'_{\varepsilon'} \in \mathcal{R}'$ for some $\varepsilon' \leq \lambda^{-1} (\varepsilon - d_{\mathsf{Spec}}(k'_1, k_2))$. Write $k'_1 = (a'_1, x'_1)$ and $k_2 = (a_2, [x_2, y_2])$, define x'_2 as in (6.2) and $k'_2 = (a_2, x'_2)$, and put $s_2 \xrightarrow{k'_2} I_2$ to in I_2 . We show that the identity relation $\mathsf{id}_{S_2} = \{(s_2, s_2) \mid s_2 \in S_2\} \subseteq S_2 \times S_2$

We show that the identity relation $\operatorname{id}_{S_2} = \{(s_2, s_2) \mid s_2 \in S_2\} \subseteq S_2 \times S_2$ witnesses $I_2 \leq_{\mathsf{m}} S_2$. Let first $s_2 \xrightarrow{k'_2} I_2$ t₂; we must have used one of the two constructions above for creating this transition. In the first case, we have $s_2 \xrightarrow{k_2} t_2$ with $k'_2 \preccurlyeq k_2$, and in the second case, we have $s_2 \xrightarrow{k_2} t_2$, hence also $s_2 \xrightarrow{k_2} t_2$, with the same property. For a transition $s_2 \xrightarrow{k_2} t_2$ on the other hand, we have introduced $s_2 \xrightarrow{k'_2} I_2$ in the second construction above, with $k'_2 \preccurlyeq k_2$.

We also want to show that the family \mathcal{R}' is a witness for $d(I_1,I_2) \leq d_{\mathsf{m}}(S_1,S_2)$. We have $(i_1^0,s_2^0) \in R'_{d_{\mathsf{m}}(S_1,S_2)} = \tilde{R} \circ R_{d_{\mathsf{m}}(S_1,S_2)}$, so let $(i_1,s_2) \in R'_{\varepsilon} \in \mathcal{R}'$ for some $\varepsilon \geq 0$. For any $i_1 \xrightarrow{k'_1} I_1$ j_1 we have $s_2 \xrightarrow{k_2} t_2$ and $s_2 \xrightarrow{k'_2} I_2$ by the first part of our construction above, with $d_{\mathsf{Spec}}(k'_1,k'_2) = d_{\mathsf{Spec}}(k'_1,k_2) \leq \varepsilon$ because of (6.3), and also $(j_1,t_2) \in R'_{\varepsilon'} \in \mathcal{R}'$ for some $\varepsilon' \leq \lambda^{-1}(\varepsilon - d_{\mathsf{Spec}}(k'_1,k_2))$. For any $s_2 \xrightarrow{k'_2} I_2$ we must have used one of the constructions above to introduce this transition, and both give us $i_1 \xrightarrow{k'_1} I_1$ j_1 with $d_{\mathsf{Spec}}(k'_1,k'_2) \leq \varepsilon$ and $(j_1,t_2) \in R'_{\varepsilon'} \in \mathcal{R}'$ for some $\varepsilon' \leq \lambda^{-1}(\varepsilon - d_{\mathsf{Spec}}(k'_1,k_2))$. \square

The fact that modal refinement only equals thorough refinement for deterministic specifications is well-known from the theory of modal transition systems [Lar89], and the special case of S_2 deterministic is important, as it can be argued [Lar89] that deterministic specifications are sufficient for applications.

6.15 Theorem. If S_2 is deterministic, then $d_{\mathsf{th}}(S_1, S_2) = d_{\mathsf{m}}(S_1, S_2)$.

Proof: If $d_{\mathsf{th}}(S_1, S_2) = \infty$, we are done by Theorem 6.14. Otherwise, let $\mathcal{R} = \{R_{\varepsilon} \mid \varepsilon \geq 0\}$ be the smallest relation family for which

- $(s_1^0, s_2^0) \in R_{d_{\mathsf{th}}(S_1, S_2)}$ and
- whenever we have $(s_1, s_2) \in R_{\varepsilon} \in \mathcal{R}$, $s_1 \stackrel{a,I_1}{\dashrightarrow} t_1$, and $s_2 \stackrel{a,I_2}{\dashrightarrow} t_2$, then $(t_1, t_2) \in R_{\lambda^{-1}(\varepsilon d_{\mathsf{Spec}}((a,I_1),(a,I_2)))} \in \mathcal{R}$.

We show below that \mathcal{R} is well-defined (also that $\varepsilon - d_{\mathsf{Spec}}((a, I_1), (a, I_2)) \geq 0$ in all cases) and a modal refinement family. We will use the convenient notation (s_1, S_1) for the WMTS S_1 with initial state s_1^0 replaced by s_1 , similarly for (s_2, S_2) .

We first show inductively that for any pair of states $(s_1, s_2) \in R_{\varepsilon} \in \mathcal{R}$ we have $d_{\mathsf{th}}((s_1, S_1), (s_2, S_2)) \leq \varepsilon$. This is obviously the case for $s_1 = s_1^0$ and $s_1 = s_2^0$, so assume now that $(s_1, s_2) \in R_{\varepsilon} \in \mathcal{R}$ is such that $d_{\mathsf{th}}((s_1, S_1), (s_2, S_2)) \leq \varepsilon$ and let $s_1 \xrightarrow{a, I_1} t_1$, $s_2 \xrightarrow{a, I_2} t_2$. Let $(q'_1, P'_1) \in [(t_1, S_1)]$ and $x_1 \in I_1$.

There is an implementation $(p_1, P_1) \in \llbracket (s_1, S_1) \rrbracket$ for which $p_1 \xrightarrow{a, x_1} q_1$ and such that $(q_1, P_1) \leq_{\mathsf{m}} (q_1', P_1')$. Now

$$d_{\mathsf{th}}((p_1, P_1), (s_2, S_2)) \le d_{\mathsf{th}}((p_1, P_1), (s_1, S_1)) + d_{\mathsf{th}}((s_1, S_1), (s_2, S_2)) \le \varepsilon,$$

hence we must have $s_2 \xrightarrow{a_2', I_2'} t_2'$ with $d_{\mathsf{Spec}}((a, x_1), (a_2', I_2')) \leq \varepsilon$. But then $a_2' = a$, hence by determinism of S_2 , $I_2 = I_2'$ and $t_2 = t_2'$.

The above considerations hold for any $x_1 \in I_1$, hence $d_{\mathsf{Spec}}((a, I_1), (a, I_2)) \leq \varepsilon$. Thus $\varepsilon - d_{\mathsf{Spec}}((a, I_1), (a, I_2)) \geq 0$, and the definition of \mathcal{R} above is justified. Now let $x_2 \in I_2$ such that $d_{\mathsf{Spec}}((a, x_1), (a, x_2)) = d_{\mathsf{Spec}}((a, x_1), (a, I_2))$, then there is an implementation $(p_2, P_2) \in \llbracket (s_2, S_2) \rrbracket$ for which $p_2 \xrightarrow{a, x_2} q_2$, and

$$\begin{split} d\big((q_1', P_1'), (q_2, P_2)\big) &\leq \lambda^{-1} \big(\varepsilon - d_{\mathsf{Spec}}((a, x_1), (a, x_2))\big) \\ &= \lambda^{-1} \big(\varepsilon - d_{\mathsf{Spec}}((a, I_1), (a, I_2))\big) \,, \end{split}$$

which, as $(q'_1, P'_1) \in [(t_1, S_1)]$ was chosen arbitrarily, entails

$$d_{\mathsf{th}}\big((s_1, S_1), (s_2, S_2)\big) \le \lambda^{-1}\big(\varepsilon - d_{\mathsf{Spec}}((a, I_1), (a, I_2))\big)$$
.

We are ready to show that \mathcal{R} is a modal refinement family. Let $(s_1,s_2) \in R_{\varepsilon} \in \mathcal{R}$ for some ε , and assume $s_1 \overset{a,I_1}{--} t_1$. Let $x \in I_1$, then there is $(p,P^x) \in \llbracket (s_1,S_1) \rrbracket$ with a transition $p \overset{m}{\longrightarrow} q$. Now $d_{\mathsf{th}}((p,P^x),(s_2,S_2)) \leq \varepsilon$, hence we have a transition $s_2 \overset{a,I_2^x}{--} t_2^x$ with $d_{\mathsf{Spec}}((a,x),(a,I_2^x)) \leq \varepsilon$. Also for any other $x' \in I_1$ we have a transition $s_2 \overset{a,I_2^x}{--} t_2^x$ with $d_{\mathsf{Spec}}((a,x),(a,I_2^x)) \leq \varepsilon$, hence by determinism of S_2 , $I_2^x = I_2^{x'}$ and $t_2^x = t_2^{x'}$. It follows that there is a unique transition $s_2 \overset{a,I_2}{--} t_2$, and as $d_{\mathsf{Spec}}((a,x),(a,I_2)) \leq \varepsilon$ for all $x \in I_1$, we

have $d_{\mathsf{Spec}}((a,I_1),(a,I_2)) \leq \varepsilon$, and $(t_1,t_2) \in R_{\lambda^{-1}(\varepsilon-d_{\mathsf{Spec}}((a,I_1),(a,I_2)))} \in \mathcal{R}$ by definition.

Now assume $s_2 \xrightarrow{a,I_2} t_2$. Let $(p_1,P_1) \in \llbracket (s_1,S_1) \rrbracket$, then we have $(p_2,P_2) \in \llbracket (s_2,S_2) \rrbracket$ with $d((p_1,P_1),(p_2,P_2)) \leq \varepsilon$. Now any $(p_2,P_2) \in \llbracket (s_2,S_2) \rrbracket$ has $p_2 \xrightarrow{a,x_2} q_2$ with $x_2 \in I_2$, thus there is also $p_1 \xrightarrow{a,x_1} q_1$ with $d_{\mathsf{Spec}}((a,x_1),(a,x_2)) \leq \varepsilon$ and $d((q_1,P_1),(q_2,P_2)) \leq \lambda^{-1}(\varepsilon - d_{\mathsf{Spec}}((a,x_1),(a,x_2)))$. This in turn implies that $s_1 \xrightarrow{a,I_1} t_1$ for some $x_1 \in I_1$. We will be done once we can show $d_{\mathsf{Spec}}((a,I_1),(a,I_2)) \leq \varepsilon$, so assume to the contrary that there is $x_1' \in I_1$ with $d_{\mathsf{Spec}}((a,x_1'),(a,I_2)) > \varepsilon$. Then there must be an implementation $(p_1',P_1') \in \llbracket (s_1,S_1) \rrbracket$ with $p_1' \xrightarrow{a,x_1'} q_1'$, hence a transition $s_2 \xrightarrow{a,I_2'} t_2'$ with $d_{\mathsf{Spec}}((a,x_1'),(a,I_2')) \leq \varepsilon$. But $I_2' = I_2$ by determinism of S_2 , a contradiction.

6.3 Relaxation

We introduce here a notion of *relaxation* which is specific to the quantitative setting. Intuitively, relaxing a specification means to weaken the quantitative constraints, while the discrete demands on which transitions may or must be present in implementations are kept. A similar notion of *strengthening* may be defined, but we do not use this here.

6.16 Definition. For WMTS S, S' and $\varepsilon \geq 0$, S' is an ε -relaxation of S if $S \leq_{\mathsf{m}} S'$ and $S' \leq_{\mathsf{m}}^{\varepsilon} S$.

Hence the quantitative constraints in S' may be more permissive than the ones in S, but no new discrete behavior may be introduced. Also note that any implementation of S is also an implementation of S', and no implementation of S' is further than ε away from an implementation of S. The following proposition relates specifications to relaxed specifications:

6.17 Proposition. If S_1' and S_2' are ε -relaxations of S_1 and S_2 , respectively, then $d_{\mathsf{m}}(S_1,S_2)-\varepsilon \leq d_{\mathsf{m}}(S_1,S_2') \leq d_{\mathsf{m}}(S_1,S_2)$ and $d_{\mathsf{m}}(S_1,S_2) \leq d_{\mathsf{m}}(S_1',S_2) \leq d_{\mathsf{m}}(S_1,S_2)+\varepsilon$.

Proof: By the triangle inequality we have

$$\begin{aligned} d_{\mathsf{m}}(S_{1}, S_{2}') &\leq d_{\mathsf{m}}(S_{1}, S_{2}) + d_{\mathsf{m}}(S_{2}, S_{2}'), \\ d_{\mathsf{m}}(S_{1}, S_{2}) &\leq d_{\mathsf{m}}(S_{1}, S_{2}') + d_{\mathsf{m}}(S_{2}', S_{2}), \\ d_{\mathsf{m}}(S_{1}, S_{2}) &\leq d_{\mathsf{m}}(S_{1}, S_{1}') + d_{\mathsf{m}}(S_{1}', S_{2}), \\ d_{\mathsf{m}}(S_{1}', S_{2}) &\leq d_{\mathsf{m}}(S_{1}', S_{1}) + d_{\mathsf{m}}(S_{1}, S_{2}). \end{aligned}$$

On the syntactic level, we can introduce the following widening operator which relaxes all quantitative constraints in a systematic manner. We write $I \pm \delta = [x - \delta, y + \delta]$ for an interval I = [x, y] and $\delta \in \mathbb{N}$.

6.18 Definition. Given $\delta \in \mathbb{N}$, the δ -widening of a WMTS S is the WMTS $S^{+\delta}$ with transitions $s \xrightarrow{a,I \pm \delta} t$ in $S^{+\delta}$ for all $s \xrightarrow{a,I} t$ in S, and $s \xrightarrow{a,I \pm \delta} t$ in $S^{+\delta}$ for all $s \xrightarrow{a,I} t$ in S.

Widening and relaxation are related as follows; note also that as widening is a global operation whereas relaxation may be achieved entirely locally, not all relaxations may be obtained as widenings.

6.19 Proposition. The δ -widening of any WMTS S is a $\frac{\delta}{1-\lambda}$ -relaxation.

Proof: For the first claim, the identity relation $\operatorname{id}_S = \{(s,s) \mid s \in S\} \subseteq S \times S$ is a witness for $S \leq_{\mathsf{m}} S^{+\delta}$: if $s \stackrel{k}{\dashrightarrow} t$, then by construction $s \stackrel{k_2}{\dashrightarrow} t \in t$ with $k \preccurlyeq k_2$, and if $s \stackrel{k_2}{\longrightarrow} t \in t$, then again by construction $s \stackrel{k}{\longrightarrow} t$ for some $k \preccurlyeq k_2$. Now to prove $d_{\mathsf{m}}(S^{+\delta}, S) \leq (1 - \lambda)^{-1}\delta$, we define a family of relations $\mathcal{R} = t$

Now to prove $d_{\mathsf{m}}(S^{+\varepsilon}, S) \leq (1-\lambda)^{-1}\delta$, we define a family of relations $\mathcal{R} = \{R_{\varepsilon} \mid \varepsilon \geq 0\}$ by $R_{\varepsilon} = \emptyset$ for $\varepsilon < (1-\lambda)^{-1}\delta$ and $R_{\varepsilon} = \mathsf{id}_S$ for $\varepsilon \geq (1-\lambda)^{-1}\delta$. We show that \mathcal{R} is a modal refinement family.

Let $(s,s) \in R_{\varepsilon}$ for some $\varepsilon \geq (1-\lambda)^{-1}\delta$, and assume $s \xrightarrow{k_2} t$. By construction there is a transition $s \xrightarrow{k} t$ with $d_{\mathsf{Spec}}(k_2,k) \leq \delta \leq \varepsilon$. Now

$$\frac{1}{\lambda} \Big(\varepsilon - d_{\mathsf{Spec}}(k_2, k) \Big) \geq \frac{1}{\lambda} \Big(\frac{\delta}{1 - \lambda} - \delta \Big) = \frac{\delta}{1 - \lambda} \geq \varepsilon$$

and $(t,t) \in R_{\varepsilon}$, which settles this part of the proof. The other direction, starting with a transition $s \xrightarrow{k} t$, is similar.

There is also an implementation-level notion which corresponds to relaxation:

- **6.20 Definition.** The ε -extended implementation semantics, for $\varepsilon \geq 0$, of a WMTS S is $[S]^{+\varepsilon} = \{I \mid I \leq_{\mathsf{m}}^{\varepsilon} S, I \text{ implementation}\}.$
- **6.21 Proposition.** If S' is an ε -relaxation of S, then $[S'] \subseteq [S]^{+\varepsilon}$.

Proof: If $I \in [S']$, then $d_{\mathsf{m}}(I,S') = 0$, hence $d_{\mathsf{m}}(I,S) \leq \varepsilon$ by Proposition 6.17, which in turn implies that $I \in [S]^{+\varepsilon}$.

The example in Figure 6.2 shows that there are WMTS S, S' such that S' is an ε -relaxation of S but the inclusion $[S'] \subseteq [S]^{+\varepsilon}$ is strict. Indeed, for $\delta = 1$ and $\lambda = .9$, we have $I \in [S]^{+(1-\lambda)^{-1}\delta}$, but $I \notin [S^{+\delta}]$.

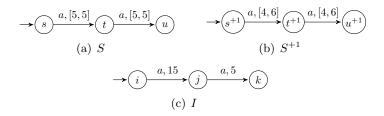


Figure 6.2: WMTS S and implementation I for which $I \in \llbracket S \rrbracket^{+(1-\lambda)^{-1}\delta}$, for $\delta = 1$ and $\lambda = .9$ (thus $(1-\lambda)^{-1}\delta = 10$), but $I \notin \llbracket S^{+\delta} \rrbracket$, so that $\llbracket S^{+\delta} \rrbracket \subsetneq \llbracket S \rrbracket^{+(1-\lambda)^{-1}\delta}$, even though $S^{+\delta}$ is a $(1-\lambda)^{-1}\delta$ -relaxation of S.

6.4 Limitations of the Quantitative Approach

In this section we turn our attention towards some of the standard operators for specification theories; determinization and logical conjunction. In the standard Boolean setting, there is indeed a determinization operator which derives the smallest deterministic overapproximation of a specification, which is useful because it enables checking thorough refinement, *cf.* Theorem 6.15. Quite surprisingly, we show that in the quantitative setting, there are problems with these notions which do not appear in the Boolean theory. More specifically, we show that there is no determinization operator which always yields a smallest deterministic overapproximation, and there is no conjunction operator which acts as a greatest lower bound.

6.22 Theorem. There is no unary operator \mathcal{D} on WMTS for which it holds that

- (6.22.1) $\mathcal{D}(S)$ is deterministic for any WMTS S,
- (6.22.2) $S \leq_{\mathsf{m}} \mathcal{D}(S)$ for any WMTS S,
- (6.22.3) $S \leq_{\mathsf{m}}^{\varepsilon} D$ implies $\mathcal{D}(S) \leq_{\mathsf{m}}^{\varepsilon} D$ for any WMTS S, any deterministic WMTS D, and any $\varepsilon \geq 0$.

Proof: There is a determinization operator \mathcal{D}' on WMTS which satisfies Properties (6.22.1) and (6.22.2) above and a weaker version of Property (6.22.3) with $\varepsilon = 0$:

(6.22.3') $S \leq_{\mathsf{m}} D$ implies $\mathcal{D}'(S) \leq_{\mathsf{m}} D$ for any WMTS S and any deterministic WMTS D

This \mathcal{D}' can be defined as follows: For a WMTS $S = (S, s_0, \dashrightarrow, \longrightarrow)$,

$$\mathcal{D}'(S) = (2^S \setminus \{\emptyset\}, \{s_0\}, {\longrightarrow}_d, {\longrightarrow}_d),$$

where 2^S is the power set of S and the transition relations $-- \rightarrow_d$ and \longrightarrow_d are defined as follows: Let $\mathcal{T} \in (2^S \setminus \{\emptyset\})$ be a state in $\mathcal{D}'(S)$. For every

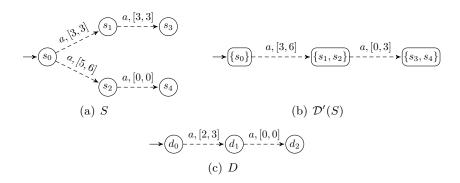


Figure 6.3: Counter-example for Theorem 6.22: $d_{\mathsf{m}}(\mathcal{D}'(S), D) = 3 + 3\lambda$ and $d_{\mathsf{m}}(S, D) = \max(3, 3\lambda) = 3$, hence $d_{\mathsf{m}}(\mathcal{D}'(S), D) \not\leq d_{\mathsf{m}}(S, D)$.

maximal, nonempty set $L_a \subseteq \{I \mid \exists s \in \mathcal{T} : s \xrightarrow{a,I} \}$ for some $a \in \Sigma$, we have $\mathcal{T} \xrightarrow{a,\bigcup L_a} \mathcal{T}_a$ where $\mathcal{T}_a = \{s' \in S \mid \exists s \in \mathcal{T}, I \in L_a : s \xrightarrow{a,I} s'\}$ and $\bigcup L_a$ is the smallest interval containing all intervals from L_a . If, moreover, for each $s \in \mathcal{T}$ we have $s \xrightarrow{a,I} s'$ for some $s' \in \mathcal{T}_a$ and some $I \in L_a$, then $\mathcal{T} \xrightarrow{a,\bigcup L_a} \mathcal{T}_a$. It is straightforward to prove that \mathcal{D}' satisfies the expected properties.

Assume now that there is an operator \mathcal{D} as in the theorem. Then for any WMTS S, $S \leq_{\mathsf{m}} \mathcal{D}'(S)$ and thus $\mathcal{D}(S) \leq_{\mathsf{m}} \mathcal{D}'(S)$ by (6.22.3), and $S \leq_{\mathsf{m}} \mathcal{D}(S)$ and hence $\mathcal{D}'(S) \leq_{\mathsf{m}} \mathcal{D}(S)$ by (6.22.3'). We finish the proof by showing that the operator \mathcal{D}' does not satisfy (6.22.3). The example in Figure 6.3 shows a WMTS S and a deterministic WMTS D for which $d_{\mathsf{m}}(\mathcal{D}'(S), D) = 3 + 3\lambda$ and $d_{\mathsf{m}}(S, D) = \max(3, 3\lambda) = 3$, hence $d_{\mathsf{m}}(\mathcal{D}'(S), D) \not\leq d_{\mathsf{m}}(S, D)$.

Likewise, the greatest-lower-bound property of logical conjunction in the Boolean setting ensures that the set of implementations of a conjunction of specifications is precisely the intersection of the implementation sets of the two specifications. Conjoining two WMTS naturally involves a partial label conjunction operator \otimes . We let $(a_1, I_1) \otimes (a_2, I_2)$ be undefined if $a_1 \neq a_2$, and otherwise

$$(a, [x_1, y_1]) \oslash (a, [x_2, y_2]) = \begin{cases} (a, [\max(x_1, x_2), \min(y_1, y_2)]) \\ & \text{if } \max(x_1, x_2) \leq \min(y_1, y_2), \\ & \text{undefined} & \text{otherwise.} \end{cases}$$

Before we show that such a conjunction operator for WMTS does not exist in general, we need to define a *pruning operator* which removes inconsistent states that naturally arise when conjoining two WMTS. The intuition is that if a WMTS S_1 requires a behavior $s_1 \xrightarrow{k_1} 1$ for which there is no may transition $s_2 \xrightarrow{k_2} 1$ such that $s_1 \otimes s_2 \otimes 1$ is defined, then the state $s_1 \otimes s_2 \otimes 1$ in the conjunction is *inconsistent* and will have to be pruned away, together with all *must*

$$(a) S \qquad (b) S_1$$

$$(c) S_2 \qquad (d) S_1 \qquad (d) S_1$$

Figure 6.4: Counter-example for Theorem 6.24: $d_{\mathsf{m}}(S, S_1) = d_{\mathsf{m}}(S, S_2) = 1$, but $d_{\mathsf{m}}(S, S_1 \wedge S_2) = \infty$.

transitions leading to it. In the definition below, pre* denotes the reflexive, transitive closure of pre.

- **6.23 Definition.** For a WMTS S, let pre : $2^S \to 2^S$ be given by $\operatorname{pre}(B) = \{s \in S \mid s \xrightarrow{k} t \in B \text{ for some } k\}$. Let $B \subseteq S$ be a set of inconsistent states. If $s^0 \notin \operatorname{pre}^*(B)$, then the pruning of S w.r.t. B is defined by $\rho^B(S) = (S_\rho, s^0, -\rightarrow_\rho, \longrightarrow_\rho)$ where $S_\rho = S \setminus \operatorname{pre}^*(B), -\rightarrow_\rho = -\rightarrow \cap (S_\rho \times \operatorname{Spec} \times S_\rho)$ and $\longrightarrow_\rho = \longrightarrow \cap (S_\rho \times \operatorname{Spec} \times S_\rho)$.
- **6.24 Theorem.** There is no partial binary operator \land on WMTS for which it holds that, for all WMTS S, S_1 , S_2 such that S_1 and S_2 are deterministic,
 - (6.24.1) whenever $S_1 \wedge S_2$ is defined, then $S_1 \wedge S_2 \leq_{\mathsf{m}} S_1$ and $S_1 \wedge S_2 \leq_{\mathsf{m}} S_2$,
 - (6.24.2) whenever $S \leq_{\mathsf{m}} S_1$ and $S \leq_{\mathsf{m}} S_2$, then $S_1 \wedge S_2$ is defined and $S \leq_{\mathsf{m}} S_1 \wedge S_2$,
 - (6.24.3) for any $\varepsilon \geq 0$, there exist $\varepsilon_1 \geq 0$ and $\varepsilon_2 \geq 0$ such that if $S_1 \wedge S_2$ is defined, $S \leq_{\mathsf{m}}^{\varepsilon_1} S_1$ and $S \leq_{\mathsf{m}}^{\varepsilon_2} S_2$, then $S \leq_{\mathsf{m}}^{\varepsilon} S_1 \wedge S_2$.

 following rules:

$$\frac{s_1 \xrightarrow{k_1} s_1' \quad s_2 \xrightarrow{k_2} s_2' \quad k_1 \otimes k_2 \text{ defined}}{(s,t) \xrightarrow{k_1 \otimes k_2} (s_1',s_2')} \qquad \frac{s_1 \xrightarrow{k_1} s_1' \quad s_2 \xrightarrow{k_2} s_2' \quad k_1 \otimes k_2 \text{ defined}}{(s_1,s_2) \xrightarrow{k_1 \otimes k_2} (s_1',s_2')}$$

$$\xrightarrow{s_1 \xrightarrow{k_1} s_2' \xrightarrow{k_2} s_2'} \qquad k_1 \otimes k_2 \text{ defined}}$$

$$\xrightarrow{(s_1,s_2) \xrightarrow{k_1 \otimes k_2} (s_1',s_2')}$$

$$\xrightarrow{s_1 \xrightarrow{k_1} \qquad (k_1 \otimes k_2 \text{ undefined for any } k_2 \text{ such that } s_2 \xrightarrow{k_2})}$$

$$\xrightarrow{(s_1,s_2) \in B}$$

$$\xrightarrow{s_2 \xrightarrow{k_2} \qquad (k_1 \otimes k_2 \text{ undefined for any } k_1 \text{ such that } s_1 \xrightarrow{k_1})}$$

$$\xrightarrow{(s_1,s_2) \in B}$$

Using these properties, one can see that for all deterministic WMTS S_1 and S_2 , $S_1 \wedge S_2 \leq_{\mathsf{m}} S_1 \wedge' S_2$ and $S_1 \wedge' S_2 \leq_{\mathsf{m}} S_1 \wedge S_2$. The WMTS depicted in Figure 6.4 then show that Property (6.24.3) cannot hold: here, $d_{\mathsf{m}}(S, S_1) = d_{\mathsf{m}}(S, S_2) = 1$, but $d_{\mathsf{m}}(S, S_1 \wedge S_2) = \infty$.

The counterexamples used in the proofs of Theorems 6.22 and 6.24 are quite general and apply to a large class of distances, rather than only to the accumulating distance discussed in this paper. Hence it can be argued that what we have exposed here is a fundamental limitation of any quantitative approach to modal specifications.

6.5 Structural Composition and Quotient

In this section we show that in our quantitative setting, notions of structural composition and quotient can be defined which obey the properties expected of such operations. In particular, structural composition satisfies independent implementability [dAH05], hence the refinement distance between structural composites can be bounded by the distances between their respective components.

First we define partial synchronization operators \oplus and \ominus on specification labels which will be used for synchronizing transitions. We let $(a_1, I_1) \oplus (a_2, I_2)$ and $(a_1, I_1) \ominus (a_2, I_2)$ be undefined if $a_1 \neq a_2$, and otherwise

$$(a, [x_1, y_1]) \oplus (a, [x_2, y_2]) = (a, [x_1 + x_2, y_1 + y_2]),$$

$$(a, [x_1, y_1]) \ominus (a, [x_2, y_2]) = \begin{cases} \text{undefined} & \text{if } x_1 - x_2 > y_1 - y_2, \\ (a, [x_1 - x_2, y_1 - y_2]) & \text{if } x_1 - x_2 \le y_1 - y_2. \end{cases}$$

Note that we use CSP-style synchronization, but other types of synchronization can easily be defined. Also, defining \oplus to add intervals (and \ominus to subtract them) is only one particular choice; depending on the application, one can also

e.g., let \oplus be intersection of intervals or some other operation. It is not difficult to see that these alternative synchronization operators would lead to properties similar to those we show here.

$$\frac{s_1 \xrightarrow{k_1} t_1 \quad s_2 \xrightarrow{k_2} t_2 \quad k_1 \oplus k_2 \text{ def.}}{(s_1, s_2) \xrightarrow{k_1 \oplus k_2} (t_1, t_2)} \qquad \frac{s_1 \xrightarrow{k_1} t_1 \quad s_2 \xrightarrow{k_2} t_2 \quad k_1 \oplus k_2 \text{ def.}}{(s_1, s_2) \xrightarrow{k_1 \oplus k_2} (t_1, t_2)}$$

Note that during the quotient construction inconsistent states can arise which are then recursively removed using the pruning operator ρ , see Definition 6.23. After a technical lemma, the next theorem shows that structural composition is well-behaved with respect to modal refinement distance in the sense that the distance between the composed systems is bounded by the distances of the individual systems. Note also the special case in the theorem of $S_1 \leq_{\mathsf{m}} S_2$ and $S_3 \leq_{\mathsf{m}} S_4$ implying $S_1 \| S_3 \leq_{\mathsf{m}} S_2 \| S_4$.

6.26 Lemma. For $k_1, k_2, k_3, k_4 \in \text{Spec}$ with $k_1 \oplus k_3$ and $k_2 \oplus k_4$ defined, we have $d_{\text{Spec}}(k_1 \oplus k_3, k_2 \oplus k_4) \leq d_{\text{Spec}}(k_1, k_2) + d_{\text{Spec}}(k_3, k_4)$.

Proof: Let $k_i = (a, [x_i, y_i])$ for all i. We have

$$\begin{split} d_{\mathsf{Spec}}(k_1,k_2) + d_{\mathsf{Spec}}(k_3,k_4) \\ &= \max(x_2 - x_1, y_1 - y_2, 0) + \max(x_4 - x_3, y_3 - y_4, 0) \\ &\geq \max\left((x_2 - x_1) + (x_4 - x_3), (y_1 - y_2) + (y_3 - y_4), 0\right) \\ &= \max\left((x_2 + x_4) - (x_1 + x_3), (y_1 + y_3) - (y_2 + y_4), 0\right) \\ &= d_{\mathsf{Spec}}(k_1 \oplus k_3, k_2 \oplus k_4). \end{split}$$

6.27 Theorem (Independent implementability). For WMTS S_1 , S_2 , S_3 , S_4 we have $d_{\mathsf{m}}(S_1 \| S_3, S_2 \| S_4) \leq d_{\mathsf{m}}(S_1, S_2) + d_{\mathsf{m}}(S_3, S_4)$.

Proof: If $d_{\mathsf{m}}(S_1, S_2) = \infty$ or $d_{\mathsf{m}}(S_3, S_4) = \infty$, we have nothing to prove. Otherwise, let $\mathcal{R}^1 = \{R^1_{\varepsilon} \subseteq S_1 \times S_2 \mid \varepsilon \geq 0\}$, $\mathcal{R}^2 = \{R^2_{\varepsilon} \subseteq S_3 \times S_4 \mid \varepsilon \geq 0\}$ be witnesses for $d_{\mathsf{m}}(S_1, S_2)$ and $d_{\mathsf{m}}(S_3, S_4)$, respectively; hence $(s^0_1, s^0_2) \in R^1_{d_{\mathsf{m}}(S_1, S_2)} \in \mathcal{R}^1$ and $(s^0_3, s^0_4) \in R^2_{d_{\mathsf{m}}(S_3, S_4)} \in \mathcal{R}^2$. Define

$$R_{\varepsilon} = \{ ((s_1, s_3), (s_2, s_4)) \in S_1 \times S_3 \times S_2 \times S_4 \mid (s_1, s_2) \in R_{\varepsilon_1}^1 \in R^1, (s_3, s_4) \in R_{\varepsilon_2}^2 \in R^2, \varepsilon_1 + \varepsilon_2 \le \varepsilon \}$$

for all $\varepsilon \geq 0$ and let $\mathcal{R} = \{R_{\varepsilon} \mid \varepsilon \geq 0\}$. We show that \mathcal{R} witnesses $d_{\mathsf{m}}(S_1||S_3, S_2||S_4) \leq d_{\mathsf{m}}(S_1, S_2) + d_{\mathsf{m}}(S_3, S_4)$.

We have $((s_1^0, s_3^0), (s_2^0, s_4^0)) \in R_{d_{\mathsf{m}}(S_1, S_2) + d_{\mathsf{m}}(S_3, S_4)} \in \mathcal{R}$. Now let

$$((s_1, s_3), (s_2, s_4)) \in R_{\varepsilon} \in \mathcal{R}$$

for some ε , then $(s_1, s_2) \in R^1_{\varepsilon_1} \in \mathcal{R}^1$ and $(s_3, s_4) \in R^2_{\varepsilon_2} \in \mathcal{R}^2$ for some $\varepsilon_1 + \varepsilon_2 \leq \varepsilon$.

Assume $(s_1, s_3) \xrightarrow{k_1 \oplus k_3} (t_1, t_3)$, then $s_1 \xrightarrow{k_1} t_1$ and $s_3 \xrightarrow{k_3} t_3$. By $(s_1, s_2) \in R^1_{\varepsilon_1} \in \mathcal{R}^1$, we have $s_2 \xrightarrow{k_2} t_2$ with $d_{\mathsf{Spec}}(k_1, k_2) \leq \varepsilon_1$ and $(t_1, t_2) \in R^1_{\varepsilon_1'} \in \mathcal{R}^1$ for some $\varepsilon_1' \leq \lambda^{-1} (\varepsilon_1 - d_{\mathsf{Spec}}(k_1, k_2))$; similarly, $s_4 \xrightarrow{k_4} t_4$ with $d_{\mathsf{Spec}}(k_3, k_4) \leq \varepsilon_2$ and $(t_3, t_4) \in R^2_{\varepsilon_2'} \in \mathcal{R}^2$ for some $\varepsilon_2' \leq \lambda^{-1} (\varepsilon_2 - d_{\mathsf{Spec}}(k_3, k_4))$. Let $\varepsilon_2' = \varepsilon_1' + \varepsilon_2'$, then the sum $k_2 \oplus k_4$ is defined, and

$$\varepsilon' \le \lambda^{-1} (\varepsilon_1 + \varepsilon_2 - (d_{\mathsf{Spec}}(k_1, k_2) + d_{\mathsf{Spec}}(k_3, k_4)))$$

$$\le \lambda^{-1} (\varepsilon - d_{\mathsf{Spec}}(k_1 \oplus k_3, k_2 \oplus k_4))$$

by Lemma 6.26. We have $(s_2, s_4) \xrightarrow{k_2 \oplus k_4} (t_2, t_4)$, $d_{\mathsf{Spec}}(k_1 \oplus k_3, k_2 \oplus k_4) \leq \varepsilon_1 + \varepsilon_2 \leq \varepsilon$ again by Lemma 6.26, and $((t_1, t_3), (t_2, t_4)) \in R_{\varepsilon'} \in \mathcal{R}$. The reverse direction, starting with a transition $(s_2, s_4) \xrightarrow{k_2 \oplus k_4} (t_2, t_4)$, is similar. \square

Again after a technical lemma, the next theorem expresses the fact that quotient is a partial inverse to structural composition. Intuitively, the theorem shows that the quotient S_1/S_2 is maximal among all WMTS S_3 with respect to any distance $S_2 \| S_3 \leq_{\mathsf{m}}^{\varepsilon} S_1$; note the special case of $S_3 \leq_{\mathsf{m}} S_1/S_2$ iff $S_2 \| S_3 \leq_{\mathsf{m}} S_1$.

6.28 Lemma. If $k_1, k_2, k_3 \in \text{Spec}$ are such that $k_1 \ominus k_2$ and $k_2 \oplus k_3$ are defined, then $d_{\text{Spec}}(k_3, k_1 \ominus k_2) = d_{\text{Spec}}(k_2 \oplus k_3, k_1)$.

Proof: We can write $k_i = (a, [x_i, y_i])$ for some $a \in \Sigma$. Then

$$\begin{split} d_{\mathsf{Spec}}(k_3,k_1\ominus k_2) &= \max\big((x_1-x_2)-x_3,y_3-(y_1-y_2),0\big) \\ &= \left\{ \begin{array}{ll} x_1-x_2-x_3 & \text{if} & x_1-x_2-x_3\geq 0,\\ & x_1-x_2-x_3\geq y_3-y_1+y_2;\\ y_3-y_1+y_2 & \text{if} & y_3-y_1+y_2\geq 0,\\ & y_3-y_1+y_2\geq x_1-x_2-x_3;\\ 0 & \text{if} & x_1-x_2-x_3\leq 0,\\ & y_3-y_1+y_2\leq 0. \end{array} \right. \end{split}$$

Similarly,

$$d_{\mathsf{Spec}}(k_2 \oplus k_3, k_1) = \max \left(x_1 - (x_2 + x_3), (y_2 + y_3) - y_1, 0 \right)$$

$$= \begin{cases} x_1 - x_2 - x_3 & \text{if} \quad x_1 - x_2 - x_3 \ge 0, \\ x_1 - x_2 - x_3 \ge y_2 + y_3 - y_1; \\ y_2 + y_3 - y_1 & \text{if} \quad y_2 + y_3 - y_1 \ge 0, \\ y_2 + y_3 - y_1 \ge x_1 - x_2 - x_3; \\ 0 & \text{if} \quad x_1 - x_2 - x_3 \le 0, \\ y_2 + y_3 - y_1 \le 0. \end{cases}$$

6.29 Theorem (Soundness and maximality of quotient). Let S_1 , S_2 and S_3 be locally consistent WMTS such that S_2 is deterministic and S_1/S_2 is defined. If $d_{\mathsf{m}}(S_3, S_1/S_2) < \infty$, then $d_{\mathsf{m}}(S_3, S_1/S_2) = d_{\mathsf{m}}(S_2 \| S_3, S_1)$.

Proof: To avoid confusion, we write $-\rightarrow$ / and \longrightarrow / for transitions in S_1/S_2 and $-\rightarrow$ || and \longrightarrow || for transitions in $S_2||S_3$. The inequality $d_{\mathsf{m}}(S_3,S_1/S_2) \geq d_{\mathsf{m}}(S_2||S_3,S_1)$ is trivial if $d_{\mathsf{m}}(S_2||S_3,S_1) = \infty$, so assume the opposite and let $\mathcal{R}^1 = \{R_{\varepsilon}^1 \subseteq S_3 \times (S_1 \times S_2 \cup \{u\}) \mid \varepsilon \geq 0\}$ be a witness for $d_{\mathsf{m}}(S_3,S_1/S_2)$. Define $R_{\varepsilon}^2 = \{((s_2,s_3),s_1) \mid (s_3,(s_1,s_2)) \in R_{\varepsilon}^1\} \subseteq S_2 \times S_3 \times S_1$ for all $\varepsilon \geq 0$, and let $\mathcal{R}^2 = \{R_{\varepsilon}^2 \mid \varepsilon \geq 0\}$. Certainly $((s_2^0,s_3^0),s_1^0) \in R_{d_{\mathsf{m}}(S_3,S_1/S_2)}^2 \in \mathcal{R}^2$, so let now $((s_2,s_3),s_1) \in R_{\varepsilon}^2 \in \mathcal{R}^2$ for some $\varepsilon \geq 0$.

Assume $(s_2, s_3) \xrightarrow{k_2 \oplus k_3} (t_2, t_3)$, then also $s_2 \xrightarrow{k_2} t_2$ and $s_3 \xrightarrow{k_3} t_3$. We have $(s_3, (s_1, s_2)) \in R^1_{\varepsilon}$, so there is $(s_1, s_2) \xrightarrow{k_1 \oplus k'_2} (t_1, t'_2)$ for which $d_{\mathsf{Spec}}(k_3, k_1 \ominus k'_2) = d_{\mathsf{Spec}}(k'_2 \oplus k_3, k_1) \leq \varepsilon$ and such that $(t_3, (t_1, t'_2)) \in R^1_{\varepsilon'} \in \mathcal{R}^1$, hence $((t'_2, t_3), t_1) \in R^2_{\varepsilon'} \in \mathcal{R}^2$, for some $\varepsilon' \leq \lambda^{-1}(\varepsilon - d_{\mathsf{Spec}}(k'_2 \oplus k_3, k_1))$. By definition of quotient we must have $s_1 \xrightarrow{k_1} t_1$ and $s_2 \xrightarrow{k'_2} t'_2$, and by determinism of S_2 , $k'_2 = k_2$ and $t'_2 = t_2$.

Assume $s_1 \xrightarrow{k_1} t_1$. We must have a transition $s_2 \xrightarrow{k_2} t_2$ for which $k_1 \ominus k_2$ is defined. Hence $(s_1, s_2) \xrightarrow{k_1 \ominus k_2} (t_1, t_2)$. This in turn implies that there is $s_3 \xrightarrow{k_3} t_3$ for which $d_{\mathsf{Spec}}(k_3, k_1 \ominus k_2) = d_{\mathsf{Spec}}(k_2 \oplus k_3, k_1) \leq \varepsilon$ and

$$(a) S_1 \xrightarrow{a, [0, 0]} \underbrace{(b) S_2} \xrightarrow{a, [0, 1]} \underbrace{(c) S_3} \xrightarrow{a, [0, 0]} \underbrace{(c) S_3}$$

$$\xrightarrow{a, [0, 0]} \underbrace{(c) S_3} \xrightarrow{a, [0, 1]} \underbrace{(c)$$

Figure 6.5: WMTS for which $d_{m}(S_{2}||S_{3}, S_{1}) \neq d_{m}(S_{3}, S_{1}/S_{2}) = \infty$.

such that $(t_3, (t_1, t_2)) \in R^1_{\varepsilon'} \in \mathcal{R}^1$, hence $((t_2, t_3), t_1) \in R^2_{\varepsilon'} \in \mathcal{R}^2$, for some $\varepsilon' \leq \lambda^{-1}(\varepsilon - d_{\mathsf{Spec}}(k_2 \oplus k_3, k_1))$, and by definition of parallel composition, $(s_2, s_3) \xrightarrow{k_2 \oplus k_3} (t_2, t_3)$.

To show that $d_{\mathsf{m}}(S_3, S_1/S_2) \leq d_{\mathsf{m}}(S_2 || S_3, S_1)$, let $\mathcal{R}^2 = \{R_{\varepsilon}^2 \subseteq S_2 \times S_3 \times S_1 \mid \varepsilon \geq 0\}$ be a witness for $d_{\mathsf{m}}(S_2 || S_3, S_1)$, define $R_{\varepsilon}^1 = \{(s_3, (s_1, s_2)) \mid ((s_2, s_3), s_1) \in R_{\varepsilon}^2\} \cup \{(s_3, u) \mid s_3 \in S_3\}$ for all $\varepsilon \geq 0$, and let $\mathcal{R}^1 = \{R_{\varepsilon}^1 \mid \varepsilon \geq 0\}$, then $(s_3^0, (s_1^0, s_2^0)) \in R_{d_{\mathsf{m}}(S_2 || S_3, S_1)}^1 \in \mathcal{R}^1$.

For any $(s_3, u) \in R^1_{\varepsilon}$ for some $\varepsilon \geq 0$, any transition $s_3 \xrightarrow{k_3} t_3$ can be matched by $u \xrightarrow{k_3} u$, and then $(t_3, u) \in R^1_0$. Let now $(s_3, (s_1, s_2)) \in R^1_{\varepsilon} \in \mathcal{R}^1$ for some $\varepsilon \geq 0$, and assume $s_3 \xrightarrow{k_3} t_3$. If $k_2 \oplus k_3$ is undefined for all transitions $s_2 \xrightarrow{k_2} t_2$, then by definition $(s_1, s_2) \xrightarrow{k_3} u$, and again $(t_3, u) \in R^1_0$. If there is a transition $s_2 \xrightarrow{k_2} t_2$ such that $k_2 \oplus k_3$ is defined, then also $(s_2, s_3) \xrightarrow{k_2 \oplus k_3} (t_2, t_3)$. Hence we have $s_1 \xrightarrow{k_1} t_1$ with $d_{\mathsf{Spec}}(k_2 \oplus k_3, k_1) \leq \varepsilon$, implying that $(s_1, s_2) \xrightarrow{k_1 \oplus k_2} (t_1, t_2)$. Hence $d_{\mathsf{Spec}}(k_3, k_1 \oplus k_2) = d_{\mathsf{Spec}}(k_2 \oplus k_3, k_1) \leq \varepsilon$. Also, $((t_2, t_3), t_1) \in R^2_{\varepsilon'} \in \mathcal{R}^2$, hence $(t_3, (t_1, t_2)) \in R^1_{\varepsilon'} \in \mathcal{R}^1$, for some $\varepsilon' \leq \lambda^{-1}(\varepsilon - d_{\mathsf{Spec}}(k_3, k_1 \oplus k_2))$.

Assume $(s_1, s_2) \xrightarrow{k_1 \ominus k_2} / (t_1, t_2)$, hence we have $s_1 \xrightarrow{k_1} t_1$ and $s_2 \xrightarrow{k_2} t_2$. It follows that $(s_2, s_3) \xrightarrow{k'_2 \oplus k_3} | (t'_2, t_3)$ with $d_{\mathsf{Spec}}(k'_2 \oplus k_3, k_1) = d_{\mathsf{Spec}}(k_3, k_1 \ominus k'_2) \leq \varepsilon$ and such that $((t'_2, t_3), t_1) \in R^2_{\varepsilon'} \in R^2$, hence $(t_3, (t_1, t'_2)) \in R^1_{\varepsilon'} \in R^1$, for some $\varepsilon' \leq \lambda^{-1}(\varepsilon - d_{\mathsf{Spec}}(k_3, k_1 \ominus k'_2))$. By definition of parallel composition we must have $s_2 \xrightarrow{k'_2} t'_2$ and $s_3 \xrightarrow{k_3} t_3$, and by determinism of $S_2, k'_2 = k_2$ and $t'_2 = t_2$.

The example of Figure 6.5 shows that the condition $d_{\mathsf{m}}(S_3, S_1/S_2) < \infty$ in Theorem 6.29 is necessary. Here $d_{\mathsf{m}}(S_2 \| S_3, S_1) = 1$, but $d_{\mathsf{m}}(S_3, S_1/S_2) = \infty$ because of inconsistency between the transitions $s_1 \xrightarrow{a,[0,0]} t_1$ and $s_2 \xrightarrow{a,[0,1]} t_2$ for which $k_1 \ominus k_2$ is defined.

As a practical application, we notice that relaxation as defined in Section 6.3 can be useful when computing quotients. The quotient construction

in Definition 6.25 introduces inconsistent states (which afterwards are pruned) whenever there is a must transition $s_1 \xrightarrow{k_1} s_1'$ such that $k_1 \ominus k_2$ is undefined for all transitions $s_2 \xrightarrow{k_2} s_2'$. Looking at the definition of \ominus , we see that this is the case if $k_1 = (a_1, [x_1, y_1])$ and $k_2 = (a_2, [x_2, y_2])$ are such that $a_1 \neq a_2$ or $x_1 - x_2 > y_1 - y_2$. In the first case, the inconsistency is of a structural nature and cannot be dealt with; but in the second case, it may be avoided by enlarging k_1 : decreasing x_1 or increasing y_1 so that now, $x_1 - x_2 \leq y_1 - y_2$.

Enlarging quantitative constraints is exactly the intuition of relaxation, thus in practical cases where we get a quotient S_1/S_2 which is "too inconsistent", we may be able to solve this problem by constructing a suitable ε -relaxation S'_1 of S_1 . Theorems 6.27 and 6.29 can then be used to ensure that also S'_1/S_2 is a relaxation of S_1/S_2 .

6.6 Conclusion

We have shown in this chapter that within the quantitative specification framework of weighted modal transition systems, refinement and implementation distances provide a useful tool for robust compositional reasoning. Note that these distances permit us not only to reason about differences between implementations and from implementations to specifications, but they also provide a means by which we can compare specifications directly at the abstract level.

We have shown that for some of the ingredients of our specification theory, namely structural composition and quotient, our formalism is a conservative extension of the standard Boolean notions. We have also noted however, that for determinization and logical conjunction, the properties of the Boolean notions are not preserved, and that this seems to be a fundamental limitation of any reasonable quantitative specification theory. We will have more to say about this in the next chapter.

7 General Quantitative Specification Theories with Modal Transition Systems¹

This chapter combines the work of the two previous chapters. It uses the general theory of linear and branching distances developed in Chapter 5 to introduce general refinement distances between *structured modal transition systems*. It then proceeds to consider quantitative properties of structural composition, quotient, and conjunction, and finishes with a logical characterization of quantitative refinement using Hennessy-Milner logic.

7.1 Structured Modal Transition Systems

We work with a poset Spec of specification labels with a partial order \leq and denote by $\operatorname{Spec}^{\infty} = \operatorname{Spec}^* \cup \operatorname{Spec}^{\omega}$ the set of finite and infinite traces over Spec. In applications, Spec may be used to model data about the behavior of a system; for specifications this may be considered as legal parameters of operation, whereas for implementations it may be thought of as observed information.

The partial order \leq is meant to model refinement of data; if $k \leq \ell$, then k is more refined (leaves fewer choices) than ℓ . The set $\mathsf{Imp} = \{k \in \mathsf{Spec} \mid k' \leq k \implies k' = k\}$ is called the set of implementation labels; these are the data which cannot be refined further. We let $[\![k]\!] = \{k' \in \mathsf{Imp} \mid k' \leq k\}$ and assume that $[\![k]\!] \neq \emptyset$ for all $k \in \mathsf{Spec}$.

When $k \not \leq \ell$, we want to be able to quantify the impact of this difference in data on the systems in question, thus circumventing the fragility of the theory. To this end, we introduce a general notion of distance on sequences of data following the approach laid out in Chapter 5.

¹This chapter is based on the journal paper [FL14a] published in Acta Informatica.

7.1.1 Trace distances

In order to build a framework for specification distances which is general enough to cover the distances commonly used, we introduce a notion of abstract trace distance which factors through a lattice on which it has a recursive characterization. We will show in Section 7.1.2 that this indeed covers the common scenarios; see also Section 5.7.

Let M be an arbitrary set and $\mathbb{L} = (\mathbb{R}_{\geq 0} \cup \{\infty\})^M$ the set of functions from M to the extended non-negative real line. Then \mathbb{L} is a complete lattice with partial order $\sqsubseteq_{\mathbb{L}}$ given by $\alpha \sqsubseteq_{\mathbb{L}} \beta$ if and only if $\alpha(x) \leq \beta(x)$ for all $x \in M$, and with an addition $\oplus_{\mathbb{L}}$ given by $(\alpha \oplus_{\mathbb{L}} \beta)(x) = \alpha(x) + \beta(x)$. The bottom element of \mathbb{L} is also the zero of $\oplus_{\mathbb{L}}$ and given by $\bot_{\mathbb{L}}(x) = 0$, and the top element is $\top_{\mathbb{L}}(x) = \infty$. We also define a metric on \mathbb{L} by $d_{\mathbb{L}}(\alpha, \beta) = \sup_{x \in M} |\alpha(x) - \beta(x)|$.

Intuitively, the lattice \mathbb{L} serves as a memory for more elaborate trace distances such as for example the limit-average distance, see Section 7.1.2. For simpler distances, it will suffice to let $M = \{*\}$ be the one-point set and thus $\mathbb{L} = \mathbb{R}_{\geq 0} \cup \{\infty\}$. We extend the notions of hemimetrics, pseudometrics and metrics to mappings $d: X \times X \to \mathbb{L}$, by replacing in their defining properties 0 by $\bot_{\mathbb{L}}$ and + by $\oplus_{\mathbb{L}}$.

Let $d: \mathsf{Imp} \times \mathsf{Imp} \to \mathbb{L}$ be a hemimetric on implementation labels. We extend d to Spec by $d(k,\ell) = \sup_{m \in [\![k]\!]} \inf_{n \in [\![\ell]\!]} d(m,n)$. Hence also this distance is asymmetric; the intuition is that any label in $[\![k]\!]$ has to be matched as good as possible in $[\![\ell]\!]$. Note that this is the Hausdorff hemimetric associated with d on implementation labels.

We will assume given an abstract $trace\ distance\ d_{\mathsf{tr}}^{\mathbb{L}}: \mathsf{Spec}^{\infty} \times \mathsf{Spec}^{\infty} \to \mathbb{L}$ which is a hemimetric and has a recursive expression using a $distance\ iterator$ function $F: \mathsf{Imp} \times \mathsf{Imp} \times \mathbb{L} \to \mathbb{L}$, see below. This will allow us to recover many of the system distances found in the literature, while preserving key results. We will need to assume that F satisfies the following properties:

- (1) F is *continuous* in the first two coordinates: $F(\cdot, n, \alpha)$ and $F(m, \cdot, \alpha)$ are continuous functions $\mathsf{Imp} \to \mathbb{L}$ for all $\alpha \in \mathbb{L}$.
- (2) F is monotone in the third coordinate: $F(m,n,\cdot): \mathbb{L} \to \mathbb{L}$ is monotone for all $m,n \in \mathsf{Imp}$.
- (3) F extends d: for all $m, n \in \text{Imp}$, $F(m, n, \perp_{\mathbb{L}}) = d(m, n)$.
- (4) Indiscernibility of identicals: $F(m, m, \alpha) = \alpha$ for all $m \in Imp$.
- (5) An extended triangle inequality: for all $m, n, o \in \text{Imp}$ and $\alpha, \beta \in \mathbb{L}$, $F(m, n, \alpha) \oplus_{\mathbb{L}} F(n, o, \beta) \supseteq_{\mathbb{L}} F(m, o, \alpha \oplus_{\mathbb{L}} \beta)$.

Note how the last two axioms are a generalization of the standard axioms for hemimetrics.

We extend F to specification labels by defining

$$F(k,\ell,\alpha) = \sup_{m \in \llbracket k \rrbracket} \inf_{n \in \llbracket \ell \rrbracket} F(m,n,\alpha).$$

Then also the extended $F: \operatorname{Spec} \times \operatorname{Spec} \times \mathbb{L} \to \mathbb{L}$ is continuous in the first two and monotone in the third coordinates. Additionally, we assume that sets of implementation labels are *closed* with respect to F in the sense that for all $k, \ell \in \operatorname{Spec}$ and $\alpha \in \mathbb{L}$ with $F(k, \ell, \alpha) \neq \top_{\mathbb{L}}$, there are $m \in [\![k]\!]$, $n \in [\![\ell]\!]$ with $F(m, \ell, \alpha) = F(k, n, \alpha) = F(k, \ell, \alpha)$. Note that this implies that the sets $[\![k]\!]$ are closed under the hemimetric d on Spec .

Axioms (4) and (5) for F above now imply that for the extension, the following hold:

- (4') For all $k, \ell \in \text{Spec}$ with $k \leq \ell$ and all $\alpha \in \mathbb{L}$, $F(k, \ell, \alpha) = \alpha$.
- (5') For all $k, \ell, m \in \operatorname{Spec}$ and $\alpha, \beta \in \mathbb{L}$,

$$F(k,\ell,\alpha) \oplus_{\mathbb{L}} F(\ell,m,\beta) \supseteq_{\mathbb{L}} F(k,m,\alpha \oplus_{\mathbb{L}} \beta).$$

Let $\varepsilon \in \mathsf{Spec}^{\infty}$ denote the empty sequence, and for any sequence $\sigma \in \mathsf{Spec}^{\infty}$, denote by σ_0 its first element and by σ^1 the tail of the sequence with the first element removed. We assume that $d_{\mathsf{tr}}^{\mathbb{L}}$ has a recursive characterization, using F, as follows:

$$d_{\mathsf{tr}}^{\mathbb{L}}(\sigma,\tau) = \begin{cases} F(\sigma_0, \tau_0, d_{\mathsf{tr}}^{\mathbb{L}}(\sigma^1, \tau^1)) & \text{if } \sigma, \tau \neq \varepsilon, \\ \top_{\mathbb{L}} & \text{if } \sigma = \varepsilon, \tau \neq \varepsilon \text{ or } \sigma \neq \varepsilon, \tau = \varepsilon, \\ \bot_{\mathbb{L}} & \text{if } \sigma = \tau = \varepsilon. \end{cases}$$
(7.1)

We remark that a recursive characterization such as the one above is quite natural. Not only does it cover all commonly used trace distances (see the examples in the next section), but recursion is central to computing, and any trace distance without a recursive characterization would strike us as being quite artificial. It is precisely this recursive characterization which allows us to lift the trace distance to *states* of specifications in Definition 7.10 below, see also Chapter 5.

In applications (see below), the lattice \mathbb{L} comes equipped with a homomorphism eval : $\mathbb{L} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ for which eval $(\bot_{\mathbb{L}}) = 0$. The actual trace distance of interest is then the composition $d_{\mathsf{tr}} = \mathsf{eval} \circ d^{\mathbb{L}}_{\mathsf{tr}}$. The triangle inequality for F implies the usual triangle inequality for d_{tr} : $d_{\mathsf{tr}}(\sigma,\tau) + d_{\mathsf{tr}}(\tau,\chi) \leq d_{\mathsf{tr}}(\sigma,\chi)$ for all $\sigma,\tau,\chi \in \mathsf{Spec}^{\infty}$, hence d_{tr} is a hemimetric on Spec^{∞} .

We need to work with distances which factor through \mathbb{L} , instead of plainly taking values in $\mathbb{R}_{\geq 0} \cup \{\infty\}$, because some distances which are useful in practice, as the ones in Examples 7.3 and 7.5 below, have no recursive characterization using $\mathbb{L} = \mathbb{R}_{\geq 0} \cup \{\infty\}$. Whether the theory works for more general intermediate lattices than $\mathbb{L} = (\mathbb{R}_{\geq 0} \cup \{\infty\})^M$ is an open question; we have had no occasion to use more general lattices in practice.

7.1.2 Examples

To give an application to the framework laid out above, we show here a few examples of specification labels and trace distances and how they fit into the framework. For a much more comprehensive application of the theory see Chapter 5.

7.1 Example. A good example of a set of specification labels is given by $\mathsf{Spec} = \Sigma \times \mathbb{I}$, where Σ is a finite set of discrete labels and $\mathbb{I} = \{[l,r] \mid l \in \mathbb{Z} \cup \{-\infty\}, r \in \mathbb{Z} \cup \{\infty\}, l \leq r\}$ is the set of extended-integer intervals. The partial order is defined by $(a, [l, r]) \preceq (a', [l', r'])$ iff $a = a', l' \leq l$ and $r' \geq r$. Hence refinement is given by restricting intervals, so that $\mathsf{Imp} = \Sigma \times \{[x, x] \mid x \in \mathbb{Z}\} \approx \Sigma \times \mathbb{Z}$.

The implementation label distance is given by

$$d((a,x),(a',x')) = \begin{cases} |x-x'| & \text{if } a = a', \\ \infty & \text{otherwise,} \end{cases}$$

so that for specification labels (a, [l, r]), (a', [l', r']),

$$\begin{split} d((a,[l,r]),(a',[l',r'])) &= \sup_{m \in [\![(a,[l,r])]\!]} \inf_{n \in [\![(a',[l',r'])]\!]} d(m,n) \\ &= \begin{cases} \max(l'-l,r-r',0) & \text{if } a=a', \\ \infty & \text{otherwise.} \end{cases} \end{split}$$

Now let $\mathbb{L} = \mathbb{R}_{\geq 0} \cup \{\infty\}$, eval = id, and

$$F(m, n, \alpha) = d(m, n) + \lambda \alpha$$

for some fixed discounting factor $\lambda \in \mathbb{R}$ with $0 < \lambda < 1$, then $d_{\mathsf{tr}}(\sigma,\tau) = \sum_{j} \lambda^{j} d(\sigma_{j},\tau_{j})$ for implementation traces σ , τ of equal length. This is the accumulating distance which we have used in Chapter 6 to develop a specification theory; we will continue this example below to show how it fits in our present context.

7.2 Example. Using the same setting as above, with $\mathsf{Spec} = \Sigma \times \mathbb{I}$, $(a, [l, r]) \preccurlyeq (a', [l', r'])$ iff a = a', $l' \leq l$ and $r' \geq r$, and d((a, x), (a', x')) = |x - x'| if a = a' and ∞ otherwise, we can instantiate F to a *point-wise* instead of accumulating distance.

Let again $\mathbb{L} = \mathbb{R}_{\geq 0} \cup \{\infty\}$ and eval = id, but now

$$F(m, n, \alpha) = \max(d(m, n), \alpha)$$
.

Then $d_{tr}(\sigma,\tau) = \sup_j d(\sigma_j,\tau_j)$ for implementation traces σ , τ of equal length, hence measuring the biggest individual difference between the traces' symbols. We will also continue this example below to show how to develop a specification theory based on the point-wise distance.

7.3 Example. Again with the same instantiations of Imp and Spec as above, we can introduce *limit-average* distance. Here we let $\mathbb{L} = (\mathbb{R}_{\geq 0} \cup \{\infty\})^{\mathbb{N}}$, eval: $\mathbb{L} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ given by $\text{eval}(\alpha) = \liminf_{j} \alpha(j)$, and

$$F(m, n, \alpha)(j) = \frac{1}{j+1}d(m, n) + \frac{j}{j+1}\alpha(j-1),$$

then $d_{\mathsf{tr}}(\sigma,\tau) = \mathsf{eval}(d^{\mathbb{L}}_{\mathsf{tr}}(\sigma,\tau)) = \liminf_{j} \frac{1}{j+1} \sum_{i=0}^{j} d(\sigma_{j},\tau_{j})$ for traces of equal length. We show below how this distance, in the framework of the present paper, gives a limit-average specification theory.

- **7.4 Example.** Examples 7.1 to 7.3 above are in a sense agnostic to the precise structure of implementation and specification labels. Indeed, the definitions only use the label distance $d: \operatorname{Imp} \times \operatorname{Imp} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$, hence Imp (and Spec) can be any set. In particular, the theory put forward here works equally well in a *multi-weighted* setting as for example in [FJLS11], where $\operatorname{Imp} = \Sigma \times \mathbb{Z}^k$ and $\operatorname{Spec} = \Sigma \times \mathbb{I}^k$ for some $k \in \mathbb{N}$.
- 7.5 Example. With the same instantiations of Imp and Spec as in Examples 7.1 to 7.3, we can introduce a distance which, instead of accumulating individual label differences, measures the long-run difference between accumulated labels. This maximum-lead distance is especially useful for real-time systems and has been considered in [HMP05, TFL10], see also Chapter 2. Unlike Examples 7.1 to 7.3, it does not use the distance d on implementation labels in the definition of the trace distance; rather it accumulates the labels itself before taking the distance.

Let $\mathbb{L}=(\mathbb{R}_{\geq 0}\cup\{\infty\})^{\mathbb{R}}$ and define $F:\mathsf{Imp}\times\mathsf{Imp}\times\mathbb{L}\to\mathbb{L}$ by

$$F((a,x),(a',x'),\alpha)(\delta) = \begin{cases} \infty & \text{if } a \neq a', \\ \max(|\delta+x-x'|,\alpha(\delta+x-x')) & \text{if } a = a'. \end{cases}$$

Define eval: $\mathbb{L} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ by eval $(\alpha) = \alpha(0)$; the maximum-lead distance assuming the lead is zero. It can then be shown that for implementation traces $\sigma = ((a_0, x_0), (a_1, x_1), \dots), \tau = ((a_0, y_0), (a_1, y_1), \dots),$

$$d_{\mathrm{tr}}(\sigma,\tau) = \mathrm{eval}(d_{\mathrm{tr}}^{\mathbb{L}}(\sigma,\tau)) = \sup_{m} \Big| \sum_{i=0}^{m} x_i - \sum_{i=0}^{m} y_i \Big|$$

is precisely the maximum-lead distance.

7.6 Example. Specification labels different from the ones above can for example be *clock constraints*, or *zones* [AD94]. For a finite set Σ , let $Spec = \Phi(\Sigma)$ be the set of closed clock constraints over Σ given by

$$\Phi(\Sigma) \ni \phi ::= a \le k \mid a \ge k \mid \phi_1 \land \phi_2 \qquad (a \in \Sigma, k \in \mathbb{N}, \phi_1, \phi_2 \in \Phi(\Sigma)).$$

Clock constraints have a natural partial order given by $\phi \preccurlyeq \phi'$ iff $\phi \Longrightarrow \phi'$. Implementation labels are then clock constraints which impose a precise value for each $a \in \Sigma$, which can be seen as functions $u : \Sigma \to \mathbb{N}$. The natural distance between such discrete clock valuations is $d(u, u') = \max_{a \in \Sigma} |u(a) - u'(a)|$, and on top of this, any interesting trace distance can be imposed using our framework.

7.1.3 Structured Modal Transition Systems

7.7 Definition. A structured modal transition system (SMTS) is a tuple $(S, s_0, --+, S, \longrightarrow_S)$ consisting of a set S of states, an initial state $s_0 \in S$, and must and may transitions $\longrightarrow_S, --+_S \subseteq S \times \operatorname{Spec} \times S$ for which it holds that for all $s \xrightarrow{k}_S s'$ there is $s \xrightarrow{\ell}_S s'$ with $k \leq \ell$.

The last condition is one of *consistency*: everything which is required is also allowed. If no confusion can arise, we will omit the subscripts S on the *must* and *may* transitions; we will also sometimes identify an SMTS $(S, s_0, -\rightarrow_S, \longrightarrow_S)$ with its state set S.

Intuitively, a may transition $s \xrightarrow{k} s'$ specifies that an implementation I of S is permitted to have a corresponding transition $i \xrightarrow{m} i'$, for any $m \in [\![k]\!]$, whereas a must transition $s \xrightarrow{\ell} s'$ postulates that I is required to implement at least one corresponding transition $i \xrightarrow{n} i'$ for some $n \in [\![\ell]\!]$. We will make this precise below.

An SMTS S is an *implementation* if $\longrightarrow_S = - \rightarrow_S \subseteq S \times \mathsf{Imp} \times S$; hence in an implementation, all optional behavior has been resolved, and all data has been refined to implementation labels.

7.8 Definition. An SMTS $(S, s_0, \dashrightarrow_S, \longrightarrow_S)$ is \mathbb{L} -deterministic, for a given lattice \mathbb{L} , if it holds for all $s \in S$, $s \xrightarrow{k_1} s_1$, $s \xrightarrow{k_2} s_2$ for which there is $k \in \mathsf{Spec}$ with $d(k, k_1) \neq \top_{\mathbb{L}}$ and $d(k, k_2) \neq \top_{\mathbb{L}}$ that $k_1 = k_2$ and $s_1 = s_2$.

Note that for the Boolean label distance given by $d(k, k') = \perp_{\mathbb{L}}$ if k = k' and $\top_{\mathbb{L}}$ otherwise, the above definition reduces to the property that if $k_1 = k_2$, then also $s_1 = s_2$, hence \mathbb{L} -determinism is a generalization of usual determinism. In our quantitative case, we need to be more restrictive: not only do we not allow distinct transitions from s with the same label, but we forbid distinct transitions with labels which have a common quantitative refinement. Despite of this, we will generally omit the \mathbb{L} and say deterministic instead of \mathbb{L} -deterministic.

7.9 Example. For the label distance d((a,x),(a',x')) = |x-x'| if a=a' and ∞ otherwise of Examples 7.1 to 7.3 and 7.5, the above condition that there exist $k \in \text{Spec}$ with $d(k,k_1) \neq \top_{\mathbb{L}}$ and $d(k,k_2) \neq \top_{\mathbb{L}}$ is equivalent, with $k_1 = (a_1,I_1)$ and $k_2 = (a_2,I_2)$, to saying that $a_1 = a_2$, hence our notion of determinism agrees with the one of the previous chapter.

A modal refinement of SMTS S, T is a relation $R \subseteq S \times T$ such that for any $(s,t) \in R$,

- whenever $s \xrightarrow{k}_{-} s'$, then also $t \xrightarrow{\ell}_{T} t'$ for some $k \leq \ell$ and $(s', t') \in R$,
- whenever $t \xrightarrow{\ell}_T t'$, then also $s \xrightarrow{k}_S s'$ for some $k \leq \ell$ and $(s', t') \in R$.

Thus any behavior which is permitted in S is also permitted in T, and any behavior required in T is also required in S. We write $S \leq_{\mathsf{m}} T$ if there is a modal refinement $R \subseteq S \times T$ with $(s_0, t_0) \in R$.

The implementation semantics of a SMTS S is the set $[S] = \{I \leq_{\mathsf{m}} S \mid I \text{ is an implementation}\}$, and we write $S \leq_{\mathsf{th}} T$ if $[S] \subseteq [T]$, saying that S thoroughly refines T. It follows by reflexivity of \leq_{m} that $S \leq_{\mathsf{m}} T$ implies $S \leq_{\mathsf{th}} T$, hence modal refinement is a syntactic over-approximation of thorough refinement.

It can be shown for standard modal transition systems that $S \leq_{\sf th} T$ does not imply $S \leq_{\sf m} T$ unless T is deterministic, see [BKLS09] and Theorem 6.15 in the previous chapter. We shall provide a quantitative generalization of this result in Theorem 7.17 below. Also, modal refinement for MTS can be decided in polynomial time, whereas deciding thorough refinement is EXPTIME-complete [BKLS09]. Intuitively, thorough refinement—inclusion of implementation sets—is the relation one really is interested in, but modal refinement provides a useful over-approximation.

7.2 Refinement Distances

We define two distances between SMTS, one at the syntactic and one at the semantic level.

7.2.1 Modal and thorough refinement distance

7.10 Definition. The modal refinement distance $d_{\mathsf{m}}: S \times T \to \mathbb{L}$ between the states of SMTS S, T is defined to be the least fixed point to the equations

$$d_{\mathsf{m}}(s,t) = \max \begin{cases} \sup_{s \xrightarrow{k} s'} \inf_{t \xrightarrow{\ell} T'} F(k,\ell,d_{\mathsf{m}}(s',t')), \\ \sup_{t \xrightarrow{\ell} T'} \inf_{s \xrightarrow{k} s} F(k,\ell,d_{\mathsf{m}}(s',t')). \end{cases}$$

We let $d_{\mathsf{m}}(S,T) = d_{\mathsf{m}}(s_0,t_0)$, and we write $S \leq_{\mathsf{m}}^{\alpha} T$ if $d_{\mathsf{m}}(S,T) \sqsubseteq_{\mathbb{L}} \alpha$.

7.11 Lemma. The modal refinement distance is well-defined and a hemimetric. Also, $S \leq_m T$ implies $d_m(S,T) = \bot_{\mathbb{L}}$.

Proof: Let $I: \mathbb{L}^{S \times T} \to \mathbb{L}^{S \times T}$ be the endofunction defined by

$$I(h)(s,t) = \max \begin{cases} \sup_{s \xrightarrow{k}_{-\rightarrow S} s'} \inf_{t \xrightarrow{\ell}_{-\rightarrow T} t'} F(k,\ell,h(s',t')), \\ \sup_{t \xrightarrow{\ell}_{-\rightarrow T} t'} \inf_{s \xrightarrow{k}_{-\rightarrow S} s'} F(k,\ell,h(s',t')). \end{cases}$$

The lattice $\mathbb{L}^{S \times T}$ is complete because \mathbb{L} is, and I is monotone because $F(k,\ell,\cdot)$: $\mathbb{L} \to \mathbb{L}$ is. By an application of Tarski's fixed point theorem [Tar55], I has a unique least fixed point which hence defines d_{m} .

The property that $d_{\mathsf{m}}(S,S) = 0$ for all SMTS S is clear, and the triangle inequality $d_{\mathsf{m}}(S,T) \oplus_{\mathbb{L}} d_{\mathsf{m}}(T,U) \supseteq_{\mathbb{L}} d_{\mathsf{m}}(S,U)$ can be shown inductively.

To show the last claim, assume $s \leq_{\mathsf{m}} t$. Then for any $s \xrightarrow{k} s'$ there is $t \xrightarrow{\ell} t'$ for which $k \leq \ell$, hence $F(k,\ell,\alpha) = \alpha$ for all $\alpha \in \mathbb{L}$ by Axiom (4'). Similarly for *must* transitions, so the fixed point equations simplify to

$$d_{\mathsf{m}}(s,t) = \max\Big(\sup_{s \to s'} \inf_{t \to t'} d_{\mathsf{m}}(s',t'), \sup_{t \to t'} \inf_{s \to s'} d_{\mathsf{m}}(s',t')\Big),$$

the least fixed point of which is $d_{\mathsf{m}}(s,t) = \bot_{\mathbb{L}}$.

One can also define a *linear distance* between states, analogous to trace inclusion. This is given by

$$d_{\mathsf{tr}}^{\mathbb{L}}(s,t) = \max\Big(\sup_{\sigma \in \mathsf{Tr}(s)} \inf_{\tau \in \mathsf{Tr}(t)} d_{\mathsf{tr}}^{\mathbb{L}}(\sigma,\tau), \sup_{\tau \in \mathsf{Tr}(t)} \inf_{\sigma \in \mathsf{Tr}(s)} d_{\mathsf{tr}}^{\mathbb{L}}(\sigma,\tau)\Big),$$

where $\mathsf{Tr}(s)$ denotes the set of (may or must) traces emanating from s. It can then be shown that $d^{\mathbb{L}}_{\mathsf{tr}}(s,t) \sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}(s,t)$ for all $s,t \in S$, see Chapter 5.

7.12 Definition. The thorough refinement distance from an SMTS S to an SMTS T is

$$d_{\mathsf{th}}(S,T) = \sup_{I \in [\![S]\!]} \inf_{J \in [\![T]\!]} d_{\mathsf{m}}(I,J),$$

and we write $S \leq_{\mathsf{th}}^{\alpha} T$ if $d_{\mathsf{th}}(S,T) \sqsubseteq_{\mathbb{L}} \alpha$.

7.13 Lemma. The thorough refinement distance is a hemimetric, and $S \leq_{\sf th} T$ implies $d_{\sf th}(S,T) = \bot_{\mathbb{L}}$.

Proof: The equality $d_{\mathsf{th}}(S,S) = \bot_{\mathbb{L}}$ is clear, and the triangle inequality $d_{\mathsf{th}}(S,T) + d_{\mathsf{th}}(T,U) \ge d_{\mathsf{th}}(S,U)$ follows like in the proof of [AB07, Lemma 3.72]. If $S \le_{\mathsf{th}} T$, then $[\![S]\!] \subseteq [\![T]\!]$ implies $d_{\mathsf{th}}(S,T) = \bot_{\mathbb{L}}$.

7.2.2 Refinement families

As is the case for ordinary (bi)simulation [Par81], there is a dual *relational* notion of refinement distance which is useful. Before we can introduce this, we need a notion similar to the *finite branching* assumption one needs to make for the case of bisimulation, *cf.* [Mil89], see also Chapter 6.

7.14 Definition. A SMTS S is said to be *compactly branching* if the sets $\{(s',k) \mid s \xrightarrow{k} s'\}, \{(s',k) \mid s \xrightarrow{k} s'\} \subseteq S \times \text{Spec}$ are compact under the symmetrized product distance $\bar{d}_{\mathsf{m}} \times \bar{d}$ for every $s \in S$.

Recall that the pseudometric $\bar{d}_{\mathsf{m}} \times \bar{d}$ is given by $\bar{d}_{\mathsf{m}} \times \bar{d}((s,k),(s',k')) = \bar{d}_{\mathsf{m}}(s,s') + \bar{d}(k,k') = \max(d_{\mathsf{m}}(s,s'),d_{\mathsf{m}}(s',s)) + \max(d(k,k'),d(k',k))$. As in Chapter 6, we will need compactness of the sets $\{(s',k) \mid s \xrightarrow{k} s'\}, \{(s',k) \mid s \xrightarrow{k} s'\} \subseteq S \times \mathsf{Spec}$ for the property that continuous functions defined on them attain their infimum and supremum, see Lemma 7.16 and its proof below.

The notion of compact branching was first introduced, for a formalism of *metric transition systems*, in [vB96]. It is a natural generalization of finite branching to a distance setting; we shall henceforth assume all our SMTS to be compactly branching.

- **7.15 Definition.** A modal refinement family from S to T, for SMTS S, T, is an \mathbb{L} -indexed family of relations $\mathcal{R} = \{R_{\alpha} \subseteq S \times T \mid \alpha \in \mathbb{L}\}$ with the property that for all $\alpha \in \mathbb{L}$ and all $(s,t) \in R_{\alpha}$,
 - whenever $s \xrightarrow{k}_{S} s'$, then there is $\beta \in \mathbb{L}$ and $(s', t') \in R_{\beta}$ for which $t \xrightarrow{\ell}_{T} t'$ and $F(k, \ell, \beta) \sqsubseteq_{\mathbb{L}} \alpha$,
 - whenever $t \xrightarrow{\ell}_T t'$, then there is $\beta \in \mathbb{L}$ and $(s',t') \in R_\beta$ for which $s \xrightarrow{k}_S s'$ and $F(k,\ell,\beta) \sqsubseteq_{\mathbb{L}} \alpha$.

Compact branching implies that refinement families \mathcal{R} are *closed* in the sense that for all $s \in S$, $t \in T$, $(s,t) \in R_{\inf\{\alpha \mid (s,t) \in R_{\alpha} \in \mathcal{R}\}} \in \mathcal{R}$. Also note how this definition is a common refinement of the notions of relaton families from Chapters 5 and 6.

7.16 Lemma. For all SMTS S, T and $\alpha \in \mathbb{L}$, $S \leq_{\mathsf{m}}^{\alpha} T$ if and only if there is a modal refinement family \mathcal{R} from S to T with $(s_0, t_0) \in R_{\alpha}$.

We say that a modal refinement family as in the lemma witnesses $S \leq_{\mathsf{m}}^{\alpha} T$; this is of course the same as saying that it witnesses $d_{\mathsf{m}}(S,T) \sqsubseteq_{\mathbb{L}} \alpha$, which we sometimes shorten to say that it witnesses $d_{\mathsf{m}}(S,T)$.

Proof: Assume first that $S \leq_{\mathsf{m}}^{\alpha} T$, thus we know that $d_{\mathsf{m}}(S,T) \sqsubseteq_{\mathbb{L}} \alpha$. We have to show that there is a modal refinement family \mathcal{R} from S to T with $(s_0, t_0) \in R_{\alpha}$. Define a family $\mathcal{R} = \{R_{\alpha'} \subseteq S \times T \mid \alpha' \in \mathbb{L}\}$ by

$$R_{\alpha'} = \{(s,t) \mid d_{\mathsf{m}}(s,t) \sqsubseteq_{\mathbb{L}} \alpha' \}$$

for every $\alpha' \in \mathbb{L}$; note that \mathcal{R} is closed in the sense above. Now let $\beta \in \mathbb{L}$ and $(s,t) \in R_{\beta}$.

- Assume $s \xrightarrow{k} s'$. By $d_{\mathsf{m}}(s,t) \sqsubseteq_{\mathbb{L}} \beta$ and the definition of $d_{\mathsf{m}}(s,t)$ it follows that $\inf_{t \xrightarrow{\ell} t'} F(k,\ell,d_{\mathsf{m}}(s',t')) \sqsubseteq_{\mathbb{L}} \beta$. As T is compactly branching and F continuous, the set $\{F(k,\ell,d_{\mathsf{m}}(s',t')) \mid t \xrightarrow{\ell} t'\}$ is compact, hence there exists a transition $t \xrightarrow{\ell} t'$ such that $F(k,\ell,d_{\mathsf{m}}(s',t')) \sqsubseteq_{\mathbb{L}} \beta$.
- Assume $t \stackrel{\ell}{\longrightarrow} t'$. By $d_{\mathsf{m}}(s,t) \sqsubseteq_{\mathbb{L}} \beta$ and the definition of $d_{\mathsf{m}}(s,t)$ it follows that $\inf_{s \stackrel{k}{\longrightarrow} s'} F(k,\ell,d_{\mathsf{m}}(s',t')) \sqsubseteq_{\mathbb{L}} \beta$. Again $\{F(k,\ell,d_{\mathsf{m}}(s',t')) \mid s \stackrel{k}{\longrightarrow} s'\}$ is a compact set, whence there exists a transition $s \stackrel{k}{\longrightarrow} s'$ such that $F(k,\ell,d_{\mathsf{m}}(s',t')) \sqsubseteq_{\mathbb{L}} \beta$.

For the other direction, assume a refinement family \mathcal{R} from S to T with $(s_0,t_0)\in R_{\alpha}$. Define $h:S\times T\to \mathbb{L}$ by $h(s,t)=\inf\{\alpha\mid (s,t)\in R_{\alpha}\}$. Then $(s,t)\in R_{\beta}$ implies that $h(s,t)\sqsubseteq_{\mathbb{L}}\beta$. Let $s\in S$ and $t\in T$, then $(s,t)\in R_{h(s,t)}$ because \mathcal{R} is closed, hence for all $s\stackrel{k}{\dashrightarrow} s'$ there is $t\stackrel{\ell}{\dashrightarrow} t'$ and $\alpha'\in\mathbb{L}$ for which $F(k,\ell,\alpha')\sqsubseteq_{\mathbb{L}}h(s,t)$ and $(s',t')\in R_{\alpha'}$, implying $h(s',t')\sqsubseteq_{\mathbb{L}}\alpha'$ and hence $F(k,\ell,h(s',t'))\sqsubseteq_{\mathbb{L}}h(s,t)$ by monotonicity and transitivity. Similarly, for all $t\stackrel{\ell}{\longrightarrow} t'$ there is $s\stackrel{k}{\longrightarrow} s'$ with $F(k,\ell,h(s',t'))\sqsubseteq_{\mathbb{L}}h(s,t)$. Hence h is a pre-fixed point for the equations in the definition of d_{m} , implying that $d_{\mathsf{m}}(s,t)\sqsubseteq_{\mathbb{L}}h(s,t)$ for all $s\in S$, $t\in T$, thus especially $d_{\mathsf{m}}(s_0,t_0)\sqsubseteq_{\mathbb{L}}\alpha$, because $(s_0,t_0)\in R_{\alpha}$ implies $h(s_0,t_0)\sqsubseteq_{\mathbb{L}}\alpha$ and $d_{\mathsf{m}}(s_0,t_0)\sqsubseteq_{\mathbb{L}}h(s_0,t_0)$. \square

7.2.3 Modal distance bounds thorough distance

The next theorem shows that the modal refinement distance overapproximates the thorough one, and that it is exact for deterministic SMTS. This is similar to the situation for standard modal transition systems [Lar89]; note [Lar89] that deterministic specifications generally suffice for applications.

7.17 Theorem. For all SMTS S, T, $d_{\mathsf{th}}(S,T) \sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}(S,T)$. If T is deterministic, then $d_{\mathsf{th}}(S,T) = d_{\mathsf{m}}(S,T)$.

The counterexample for the Boolean version of the second result given in [BKLS09] also works in our setting, to show that there exist (necessarily nondeterministic) SMTS S, T for which $d_{\mathsf{th}}(S,T) = \bot_{\mathbb{L}}$, but $d_{\mathsf{m}}(S,T) = \top_{\mathbb{L}}$.

Proof: For the first claim, if $d_{\mathsf{m}}(S,T) = \top_{\mathbb{L}}$, we have nothing to prove. Otherwise, let $\mathcal{R} = \{R_{\alpha} \subseteq S \times T \mid \alpha \in \mathbb{L}\}$ be a modal refinement family which witnesses $d_{\mathsf{m}}(S,T)$, then $(s_0,t_0) \in R_{d_{\mathsf{m}}(S,T)}$. Let $I \in \llbracket S \rrbracket$; we will expose $J \in \llbracket T \rrbracket$ for which $d_{\mathsf{m}}(I,J) \sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}(S,T)$.

Let $\tilde{R} \subseteq I \times S$ be a witness for $I \leq_{\mathbf{m}} S$, define $R'_{\alpha} = \tilde{R} \circ R_{\alpha} \subseteq I \times T$ for all $\alpha \in \mathbb{L}$, and let $\mathcal{R}' = \{R'_{\alpha} \mid \alpha \in \mathbb{L}\}$. We let the states of J be J = T, with $j_0 = t_0$, and define $\dashrightarrow_J = \longrightarrow_J$ as follows:

For any $i \xrightarrow{m}_I i'$ and any $t \in T$ for which $(i,t) \in R'_{\alpha} \in \mathcal{R}'$ for some $\alpha \in \mathbb{L}$, $\alpha \neq \top_{\mathbb{L}}$, we have $t \xrightarrow{\ell}_T t'$ with $(i',t') \in R'_{\beta} \in \mathcal{R}'$ for some $\beta \in \mathbb{L}$ with $F(m,\ell,\beta) \sqsubseteq_{\mathbb{L}} \alpha$. As $\llbracket \ell \rrbracket$ is closed under F, there is $n \in \llbracket \ell \rrbracket$ for which $F(m,n,\beta) = F(m,\ell,\beta)$, and we add a transition $t \xrightarrow{m}_J t'$ to J.

Similarly, for any $t \xrightarrow{\ell}_T t'$ and any $i \in I$ for which $(i,t) \in R'_{\alpha} \in \mathcal{R}'$ for some $\alpha \in \mathbb{L}$, $\alpha \neq \top_{\mathbb{L}}$, we have $i \xrightarrow{m}_I i'$ with $(i',t') \in R'_{\beta}$ for some $\beta \in \mathbb{L}$ with $F(m,\ell,\beta) \sqsubseteq_{\mathbb{L}} \alpha$. Using again closedness of $[\![\ell]\!]$, we find $n \in [\![\ell]\!]$ for which $F(m,n,\beta) = F(m,\ell,\beta)$ and add a transition $t \xrightarrow{n}_J t'$ to J.

We show that the identity relation $\{(t,t) \mid t \in T\} \subseteq J \times T$ witnesses $J \leq_{\mathsf{m}} T$. Let first $t \xrightarrow{n}_J t'$; we must have used one of the two constructions above for creating this transition. In the first case, there is $t \xrightarrow{\ell}_T t'$ with $n \in [\![\ell]\!]$, and in the second case, there is $t \xrightarrow{\ell}_T t'$, hence also $t \xrightarrow{\ell'}_T t'$ with $\ell \preccurlyeq \ell'$, thus $n \in [\![\ell]\!] \subseteq [\![\ell']\!]$. Now let $t \xrightarrow{\ell}_T t'$, then the second construction above has introduced $t \xrightarrow{n}_J t'$ with $n \in [\![\ell]\!]$.

To finish the proof, we show that the family \mathcal{R}' is a witness for $d_{\mathsf{m}}(I,J) \sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}(S,T)$. First, $(i_0,s_0) \in \tilde{R}$ and $(s_0,t_0) \in R_{d_{\mathsf{m}}(S,T)}$ imply $(i_0,t_0) \in R'_{d_{\mathsf{m}}(S,T)}$. Let $(i,t) \in R'_{\alpha} \in \mathcal{R}'$ for some $\alpha \in \mathbb{L}$, $\alpha \neq \top_{\mathbb{L}}$, and assume first $i \xrightarrow{m}_{I} i'$. Then $t \xrightarrow{-}_{-}^{\ell}_{T} t'$ and $t \xrightarrow{m}_{J} t'$ by the first part of our above construction, and $(i',t') \in R'_{\beta}$ with $F(m,n,\beta) \sqsubseteq_{\mathbb{L}} F(m,\ell,\beta) \sqsubseteq_{\mathbb{L}} \alpha$. For the converse, a transition $t \xrightarrow{m}_{J} t'$ must have been introduced above, and in both cases, $i \xrightarrow{m}_{I} i'$ with $(i',t') \in R'_{\beta}$ and $F(m,n,\beta) \sqsubseteq_{\mathbb{L}} F(m,\ell,\beta) \sqsubseteq_{\mathbb{L}} \alpha$.

Now to the proof of the second assertion of the theorem. If $d_{\mathsf{th}}(S,T) = \top_{\mathbb{L}}$, we are done. Otherwise we inductively construct a relation family $\mathcal{R} = \{R_{\alpha} \subseteq S \times T \mid \alpha \in \mathbb{L}\}$ which satisfies $d_{\mathsf{th}}((s,S),(t,T)) \sqsubseteq \alpha$ for any $(s,t) \in R_{\alpha}$, as follows: Begin by letting $R_{\alpha} = \{(s_0,t_0)\}$ for all $\alpha \supseteq_{\mathbb{L}} d_{\mathsf{th}}(S,T)$, and let now $(s,t) \in R_{\alpha}$ with $d_{\mathsf{th}}((s,S),(t,T)) \sqsubseteq \alpha \neq \top_{\mathbb{L}}$.

Let $s \xrightarrow{k}_S s'$ and $t \xrightarrow{\ell}_T t'$ such that $d(k,\ell) \neq \top_{\mathbb{L}}$. Let $(i',I') \in \llbracket (s',S) \rrbracket$ and $m \in \llbracket k \rrbracket$, then there is $(i,I) \in \llbracket (s,S) \rrbracket$ for which $i \xrightarrow{m}_I i''$ and $(i'',I) \leq_{\mathsf{m}} (i',I')$. By the triangle inequality we have $d_{\mathsf{th}}((i,I),(t,T)) \sqsubseteq_{\mathbb{L}} d_{\mathsf{th}}((i,I),(s,S)) \oplus_{\mathbb{L}} d_{\mathsf{th}}((s,S),(t,T)) \sqsubseteq_{\mathbb{L}} \alpha$, hence there is $t \xrightarrow{\ell'} t''$ for which $d(m,\ell') \sqsubseteq_{\mathbb{L}} \alpha$. But we also have $d(m,\ell) \sqsubseteq_{\mathbb{L}} d(m,k) \oplus_{\mathbb{L}} d(k,\ell) = d(k,\ell) \neq \top_{\mathbb{L}}$, so by determinism of T it follows that $\ell = \ell'$ and t' = t''.

As $m \in \llbracket k \rrbracket$ was chosen arbitrarily above, we have $d(m,\ell) \sqsubseteq_{\mathbb{L}} \alpha$ for all $m \in \llbracket k \rrbracket$, hence $d(k,\ell) = F(k,\ell,\perp_{\mathbb{L}}) \sqsubseteq \alpha$. Let $B = \{\beta' \in \mathbb{L} \mid F(k,\ell,\beta') \sqsubseteq_{\mathbb{L}} \alpha\}$

and $\beta = \sup B$, then $F(k, \ell, \beta) \sqsubseteq_{\mathbb{L}} \alpha$ as $\bot_{\mathbb{L}} \in S$. Add (s', t') to R_{γ} for all $\gamma \sqsupseteq_{\mathbb{L}} \beta$.

We miss to show that $d_{\mathsf{th}}((s',S),(t',T)) \sqsubseteq_{\mathbb{L}} \beta$. By $d_{\mathsf{th}}((s,S),(t,T)) \sqsubseteq_{\mathbb{L}} \alpha$ we must have $(j,J) \in [\![(t,T)]\!], j \xrightarrow{n}_J j'$, and an element $\beta' \in \mathbb{L}$ for which $d_{\mathsf{m}}((i',I'),(j',J)) \sqsubseteq_{\mathbb{L}} \beta'$ and $F(m,n,\beta') \sqsubseteq_{\mathbb{L}} \alpha$. Then

$$F(k,\ell,\beta') = F(m,\ell,\beta') \sqsubseteq_{\mathbb{L}} F(m,n,\beta') \sqsubseteq_{\mathbb{L}} \alpha,$$

hence $\beta' \in B$, implying that $d_{\mathsf{th}}((s', S), (t', T)) \sqsubseteq_{\mathbb{L}} \beta' \sqsubseteq_{\mathbb{L}} \beta$.

We show that \mathcal{R} is a refinement family which witnesses $d_{\mathsf{m}}(S,T)$. Let $(s,t) \in R_{\alpha} \in \mathcal{R}$ for some $\alpha \in \mathbb{L}$ and assume $s \stackrel{k}{-} \rightarrow_{S} s'$. Let $m \in \llbracket k \rrbracket$, then there is $(i,I) \in \llbracket (s,S) \rrbracket$ with $i \stackrel{m}{\longrightarrow}_{I} i'$. As $d_{\mathsf{th}}((i,I),(t,T)) \sqsubseteq_{\mathbb{L}} \alpha$, this implies that there is $t \stackrel{\ell}{-} \rightarrow_{T} t'$ with $d(m,\ell) \sqsubseteq_{\mathbb{L}} \alpha$. Also for any other $m' \in \llbracket k \rrbracket$ we have $t \stackrel{\ell'}{-} \rightarrow_{T} t''$ with $d(m,\ell') \sqsubseteq_{\mathbb{L}} \alpha$, hence $\ell = \ell'$ and t' = t'' by determinism. As m was chosen arbitrarily, we have $d(m,\ell) \sqsubseteq \alpha$ for all $m \in \llbracket k \rrbracket$, hence $d(k,\ell) = F(k,\ell,\perp_{\mathbb{L}}) \sqsubseteq \alpha$. By construction of \mathcal{R} , $(s',t') \in R_{\beta}$ for $\beta = \sup\{\beta' \in \mathbb{L} \mid F(k,\ell,\beta') \sqsubseteq_{\mathbb{L}} \alpha\}$.

Now assume $t \xrightarrow{\ell}_T t'$. Let $(i,I) \in \llbracket (s,S) \rrbracket$, then we have $(j,J) \in \llbracket (t,T) \rrbracket$ with $d_{\mathbf{m}}((i,I),(j,J)) \sqsubseteq_{\mathbb{L}} \alpha$. We must have $j \xrightarrow{n}_J j'$ with $n \in \llbracket \ell \rrbracket$, hence there are $i \xrightarrow{m}_I i'$ and $\beta' \in \mathbb{L}$ with $d_{\mathbf{m}}((i',I),(j',J)) \sqsubseteq_{\mathbb{L}} \beta'$ and $F(m,n,\beta') \sqsubseteq_{\mathbb{L}} \alpha$.

The above considerations hold for all $(i, I) \in \llbracket (s, S) \rrbracket$, hence there is $k \in \mathbb{L}$ with $m \in \llbracket k \rrbracket$, $s \xrightarrow{k}_S s'$, and $F(k, \ell, \beta') = F(m, \ell, \beta')$. But then $F(k, \ell, \beta') \sqsubseteq_{\mathbb{L}} F(m, n, \beta') \sqsubseteq_{\mathbb{L}} \alpha$, hence by construction of \mathcal{R} , $(s', t') \in R_{\beta}$ for $\beta = \sup\{\beta' \in \mathbb{L} \mid F(k, \ell, \beta') \sqsubseteq_{\mathbb{L}} \alpha\}$.

7.2.4 Quantitative relaxation

In a quantitative framework, it can be useful to be able to relax and strengthen specifications during the development process. Which precise relaxations and strengthenings one wishes to apply will depend on the actual application, but we can here show three general relaxations which differ from each other in the level of the theory at which they are applied. For $\alpha \in \mathbb{L}$ and SMTS S, T,

- T is an α -widening of S if there is a relation $R \subseteq S \times T$ for which $(s_0, t_0) \in R$ and such that for all $(s, t) \in R$, $s \stackrel{k}{\dashrightarrow}_S s'$ if and only if $t \stackrel{\ell}{\longrightarrow}_T t'$, and $s \stackrel{k}{\longrightarrow}_S s'$ if and only if $t \stackrel{\ell}{\longrightarrow}_T t'$, for $k \leq \ell$, $d(\ell, k) \sqsubseteq_{\mathbb{L}} \alpha$, and $(s', t') \in R$;
- T is an α -relaxation of S if $S \leq_{\mathsf{m}} T$ and $T \leq_{\mathsf{m}}^{\alpha} S$;
- the α -extended implementation semantics of S is

$$[\![S]\!]^{+\alpha} = \{I \leq_{\mathsf{m}}^{\alpha} S \mid I \text{ implementation}\}.$$

All three notions have also been introduced for the special case of integer weights in Section 6.3; but note that the notion of widening presented here is more synthetic than the one of the previous chapter.

The notion of α -widening is entirely syntactic: up to unweighted bisimulation, T is the same as S, but transition labels in T can be α "wider" than in S (hence also $S \leq_{\mathsf{m}} T$). The second notion, α -relaxation, works at the level of semantics of specifications, whereas the last notion is at implementation level. A priori, there is no relation between the syntactic and semantic notions, even though one can be established in some special cases.

7.18 Example. For the accumulated distance with discounting factor λ , any α -widening is also an $\frac{\alpha}{1-\lambda}$ -relaxation, see Proposition 6.19. This is due to the fact that for traces $\sigma, \tau \in \mathsf{Spec}^{\infty}$ with $d(\sigma_j, \tau_j) \leq \alpha$ for all j, we have $\sum_j \lambda^j d(\sigma_j, \tau_j) \leq \sum_j \lambda^j \alpha \leq (1-\lambda)^{-1} \alpha$ by convergence of the geometric series.

For the point-wise distance, it is easy to see that any α -widening is also an α -relaxation, and the same holds for the limit-average distance:

$$\liminf_{j} \frac{1}{j+1} \sum_{i=0}^{j} d(\sigma_i, \tau_i) \le \liminf_{j} \frac{1}{j+1} j\alpha = \alpha$$

- **7.19 Example.** For the maximum-lead distance on the other hand, it is easy to expose cases of α -widenings which are *not* β -relaxations for any β . One example consists of two one-state SMTS S, T with loops $s_0 \xrightarrow{a,1} s_0$ and $t_0 \xrightarrow{a,[0,2]} t_0$; then T is an α -widening of S for $\alpha(\delta) = |\delta + 1|$, but $d_{\mathbf{m}}(T,S) = \top_{\mathbb{L}}$.
- **7.20 Proposition.** If T is an α -relaxation of S, then $[T] \subseteq [S]^{+\alpha}$.

It can be shown for special cases that the inclusion in the proposition is strict, see Section 6.3; for its proof one only needs the fact that $d_{\mathsf{m}}(I,S) \sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}(I,T) \oplus_{\mathbb{L}} d_{\mathsf{m}}(I,S) \sqsubseteq_{\mathbb{L}} \alpha$ for all $I \in \llbracket T \rrbracket$.

Also of interest is the relation between relaxations of different specifications. An easy application of the triangle inequality for d_{m} shows that the distance between relaxations is bounded by the sum of the relaxation constants and the unrelaxed systems' distances:

7.21 Proposition. Let T be an α -relaxation of S and T' an α' -relaxation of S'. Then $d_{\mathsf{m}}(T,T') \sqsubseteq_{\mathbb{L}} \alpha \oplus_{\mathbb{L}} d_{\mathsf{m}}(S,S')$ and $d_{\mathsf{m}}(T',T) \sqsubseteq_{\mathbb{L}} \alpha' \oplus_{\mathbb{L}} d_{\mathsf{m}}(S',S)$.

7.3 Structural Composition and Quotient

We now introduce the different operations on SMTS which make up a specification theory. Firstly, we are interested in composing specifications S, S' into a specification S||S'| by synchronizing on shared actions. Secondly, we need a quotient operator which solves equations of the form $S||X| \equiv T$, that is, the quotient synthesizes the most general specification $T \oslash S$ which describes all SMTS X satisfying the above equation.

7.3.1 Structural composition

To structurally compose SMTS, we assume given a generic partial *label com*position operator \oplus : Spec \times Spec \rightarrow Spec which specifies which labels can synchronize, *cf.* [WN95]. We will need to assume the following property:

• for all $\ell, \ell' \in \mathsf{Spec}$, $(\exists k \in \mathsf{Spec} : d(k, \ell) \neq \top_{\mathbb{L}}, d(k, \ell') \neq \top_{\mathbb{L}}) \Leftrightarrow (\exists m \in \mathsf{Spec} : \ell \oplus m, \ell' \oplus m \text{ are defined}).$

This operator permits to compose labels at transitions which are executed in parallel; the property required relates composability to distances in such a way that two labels have a common quantitative refinement if and only if they have a common synchronization. This is quite natural and holds for all our examples, and is needed to relate determinism to composition in the proof of Theorem 7.29 below.

Additionally, we must assume that there exists a function $P : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ which allows us to infer bounds on distances on synchronized labels. We assume that P is monotone in both coordinates, has $P(\bot_{\mathbb{L}}, \bot_{\mathbb{L}}) = \bot_{\mathbb{L}}$, $P(\alpha, \top_{\mathbb{L}}) = P(\top_{\mathbb{L}}, \alpha) = T_{\mathbb{L}}$ for all $\alpha \in \mathbb{L}$, and that

$$F(k \oplus k', \ell \oplus \ell', P(\alpha, \alpha')) \sqsubseteq_{\mathbb{L}} P(F(k, \ell, \alpha), F(k', \ell', \alpha'))$$

for all $k, \ell, k', \ell' \in \mathsf{Spec}$ and $\alpha, \alpha' \in \mathbb{L}$ for which $k \oplus k'$ and $\ell \oplus \ell'$ are defined. Hence $d(k \oplus k', \ell \oplus \ell') \sqsubseteq_{\mathbb{L}} P(d(k, \ell), d(k', \ell'))$ for all such $k, \ell, k', \ell' \in \mathsf{Spec}$, thus P indeed bounds distances of synchronized labels.

Intuitively, P gives us a *uniform bound* on label composition: distances between composed labels can be bounded above using P and the individual labels' distances.

7.22 Definition. The structural composition of two SMTS S and T is the SMTS $S || T = (S \times T, (s_0, t_0), \cdots, s_{\parallel T}, \longrightarrow_{S \parallel T})$ with transitions defined as follows:

$$\frac{s \xrightarrow{k}_{S} s' \quad t \xrightarrow{\ell}_{T} t' \quad k \oplus \ell \text{ defined}}{(s,t) \xrightarrow{k \oplus \ell}_{S\parallel T} (s',t')} \qquad \frac{s \xrightarrow{k}_{S} s' \quad t \xrightarrow{\ell}_{T} t' \quad k \oplus \ell \text{ defined}}{(s,t) \xrightarrow{k \oplus \ell}_{S\parallel T} (s',t')}$$

The next theorem shows that structural composition supports quantitative $independent\ implementability$: the distance between structural compositions can bounded above using P and the distances between the individual components.

7.23 Theorem. For all SMTS S, T, S' and T' with $d_{\mathsf{m}}(S\|S', T\|T') \neq \top_{\mathbb{L}}$, $d_{\mathsf{m}}(S\|S', T\|T') \sqsubseteq_{\mathbb{L}} P(d_{\mathsf{m}}(S, T), d_{\mathsf{m}}(S', T'))$.

Proof: Let $\mathcal{R} = \{R_{\alpha} \subseteq S \times T \mid \alpha \in \mathbb{L}\}, \mathcal{R}' = \{R'_{\alpha} \subseteq S' \times T' \mid \alpha \in \mathbb{L}\}$ be witnesses for $d_{\mathsf{m}}(S,T)$ and $d_{\mathsf{m}}(S',T')$, respectively, and define

$$R_{\beta}^{\parallel} = \{ ((s, s'), (t, t')) \in S \times S' \times T \times T' \mid \exists \alpha, \alpha' \in \mathbb{L} : (s, t) \in R_{\alpha} \in \mathcal{R}, (s', t') \in R'_{\alpha'} \in \mathcal{R}', P(\alpha, \alpha') \sqsubseteq_{\mathbb{L}} \beta \}$$

for all $\beta \in \mathbb{L}$. We show that $\mathcal{R}^{\parallel} = \{R_{\beta}^{\parallel} \mid \beta \in L\}$ witnesses $d_{\mathsf{m}}(S \parallel S', T \parallel T') \sqsubseteq_{\mathbb{L}} P(d_{\mathsf{m}}(S,T),d_{\mathsf{m}}(S',T'))$.

First, $((s_0, s'_0), (t_0, t'_0)) \in R_{P(d_{\mathsf{m}}(S,T), d_{\mathsf{m}}(S',T'))}^{\parallel}$. Let now $\beta \in \mathbb{L} \setminus \{\top_{\mathbb{L}}\}$ and $((s, s'), (t, t')) \in R_{\beta}^{\parallel} \in \mathcal{R}^{\parallel}$, then we have $\alpha, \alpha' \in \mathbb{L} \setminus \{\top_{\mathbb{L}}\}$ with $(s, t) \in R_{\alpha} \in \mathcal{R}$, $(s', t') \in R'_{\alpha'} \in \mathcal{R}'$, and $P(\alpha, \alpha') \sqsubseteq_{\mathbb{L}} \beta$.

Let $(s,s') \xrightarrow{k \oplus k'} S_{\parallel S'} (\bar{s},\bar{s}')$, then $s \xrightarrow{k} S_{\bar{s}} \bar{s}$ and $s' \xrightarrow{k'} S'_{\bar{s}'} \bar{s}'$. As $(s,t) \in R_{\alpha} \in \mathcal{R}$, we have $t \xrightarrow{\ell} T_{\bar{t}} \bar{t}$ and $\bar{\alpha} \in \mathbb{L}$ with $(\bar{s},\bar{t}) \in R_{\bar{\alpha}} \in \mathcal{R}$ and $F(k,\ell,\bar{\alpha}) \sqsubseteq_{\mathbb{L}} \alpha$. Similarly, $(s',t') \in R'_{\alpha'} \in \mathcal{R}'$ implies that there is $t' \xrightarrow{\ell'} T'_{\bar{t}} \bar{t}'$ and $\bar{\alpha}' \in \mathbb{L}$ with $(\bar{s}',\bar{t}') \in R'_{\bar{\alpha}'} \in \mathcal{R}'$ and $F(k',\ell',\bar{\alpha}') \sqsubseteq_{\mathbb{L}} \alpha'$.

Now if the composition $\ell \oplus \ell'$ is undefined, then $d_{\mathsf{m}}(S \| S', T \| T') = \top_{\mathbb{L}}$. If it is defined, then we have $(t, t') \xrightarrow{\ell \oplus \ell'}_{T \| T'} (\bar{t}, \bar{t}')$ by definition of $S \| S'$. Also, $(\bar{t}, \bar{t}') \in R_{P(\bar{\alpha}, \bar{\alpha}')}^{\parallel} \in \mathcal{R}^{\parallel}$ and

$$F(k \oplus k', \ell \oplus \ell', P(\bar{\alpha}, \bar{\alpha}')) \sqsubseteq_{\mathbb{L}} P(F(k, \ell, \bar{\alpha}), F(k', \ell', \bar{\alpha}')) \sqsubseteq_{\mathbb{L}} P(\alpha, \alpha').$$

The reverse direction, assuming a transition $(t,t') \xrightarrow{\ell \oplus \ell'}_{T||T'} (\bar{t},\bar{t}')$, is similar.

7.24 Example. One popular label synchronization operator for the set $\mathsf{Spec} = \Sigma \times \mathbb{I}$ from our examples, also used in Chapter 6, is given by adding interval boundaries, viz.

$$(a,[l,r]) \oplus (a',[l',r']) = \begin{cases} (a,[l+l',r+r']) & \text{if } a=a',\\ \text{undefined} & \text{otherwise.} \end{cases}$$

It can then be shown that

$$d(k \oplus k', \ell \oplus \ell') \le d(k, \ell) + d(k', \ell') \tag{7.2}$$

for all $k, \ell, k', \ell' \in \mathsf{Spec}$ for which $k \oplus k'$ and $\ell \oplus \ell'$ are defined.

For the accumulating distance, (7.2) implies that \oplus is bounded above by $P(\alpha, \alpha') = \alpha + \alpha'$:

$$F(k \oplus k', \ell \oplus \ell', \alpha + \alpha') = d(k \oplus k', \ell \oplus \ell') + \lambda(\alpha + \alpha')$$

$$\leq d(k, \ell) + \lambda\alpha + d(k', \ell') + \lambda\alpha'$$

$$= F(k, \ell, \alpha) + F(k', \ell', \alpha')$$

Theorem 7.23 thus specializes to Theorem 6.27: $d_{\mathsf{m}}(S \| S', T \| T') \leq d_{\mathsf{m}}(S, T) + d_{\mathsf{m}}(S', T')$ for all SMTS S, T, S', T'.

7.25 Example. Also for the point-wise distance, a bound is given by $P(\alpha, \alpha') = \alpha + \alpha'$:

$$F(k \oplus k', \ell \oplus \ell', \alpha + \alpha') = \max(d(k \oplus k', \ell \oplus \ell'), \alpha + \alpha')$$

$$\leq \max(d(k, \ell) + d(k', \ell'), \alpha + \alpha')$$

$$\leq \max(d(k, \ell), \alpha) + \max(d(k', \ell'), \alpha')$$

$$= F(k, \ell, \alpha) + F(k', \ell', \alpha'),$$

the last inequality because of distributivity of addition over maximum. Thus also here, $d_{\mathsf{m}}(S||S',T||T') \leq d_{\mathsf{m}}(S,T) + d_{\mathsf{m}}(S',T')$ for all SMTS S,T,S',T'.

7.26 Example. For the limit-average distance, a similar bound $P(\alpha, \alpha') = \alpha \oplus_{\mathbb{L}} \alpha'$ works: For all $j \in \mathbb{N}$,

$$F(k \oplus k', \ell \oplus \ell', \alpha \oplus_{\mathbb{L}} \alpha')(j)$$

$$= \frac{1}{j+1} d(k \oplus k', \ell \oplus \ell') + \frac{j}{j+1} (\alpha(j-1) + \alpha'(j-1))$$

$$\leq \frac{1}{j+1} d(k, \ell) + \frac{j}{j+1} \alpha(j-1) + \frac{1}{j+1} d(k', \ell') + \frac{j}{j+1} \alpha'(j-1)$$

$$= F(k, \ell, \alpha)(j) + F(k', \ell', \alpha')(j).$$

Hence also for the limit-average distance, we have $d_{\mathsf{m}}(S||S',T||T') \leq d_{\mathsf{m}}(S,T) + d_{\mathsf{m}}(S',T')$ for all SMTS S,T,S',T'.

7.27 Example. In a real-time setting, a label synchronization operator which uses intersection of intervals instead of addition has been used [FL12, BLPR09]. That is,

$$(a,[l,r]) \oplus (a',[l',r']) = \begin{cases} (a,[\max(l,l'),\min(r,r')]) \\ & \text{if } a=a',\max(l,l') \leq \min(r,r'), \\ & \text{undefined} \end{cases}$$

We show that for the maximum-lead distance, \oplus is bounded above by $P(\alpha, \alpha') = \max(\alpha, \alpha')$, that is,

$$F(k \oplus k', \ell \oplus \ell', \max(\alpha, \alpha'))(d) \leq \max(F(k, \ell, \alpha)(d), F(k', \ell', \alpha')(d))$$
.

Applying the definition of F, we see that this is equivalent to

$$\sup_{p \in \llbracket k \oplus k' \rrbracket} \inf_{q \in \llbracket \ell \oplus \ell' \rrbracket} \max(|d+p-q|, \max(\alpha(d+p-q), \alpha'(d+p-q)))$$

$$\leq \max \begin{cases} \sup_{m \in \llbracket k \rrbracket} \inf_{n \in \llbracket \ell \rrbracket} \max(|d+m-n|, \alpha(d+m-n)) \\ \sup_{m' \in \llbracket k' \rrbracket} \inf_{n' \in \llbracket \ell' \rrbracket} \max(|d+m'-n'|, \alpha'(d+m'-n')); \end{cases}$$

note that we are abusing notation by identifying p = (a, x) with x etc. This inequality in turn is equivalent to

$$\max \begin{cases} \sup_{p \in \llbracket k \oplus k' \rrbracket} \inf_{q \in \llbracket \ell \oplus \ell' \rrbracket} |d+p-q| \\ \sup_{\text{sup inf}} \inf_{p \in \llbracket k \oplus k' \rrbracket} \alpha(d+p-q) \\ \sup_{p \in \llbracket k \oplus k' \rrbracket} \inf_{q \in \llbracket \ell \oplus \ell' \rrbracket} \alpha'(d+p-q) \end{cases} \leq \max \begin{cases} \sup_{m \in \llbracket k \rrbracket} \inf_{n \in \llbracket \ell \rrbracket} |d+m-n| \\ \sup_{m \in \llbracket k \rrbracket} \inf_{n \in \llbracket \ell \rrbracket} \alpha(d+m-n) \\ \sup_{m \in \llbracket k \rrbracket} \inf_{n \in \llbracket \ell' \rrbracket} \alpha'(d+m-n') \\ \sup_{p \in \llbracket k \oplus k' \rrbracket} \inf_{q \in \llbracket \ell \oplus \ell' \rrbracket} \alpha'(d+p-q) \end{cases}$$

In this expression, the first line on the left-hand side is bounded by the right-hand side's first line, the second line on the left by the second line on the right, and the left-hand side's last line by the last line of the right-hand side, so that altogether, it holds. Theorem 7.23 then translates to $d_{\mathsf{m}}(S||S',T||T') \leq \max (d_{\mathsf{m}}(S,T),d_{\mathsf{m}}(S',T'))$.

7.3.2 Quotient

For *quotients* of SMTS, we need a partial label operator \oslash : Spec \times Spec \rightharpoonup Spec for which it holds that

- for all $k, \ell, m \in \mathsf{Spec}$, $\ell \oslash k$ is defined and $m \preccurlyeq \ell \oslash k$ if and only if $k \oplus m$ is defined and $k \oplus m \preccurlyeq \ell$;
- for all $\ell, \ell' \in \mathsf{Spec}$, $(\exists k \in \mathsf{Spec} : d(k, \ell) \neq \top_{\mathbb{L}}, d(k, \ell') \neq \top_{\mathbb{L}}) \Leftrightarrow (\exists m \in \mathsf{Spec} : m \oslash \ell, m \oslash \ell' \text{ are defined}).$

The first condition ensures that \oslash is adjoint to \oplus , and the second relates it to distances just as we did for \oplus above. Extending the first condition, we say that

- \oslash is quantitatively well-behaved if it holds for all $k, \ell, m \in \mathsf{Spec}$ that $\ell \oslash k$ is defined and $d(m, \ell \oslash k) \neq \top_{\mathbb{L}}$ if and only if $k \oplus m$ is defined and $d(k \oplus m, \ell) \neq \top_{\mathbb{L}}$, and in that case, $F(m, \ell \oslash k, \alpha) \sqsupseteq_{\mathbb{L}} F(k \oplus m, \ell, \alpha)$ for all $\alpha \in \mathbb{L}$, and that
- \oslash is quantitatively exact if the inequality can be sharpened to $F(m, \ell \oslash k, \alpha) = F(k \oplus m, \ell, \alpha)$.

Both of these are useful quantitative generalization of the adjunction between \oslash and \oplus ; we will see examples below of quantitatively exact and quantitatively well-behaved label quotients.

In the definition of quotient below, we denote by $\rho_B(S)$ the *pruning* of a SMTS S with respect to the states in $B \subseteq S$, see Section 6.4.

7.28 Definition. For SMTS S, T, the quotient of T by S is the SMTS $T/S = \rho_B(T \times S \cup \{u\}, (t_0, s_0), -\rightarrow_{T/S}, \longrightarrow_{T/S})$ given as follows (if it exists):

$$\frac{t\stackrel{\ell}{--} \downarrow_T t' \quad s \stackrel{k}{--} \downarrow_S s' \quad \ell \oslash k \text{ defined}}{(t,s)\stackrel{\ell \oslash k}{--} \downarrow_{T/S} (t',s')} \qquad \frac{t\stackrel{\ell}{--} \downarrow_T t' \quad s \stackrel{k}{\longrightarrow}_S s' \quad \ell \oslash k \text{ defined}}{(t,s)\stackrel{\ell \oslash k}{\longrightarrow}_{T/S} (t',s')}$$

$$\frac{t\stackrel{\ell}{\longrightarrow}_T t' \quad \forall s \stackrel{k}{\longrightarrow}_S s' : \ell \oslash k \text{ undefined}}{(t,s) \in B}$$

$$\underline{m \in \text{Spec}} \quad \forall s \stackrel{k}{--} \downarrow_S s' : k \oplus m \text{ undefined}}_{(t,s)\stackrel{m}{--} \downarrow_{T/S} u} \qquad \underline{m \in \text{Spec}}_{u\stackrel{m}{--} \downarrow_{T/S} u}$$

In the above definition, u is a new universal state from which everything is allowed and nothing required (last SOS rule). This state is reached from a quotient state (t, s) under label m whenever there is no may transition from s with whose label m can synchronize (next-to-last SOS rule), because in that case, any transition in the quotient will be canceled in the structural composition (cf. Theorem 7.29 below), and we need the quotient to be maximal. Similarly, if t specifies a must transition under a label ℓ which cannot be matched by any transition from s, then the quotient state (t,s) is inconsistent; hence we add it to B and remove it when pruning.

The next theorem shows that under certain standard conditions, quotient is *sound* and *maximal* with respect to structural composition.

- **7.29 Theorem.** Let S, T, X be SMTS such that S is deterministic and T/S exists. Then $X \leq_{\mathbf{m}} T/S$ if and only if $S || X \leq_{\mathbf{m}} T$. Also,
 - if \oslash is quantitatively well-behaved, then $d_{\mathsf{m}}(X,T/S) \supseteq_{\mathbb{L}} d_{\mathsf{m}}(S||X,T)$;
 - if \oslash is quantitatively exact and $d_{\mathsf{m}}(X,T/S) \neq \top_{\mathbb{L}}$, then $d_{\mathsf{m}}(X,T/S) = d_{\mathsf{m}}(S||X,T)$.

The (Boolean) property that $X \leq_{\mathsf{m}} T/S$ iff $S \| X \leq_{\mathsf{m}} T$ implies uniqueness of quotient [FLW11]. For the quantitative generalizations, the property induced by a well-behaved \oslash means that distances to the quotient bound distances of structural compositions, which can be useful in further calculations; similarly for exact \oslash . Note that uniqueness implies that if a certain instantiation of our framework admits a quotient which is not quantitatively well-behaved, there is no hope that one can find another one which is.

Proof: The proof that $X \leq_{\mathsf{m}} T/S$ if and only if $S \| X \leq_{\mathsf{m}} T$ is in [BJL⁺12a]. For the other properties, assume first \emptyset to be quantitatively well-behaved; we show that $d_{\mathsf{m}}(S \| X, T) \sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}(X, T/S)$. If $d_{\mathsf{m}}(X, T/S) = \top_{\mathbb{L}}$, there is nothing to prove, so assume $d_{\mathsf{m}}(X, T/S) \neq \top_{\mathbb{L}}$ and let $\mathcal{R} = \{R_{\alpha} \subseteq X \times (T \times S \cup \{u\})\}$ be a witness for $d_{\mathsf{m}}(X, T/S)$. Define $R'_{\alpha} = \{((s, x), t) \mid (x, (t, s)) \in R_{\alpha}\} \subseteq$

 $S \times X \times T$ for all $\alpha \in \mathbb{L}$ and collect these to a family $\mathcal{R}' = \{R'_{\alpha} \mid \alpha \in \mathbb{L}\}$. We show that \mathcal{R}' is a witness for $d_{\mathsf{m}}(S||X,T) \sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}(X,T/S)$.

We have $((s_0, x_0), t_0) \in R'_{d_{\mathfrak{m}}(X, T/S)} \in \mathcal{R}'$, so let $\alpha \in \mathbb{L}$ and $((s, x), t) \in R'_{\alpha} \in \mathcal{R}'$, and assume first that $(s, x) \xrightarrow{k \oplus m} S_{\parallel X} (s', x')$. Then $s \xrightarrow{k} S'$ and $x \xrightarrow{m} X'$ by definition of $S \parallel X$. Now $(x, (t, s)) \in R_{\alpha} \in \mathcal{R}$ implies that there is $(t, s) \xrightarrow{\ell \oslash k'} (t', s'')$ and $\alpha' \in \mathbb{L}$ for which $F(m, \ell \oslash k', \alpha') \sqsubseteq_{\mathbb{L}} \alpha$ and $(x', (t', s'')) \in R_{\alpha'} \in \mathcal{R}$. But then also $((s'', x'), t') \in R'_{\alpha} \in \mathcal{R}'$, hence $k' \oplus m$ is defined and $F(k' \oplus m, \ell, \alpha') \sqsubseteq_{\mathbb{L}} F(m, \ell \oslash k', \alpha') \sqsubseteq_{\mathbb{L}} \alpha$.

Now $k \oplus m$ and $k' \oplus m$ being defined implies that there is k'' for which $d(k'',k) \neq \top_{\mathbb{L}}$ and $d(k'',k') \neq \top_{\mathbb{L}}$, and by definition of T/S, $s \xrightarrow{k'} s''$. As S is deterministic, this implies k = k' and s' = s''. Hence $((s',x'),t') \in R'_{\alpha'} \in \mathcal{R}'$ and $F(k \oplus m, \ell, \alpha') \sqsubseteq_{\mathbb{L}} \alpha$.

Assume now that $t \xrightarrow{\ell}_T t'$. We must have $s \xrightarrow{k}_S s'$ for which $\ell \oslash k$ is defined, for otherwise $(t,s) \in B$ and hence (t,s) would have been pruned in T/S. Thus $(t,s) \xrightarrow{\ell \oslash k}_{T/S} (t',s')$, which by $(x,(t,s)) \in R_\alpha \in \mathcal{R}$ implies that there is $x \xrightarrow{m}_X x'$ and $\alpha' \in \mathbb{L}$ for which $F(m,\ell \oslash k,\alpha') \sqsubseteq_{\mathbb{L}} \alpha$ and $(x',(t',s')) \in R_{\alpha'} \in \mathcal{R}$, hence $((s',x'),t') \in R'_{\alpha'} \in \mathcal{R}'$. But then $k \oplus m$ is defined and $F(k \oplus m,\ell,\alpha') \sqsubseteq_{\mathbb{L}} F(m,\ell \oslash k,\alpha') \sqsubseteq_{\mathbb{L}} \alpha$, and $(s,x) \xrightarrow{k \oplus m}_{S \parallel X} (s',x')$.

In order to prove the theorem's last claim, let \oslash be quantitatively exact. To show that $d_{\mathsf{m}}(X,T/S) \sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}(S \| X,T)$, assume that $d_{\mathsf{m}}(S \| X,T) \neq \top_{\mathbb{L}}$ (otherwise there is nothing to prove), let $\mathcal{R} = \{R_{\alpha} \subseteq S \times X \times T \mid \alpha \in \mathbb{L}\}$ be a witness for $d_{\mathsf{m}}(S \| X,T)$, and define $R'_{\alpha} = \{(x,(t,s)) \mid ((s,x),t) \in R_{\alpha}\} \cup \{(x,u) \mid x \in X\} \subseteq X \times (T \times S \cup \{u\}) \text{ for all } \alpha \in \mathbb{L}$. We show that $\mathcal{R}' = \{R'_{\alpha} \mid \alpha \in \mathbb{L}\}$ is a witness for $d_{\mathsf{m}}(X,T/S) \sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}(S \| X,T)$.

We have $(x_0, (t_0, s_0)) \in R'_{d_{\mathsf{m}}(S||X,T)} \in \mathcal{R}'$. Let $\alpha \in \mathbb{L}$, $(x, u) \in R'_{\alpha} \in \mathcal{R}'$ and $x \xrightarrow{m}_{X} x'$, then also $u \xrightarrow{m}_{T/S} u$, $F(m, m, \bot_{\mathbb{L}}) \sqsubseteq \alpha$, and $(x', u) \in R'_{\bot_{\mathbb{L}}} \in \mathcal{R}'$. Now let $(x, (t, s)) \in R'_{\alpha} \in \mathcal{R}'$ and $x \xrightarrow{m}_{X} x'$. If $k \oplus m$ is undefined for all $s \xrightarrow{k}_{S} s'$, then by definition of T/S, $(t, s) \xrightarrow{m}_{T/S} u$, $F(m, m, \bot_{\mathbb{L}}) \sqsubseteq \alpha$, and $(x', u) \in R'_{\bot_{\mathbb{L}}} \in \mathcal{R}'$.

If there is a transition $s \xrightarrow{k}_S s'$ for which $k \oplus m$ is defined (by determinism there can be at most one), then also $(s,x) \xrightarrow{k \oplus m}_{--} s_{\parallel X} (s',x')$. As $((s,x),t) \in R_{\alpha} \in \mathcal{R}$, we must have $t \xrightarrow{\ell} t'$ and $\alpha' \in \mathbb{L}$ with $F(k \oplus m, \ell, \alpha') \sqsubseteq_{\mathbb{L}} \alpha$ and $((s',x'),t') \in R_{\alpha'} \in \mathcal{R}$, hence $(x',(t',s')) \in R'_{\alpha'} \in \mathcal{R}'$. Then $\ell \oslash k$ is defined and $F(m,\ell \oslash k,\alpha') \sqsubseteq_{\mathbb{L}} \alpha$, and by definition of T/S, $(t,s) \xrightarrow{\ell \oslash k}_{--} t_{/S} (t',s')$.

Now assume that $(t,s) \xrightarrow{\ell \oslash k}_{T/S} (t',s')$, then $t \xrightarrow{\ell}_{T} t'$ and $s \xrightarrow{k}_{S} s'$ by definition of T/S. By $((s,x),t) \in R_{\alpha} \in \mathcal{R}$, we have $(s,x) \xrightarrow{k' \oplus m}_{S||X} (s'',x')$ and $\alpha' \in \mathbb{L}$ with $F(k' \oplus m, \ell, \alpha') \sqsubseteq_{\mathbb{L}} \alpha$ and $((s'',x'),t') \in R_{\alpha'} \in \mathcal{R}$. This in turn implies that $s \xrightarrow{k'}_{S} s''$ and $x \xrightarrow{m}_{X} s'$ by definition of S||X. We also see that

Figure 7.1: Quotient $[l', r'] \oslash [l, r]$ of intervals in Example 7.31, six cases. Top bar: [l, r]; middle bar: [l', r']; bottom bar: quotient. Note that for the two cases on the right, quotient is undefined.

$$\ell \oslash k'$$
 is defined, which by determinism of S entails $k = k'$ and $s' = s''$. Hence $F(k \oplus m, \ell, \alpha') \sqsubseteq_{\mathbb{L}} \alpha$ and $(x', (t', s')) \in R'_{\alpha'} \in \mathcal{R}'$.

7.30 Example. For the label synchronization operator for $Spec = \Sigma \times \mathbb{I}$ given by adding interval boundaries, a quotient can be defined by

$$(a',[l',r'])\oslash(a,[l,r])=\begin{cases} (a,[l'-l,r'-r]) & \text{if } a=a' \text{ and } l'-l\leq r'-r,\\ \text{undefined} & \text{otherwise.} \end{cases}$$

It can then be shown that $d(m, \ell \oslash k) = d(k \oplus m, \ell)$ for all $k, \ell, m \in \mathsf{Spec}$ for which both $\ell \oslash k$ and $k \oplus m$ are defined, see Chapter 6. From this it easily follows that both for the accumulating, the point-wise, and the limit-average distance, \oslash is quantitatively exact, hence for all three distances, Theorem 7.29 specializes to the theorem that $d_{\mathsf{m}}(X, T/S) = d_{\mathsf{m}}(S || X, T)$ for all SMTS S, T, X for which S is deterministic, T/S exists and $d_{\mathsf{m}}(X, T/S) \neq \infty$. For the accumulating distance, this is Theorem 6.29.

7.31 Example. For the variant of the operator \bigcirc which uses intersection of intervals instead of addition, a quotient can be defined as follows:

$$(a',[l',r']) \oslash (a,[l,r]) = \begin{cases} \text{undefined} & \text{if } a \neq a' \text{ ,} \\ (a,[l',\infty]) & \text{if } a = a' \text{ and } l < l' \leq r \leq r' \text{ ,} \\ (a,[l',r']) & \text{if } a = a' \text{ and } l < l' \leq r' < r \text{ ,} \\ \text{undefined} & \text{if } a = a' \text{ and } l \leq r < l' \leq r' \text{ ,} \\ (a,[0,\infty]) & \text{if } a = a' \text{ and } l' \leq l \leq r \leq r' \text{ ,} \\ (a,[0,r']) & \text{if } a = a' \text{ and } l' \leq l \leq r < r' \text{ ,} \\ \text{undefined} & \text{if } a = a' \text{ and } l' \leq r' < l \leq r \text{ .} \end{cases}$$

The intuition is that to obtain the maximal solution [p,q] to an equation $[l,r] \oplus [p,q] \sqsubseteq_{\mathsf{Spec}} [l',r']$, whether p and q must restrain the interval in the intersection, or can be 0 and ∞ , respectively, depends on the position of [l,r] relative to [l',r'], cf. Figure 7.1.

It can be shown [FL12] that for the maximum-lead distance, this variant of ⊘ is quantitatively well-behaved, but not quantitatively exact. Theorem 7.29 hence translates to the fact that for all SMTA S, T, X for which S is deterministic and T/S exists, $d_{\mathsf{m}}(X, T/S) \geq d_{\mathsf{m}}(S||X, T)$.

7.4 Conjunction

Conjunction of SMTS can be used to merge two specifications into one. Let \emptyset : Spec \times Spec \to Spec be a partial label operator for which it holds that

- for all $k, \ell \in \mathsf{Spec}$, if $k \otimes \ell$ is defined, then $k \otimes \ell \leq k, k \otimes \ell \leq \ell$, and
- for all $\ell, \ell' \in \mathsf{Spec}$, $(\exists k \in \mathsf{Spec} : d(k, \ell) \neq \top_{\mathbb{L}}, d(k, \ell') \neq \top_{\mathbb{L}}) \Leftrightarrow (\exists m \in \mathsf{Spec} : \ell \otimes m, \ell' \otimes m \text{ are defined}).$

The first requirement above ensures that conjunction acts as a lower bound, and the second one relates it to distances such that two labels have a common quantitative refinement if and only if they have a common conjunction. One also usually wants conjunction to be a *greatest* lower bound; we say that \emptyset is conjunctively compositional if it holds for all $k, \ell, m \in \mathsf{Spec}$ for which $m \leq k$ and $m \leq \ell$ that also $k \otimes \ell$ is defined and $m \leq k \otimes \ell$.

As a quantitative generalization, and analogously to what we did for structural composition, we say that \otimes is *conjunctively bounded* by a function $C: \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ if C is monotone in both coordinates, has $C(\bot_{\mathbb{L}}, \bot_{\mathbb{L}}) = \bot_{\mathbb{L}}$, $C(\alpha, \top_{\mathbb{L}}) = C(\top_{\mathbb{L}}, \alpha) = \top_{\mathbb{L}}$ for all $\alpha \in \mathbb{L}$, and if it holds for all $k, \ell, m \in \mathsf{Spec}$ for which $d(m, k) \neq \top_{\mathbb{L}}$ and $d(m, \ell) \neq \top_{\mathbb{L}}$ that $k \otimes \ell$ is defined and

$$F(m, k \otimes \ell, C(\alpha, \alpha')) \sqsubseteq_{\mathbb{L}} C(F(m, k, \alpha), F(m, \ell, \alpha'))$$

for all $\alpha, \alpha' \in \mathbb{L}$. Note that this implies that $d(m, k \otimes \ell) \sqsubseteq_{\mathbb{L}} C(d(m, k), d(m, \ell))$, hence conjunctive boundedness implies conjunctive compositionality. Like P for structural composition, C gives a uniform bound on label conjunction.

7.32 Definition. The *conjunction* of two SMTS S and T is the SMTS $S \wedge T = \rho_B(S \times T, (s_0, t_0), -- \star_{S \wedge T}, \longrightarrow_{S \wedge T})$ given as follows:

$$\frac{s \xrightarrow{k}_S s' \quad t \xrightarrow{\ell}_T t' \quad k \otimes \ell \text{ defined}}{(s,t) \xrightarrow{k \otimes \ell}_{S \wedge T} (s',t')} \qquad \frac{s \xrightarrow{k}_S s' \quad t \xrightarrow{\ell}_T t' \quad k \otimes \ell \text{ defined}}{(s,t) \xrightarrow{k \otimes \ell}_{S \wedge T} (s',t')} \\ \qquad \frac{s \xrightarrow{k}_{-\to S} s' \quad t \xrightarrow{-\to_T} t' \quad k \otimes \ell \text{ defined}}{(s,t) \xrightarrow{k \otimes \ell}_{S \wedge T} (s',t')} \\ \frac{s \xrightarrow{k}_S s' \quad \forall t \xrightarrow{-\to_T} t' : k \otimes \ell \text{ undef.}}{(s,t) \in B} \qquad \frac{t \xrightarrow{\ell}_T t' \quad \forall s \xrightarrow{k}_S s' : k \otimes \ell \text{ undef.}}{(s,t) \in B}$$

Note that like for quotient, conjunction of SMTS may give inconsistent states which need to be pruned after. As seen in the last two SOS rules above, this is the case when one SMTS specifies a *must* transition with which

the other SMTS cannot synchronize; then, the demand on implementations would be that they simultaneously *must* and *cannot* have a transition, which of course is unsatisfiable.

The next theorem shows the precise conditions under which conjunction is a greatest lower bound. Note that the greatest-lower-bound condition $U \leq_{\mathsf{m}} S$, $U \leq_{\mathsf{m}} T \Rightarrow U \leq_{\mathsf{m}} S \wedge T$ entails uniqueness.

- **7.33 Theorem.** Let S, T, U be SMTS. If $S \wedge T$ is defined, then $S \wedge T \leq_{\mathsf{m}} S$ and $S \wedge T \leq_{\mathsf{m}} T$. If, additionally, S or T are deterministic, then:
 - If \otimes is conjunctively compositional, $U \leq_m S$, and $U \leq_m T$, then $S \wedge T$ is defined and $U \leq_m S \wedge T$.
 - If \otimes is conjunctively bounded by C, $d_{\mathsf{m}}(U,S) \neq \top_{\mathbb{L}}$, and $d_{\mathsf{m}}(U,T) \neq \top_{\mathbb{L}}$, then $S \wedge T$ is defined and $d_{\mathsf{m}}(U,S \wedge T) \sqsubseteq_{\mathbb{L}} C(d_{\mathsf{m}}(U,S),d_{\mathsf{m}}(U,T))$.

Proof: The proof of the two first claims is in [BJL⁺12a]. For the third claim, let $\mathcal{R} = \{R_{\alpha} \subseteq U \times S \mid \alpha \in \mathbb{L}\}$ and $\mathcal{R}' = \{R'_{\alpha} \subseteq U \times T \mid \alpha \in \mathbb{L}\}$ be relation families witnessing $d_{\mathsf{m}}(U,S)$ and $d_{\mathsf{m}}(U,T)$, respectively, define $R'_{\beta} = \{(u,(s,t)) \mid \exists \alpha, \alpha' \in \mathbb{L} : (u,s) \in R_{\alpha}, (u,t) \in R'_{\alpha'}, C(\alpha,\alpha') \sqsubseteq_{\mathbb{L}} \beta\} \subseteq U \times S \times T$ for all $\beta \in \mathbb{L}$, and let $\mathcal{R}^{\wedge} = \{R'_{\beta} \mid \beta \in \mathbb{L}\}$. We show that \mathcal{R}^{\wedge} is a witness for $d_{\mathsf{m}}(U,S \wedge T) \sqsubseteq_{\mathbb{L}} C(d_{\mathsf{m}}(U,S), d_{\mathsf{m}}(U,T))$.

We have $(u_0, (s_0, t_0)) \in R^{\wedge}_{C(d_{\mathsf{m}}(U,S),d_{\mathsf{m}}(U,T))} \in \mathcal{R}^{\wedge}$. Let $\beta \in \mathbb{L} \setminus \{\bot_{\mathbb{L}}\}$ and $(u, (s,t)) \in R^{\wedge}_{\beta} \in \mathcal{R}^{\wedge}$, then we have $\alpha, \alpha' \in \mathbb{L} \setminus \{\bot_{\mathbb{L}}\}$ with $(u,s) \in R_{\alpha} \in \mathcal{R}$, $(u,t) \in R'_{\alpha'} \in \mathcal{R}'$, and $C(\alpha, \alpha') \sqsubseteq_{\mathbb{L}} \beta$.

Assume $u \xrightarrow{m} U u'$, then there exist $s \xrightarrow{k} S s'$ and $\bar{\alpha} \in \mathbb{L}$ for which $(u', s') \in R_{\bar{\alpha}} \in \mathcal{R}$ and $F(m, k, \bar{\alpha}) \sqsubseteq_{\mathbb{L}} \alpha$, and similarly $t \xrightarrow{\ell} T t'$ and $\bar{\alpha}'$ with $(u', t') \in R'_{\bar{\alpha}'} \in \mathcal{R}'$ and $F(m, \ell, \bar{\alpha}') \sqsubseteq \alpha'$. Then $d(m, k) \neq \top_{\mathbb{L}}$ and $d(m, \ell) \neq \top_{\mathbb{L}}$, so by conjunctive boundedness $k \otimes \ell$ is defined, and $(s, t) \xrightarrow{k \otimes \ell} (s', t')$ by definition of $S \wedge T$. Also, $(u', (s', t')) \in R_{C(\bar{\alpha}, \bar{\alpha}')}^{\wedge} \in \mathcal{R}^{\wedge}$ and $F(m, k \otimes \ell, C(\bar{\alpha}, \bar{\alpha}')) \sqsubseteq_{\mathbb{L}} C(F(m, k, \bar{\alpha}), F(m, \ell, \bar{\alpha}')) \sqsubseteq_{\mathbb{L}} C(\alpha, \alpha')$.

Assume $(s,t) \xrightarrow{k \otimes \ell}_{S \wedge T} (s',t')$, then $s \xrightarrow{k}_{S} s'$ and $t \xrightarrow{\ell}_{T} t'$ by definition of $S \wedge T$. We can without loss of generality postulate that T is deterministic. The fact that $(u,s) \in R_{\alpha} \in \mathcal{R}$ implies that there are $u \xrightarrow{m}_{U} u'$ and $\bar{\alpha} \in \mathbb{L}$ for which $(u',s') \in R_{\bar{\alpha}} \in \mathcal{R}$ and $F(m,k,\bar{\alpha}) \sqsubseteq_{\mathbb{L}} \alpha$. We must also have $u \xrightarrow{m'}_{U} u'$ for some $m' \supseteq_{\mathsf{Spec}} m$, and then $(u,t) \in R'_{\bar{\alpha}} \in \mathcal{R}'$ implies that there exist $t \xrightarrow{\ell'}_{T} t''$ and $\bar{\alpha}' \in \mathbb{L}$ with $(u',t'') \in R'_{\bar{\alpha}'} \in \mathcal{R}'$ and $F(m',\ell',\bar{\alpha}') \sqsubseteq_{\mathbb{L}} \alpha'$.

The triangle inequality for F gives

$$F(m, \ell', \bar{\alpha}') \sqsubseteq_{\mathbb{L}} F(m, m', \perp_{\mathbb{L}}) \oplus_{\mathbb{L}} F(m', \ell', \bar{\alpha}') \sqsubseteq_{\mathbb{L}} \alpha',$$

hence $d(m,\ell') \neq \top_{\mathbb{L}}$. Together with $d(m,k) \neq \top_{\mathbb{L}}$, conjunctive boundedness allows us to conclude that $k \otimes \ell'$ is defined, but then both $k \otimes \ell$ and $k \otimes \ell'$ are defined, hence by determinism of T, $\ell = \ell'$ and t' = t''.

7.34 Example. For the set $\mathsf{Spec} = \Sigma \times \mathbb{I}$ from our examples, the unique compositional conjunction operator on is given, on labels, by intersection of intervals:

$$(a,[l,r]) \oslash (a',[l',r']) = \begin{cases} (a,[\max(l,l'),\min(r,r')]) \\ & \text{if } a=a',\max(l,l') \leq \min(r,r'), \\ & \text{undefined} & \text{otherwise.} \end{cases}$$

We can easily show that \emptyset is *not* conjunctively bounded: with m = (a, [2, 2]), k = (a, [0, 1]) and $\ell = (a, [3, 4])$, we have $d(m, k) = d(m, \ell) = 1$, but $k \otimes \ell$ is not defined. Noting that this statement does not involve the distance iterator F, we conclude that neither accumulating, point-wise nor limit-average distance admit a bounded conjunction operator. For the accumulating distance, this statement is Theorem 6.24.

To deal with the problem that, as in the above example, conjunction may not be conjunctively bounded, we introduce another, weaker, property which ensures some compatibility of conjunction with distances. We say that \otimes is relaxed conjunctively bounded by a function $C: \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ if C is monotone in both coordinates, has $C(\bot_{\mathbb{L}}, \bot_{\mathbb{L}}) = \bot_{\mathbb{L}}$, $C(\alpha, \top_{\mathbb{L}}) = C(\top_{\mathbb{L}}, \alpha) = \top_{\mathbb{L}}$ for all $\alpha \in \mathbb{L}$, and such that for all $k, \ell \in \mathsf{Spec}$ for which there is $m \in \mathsf{Spec}$ with $d(m, k) \neq \top_{\mathbb{L}}$ and $d(m, \ell) \neq \top_{\mathbb{L}}$, there exist $k', \ell' \in \mathsf{Spec}$ with $k' \otimes \ell'$ defined, $k \leq k'$, $\ell \leq \ell'$, $d(k', k) \neq \top_{\mathbb{L}}$, and $d(\ell', \ell) \neq \top_{\mathbb{L}}$, such that for all $m' \in \mathsf{Spec}$, $\alpha, \alpha' \in \mathbb{L}$,

$$F(m', k' \otimes \ell', C(\alpha, \alpha')) \sqsubseteq_{\mathbb{L}} C(F(m', k, \alpha), F(m', \ell, \alpha')). \tag{7.3}$$

The following theorem shows that relaxed boundedness of \otimes entails a similar property for SMTS conjunction.

7.35 Theorem. Let S, T be SMTS with S or T deterministic and \otimes relaxed conjunctively bounded by C. If there is an SMTS U for which $d_{\mathsf{m}}(U,S) \neq \top_{\mathbb{L}}$ and $d_{\mathsf{m}}(U,T) \neq \top_{\mathbb{L}}$, then there exist β - and γ -widenings S' of S and T' of T such that $S' \wedge T'$ is defined, and $d_{\mathsf{m}}(U',S' \wedge T') \sqsubseteq_{\mathbb{L}} C(d_{\mathsf{m}}(U',S),d_{\mathsf{m}}(U',T))$ for all SMTS U'.

Proof: We start by constructing S' and T', almost as in the proof of the third claim of Theorem 7.33. The states of S' and T' will be the same as for S and T, and we start by letting $\beta = \bot_{\mathbb{L}}$, $\gamma = \bot_{\mathbb{L}}$.

Let U fulfill $d_{\mathsf{m}}(U,S) \neq \top_{\mathbb{L}}$ and $d_{\mathsf{m}}(U,T) \neq \top_{\mathbb{L}}$, let $\mathcal{R} = \{R_{\alpha} \subseteq U \times S \mid \alpha \in \mathbb{L}\}$ and $\mathcal{R}' = \{R'_{\alpha} \subseteq U \times T \mid \alpha \in \mathbb{L}\}$ be relation families witnessing $d_{\mathsf{m}}(U,S)$ and $d_{\mathsf{m}}(U,T)$, respectively, define $R^{\wedge}_{\eta} = \{(u,(s,t)) \mid \exists \alpha, \alpha' \in \mathbb{L} : (u,s) \in R_{\alpha}, (u,t) \in R'_{\alpha'}, C(\alpha,\alpha') \sqsubseteq_{\mathbb{L}} \eta\} \subseteq U \times S \times T$ for all $\eta \in \mathbb{L}$, and let $\mathcal{R}^{\wedge} = \{R^{\wedge}_{\eta} \mid \eta \in \mathbb{L}\}$.

 $\mathcal{R}^{\wedge} = \{R_{\eta}^{\wedge} \mid \eta \in \mathbb{L}\}.$ Now let $\eta \in \mathbb{L} \setminus \{\top_{\mathbb{L}}\}$ and $(u, (s, t)) \in R_{\eta}^{\wedge} \in \mathcal{R}^{\wedge}$, then we have $\alpha, \alpha' \in \mathbb{L} \setminus \{\bot_{\mathbb{L}}\}$ with $(u, s) \in R_{\alpha} \in \mathcal{R}$, $(u, t) \in R'_{\alpha'} \in \mathcal{R}'$, and $C(\alpha, \alpha') \sqsubseteq_{\mathbb{L}} \eta$.

Let $u \xrightarrow{m}_{U} u'$, then also $s \xrightarrow{k}_{S} s'$ and $t \xrightarrow{\ell}_{T} t'$, and there are $\bar{\alpha}, \bar{\alpha}' \in \mathbb{L} \setminus \{\top_{\mathbb{L}}\}$ with $F(m, k, \bar{\alpha}) \sqsubseteq_{\mathbb{L}} \alpha$ and $F(m, \ell, \bar{\alpha}') \sqsubseteq_{\mathbb{L}} \alpha'$. Hence $d(m, k) \neq \top_{\mathbb{L}}$ and $d(m, \ell) \neq \top_{\mathbb{L}}$, and by relaxed conjunctive boundedness we have $k', \ell' \in \mathsf{Spec}$ with $k \leq k', \ell \leq \ell', d(k', k) \neq \top_{\mathbb{L}}, d(\ell', \ell) \neq \top_{\mathbb{L}}$, and $k' \otimes \ell'$ defined. We add the transitions $s \xrightarrow{k'}_{S'} s', t \xrightarrow{\ell'}_{T'} t'$ to S' and T' and update $\beta := \max(\beta, d(k', k)), \gamma := \max(\gamma, d(\ell', \ell))$.

As the sets $\{k \in \mathsf{Spec} \mid s \stackrel{k}{\dashrightarrow}_S s'\}$, $\{\ell \in \mathsf{Spec} \mid t \stackrel{\ell}{\dashrightarrow}_T t'\}$ are compact, the above process converges to some $\beta, \gamma \neq \top_{\mathbb{L}}$. The *must* transitions we just copy from S to S' and from T to T', and then S' is a β -widening of S and S' is a S-widening of S.

We must show that S' and T' satisfy the properties claimed. By construction $S' \wedge T'$ is defined, so let U' be an SMTS with $d_{\mathsf{m}}(U',S) \neq \top_{\mathbb{L}}$ and $d_{\mathsf{m}}(U',T) \neq \top_{\mathbb{L}}$ (otherwise we have nothing to prove). We must show that $d_{\mathsf{m}}(U',S'\wedge T') \sqsubseteq_{\mathbb{L}} C(d_{\mathsf{m}}(U',S),d_{\mathsf{m}}(U',T))$. Let $\mathcal{R} = \{R_{\alpha} \subseteq U' \times S \mid \alpha \in \mathbb{L}\}$ and $\mathcal{R}' = \{R'_{\alpha} \subseteq U' \times T \mid \alpha \in \mathbb{L}\}$ be relation families witnessing $d_{\mathsf{m}}(U',S)$ and $d_{\mathsf{m}}(U',T)$, respectively, define $R_{\eta}^{\wedge'} = \{(u',(s,t)) \mid \exists \alpha,\alpha' \in \mathbb{L} : (u',s) \in R_{\alpha},(u',t) \in R'_{\alpha'},C(\alpha,\alpha') \sqsubseteq_{\mathbb{L}} \eta\} \subseteq U' \times S \times T$ for all $\eta \in \mathbb{L}$, and let $\mathcal{R}^{\wedge'} = \{R_{\eta}^{\wedge'} \mid \eta \in \mathbb{L}\}$.

We have $(u'_0, (s_0, t_0)) \in R_{C(d_{\mathsf{m}}(U', S), d_{\mathsf{m}}(U', T))}^{\wedge'} \in \mathcal{R}^{\wedge'}$. Let $\eta \in \mathbb{L} \setminus \{\bot_{\mathbb{L}}\}$ and $(u', (s, t)) \in R_{\eta}^{\wedge'}$, then we have $\alpha, \alpha' \in \mathbb{L} \setminus \{\bot_{\mathbb{L}}\}$ with $(u', s) \in R_{\alpha} \in \mathcal{R}$, $(u', t) \in R'_{\alpha'} \in \mathcal{R}'$, and $C(\alpha, \alpha') \sqsubseteq_{\mathbb{L}} \eta$. Let $u' \xrightarrow{m}_{U'} u''$, then also $s \xrightarrow{k}_{S} s'$ and $t \xrightarrow{\ell}_{T} t'$, and there are $\bar{\alpha}, \bar{\alpha}' \in \mathbb{L} \setminus \{\top_{\mathbb{L}}\}$ with $(u'', s') \in R_{\bar{\alpha}}, (u'', t') \in R'_{\bar{\alpha}'}, F(m, k, \bar{\alpha}) \sqsubseteq_{\mathbb{L}} \alpha$, and $F(m, \ell, \bar{\alpha}') \sqsubseteq_{\mathbb{L}} \alpha'$.

By construction of S' and T', we have $s \xrightarrow{k'}_{S'} s'$ and $t \xrightarrow{\ell'}_{T'} t'$ with $k \leq k'$, $\ell \leq \ell'$, $d(k',k) \sqsubseteq_{\mathbb{L}} \beta$, and $d(\ell',\ell) \sqsubseteq_{\mathbb{L}} \gamma$, and such that $k' \otimes \ell'$ is defined. Also, $(u'',(s',t')) \in R_{C(\bar{\alpha},\bar{\alpha}')}^{\wedge'}$ and

$$F(m, k' \otimes \ell', C(\bar{\alpha}, \bar{\alpha}')) \sqsubseteq_{\mathbb{L}} C(F(m, k, \bar{\alpha}), F(m, \ell, \bar{\alpha}')) \sqsubseteq_{\mathbb{L}} C(\alpha, \alpha').$$

The other direction of the proof, starting with a transition $(s,t) \xrightarrow{k \oplus \ell} S' \wedge T'$ (s',t'), is an exact copy of the corresponding part of the proof of Theorem 7.33.

- **7.36 Example.** For the set $\mathsf{Spec} = \Sigma \times \mathbb{I}$ from our examples, the following lemma shows a one-step version of relaxed conjunctive boundedness.
- **7.37 Lemma.** For all $k, \ell \in \mathsf{Spec}$ for which there is $m \in \mathsf{Spec}$ with $d(m, k) \neq \infty$ and $d(m, \ell) \neq \infty$, there exist $k', \ell' \in \mathsf{Spec}$ with $k \preccurlyeq k', \ell \preccurlyeq \ell', d(k', k) \neq \infty$, $d(\ell', \ell) \neq \infty$, and $k' \otimes \ell'$ defined, and then $d(m', k' \otimes \ell') \leq \max(d(m', k), d(m', \ell))$ for all $m' \in \mathsf{Spec}$.

Proof: Let $k, \ell \in \mathsf{Spec}$ such that there is $m \in \mathsf{Spec}$ with $d(m, k) \neq \infty$ and $d(m, \ell) \neq \infty$. This implies that k = (a, [l, r]) and $\ell = (a, [l', r'])$ for some

 $a \in \Sigma$, $k, l, k', l' \in \mathbb{Z} \cup \{-\infty, \infty\}$. Without loss of generality we can assume that $l \leq l'$.

If $r \geq l'$, then $k \otimes \ell = (a, [l', r])$ is defined, and we take k' = k, $\ell' = \ell$. Now let $m' = (a', [l'', r'']) \in \mathsf{Spec}$. If $a' \neq a$, the property to prove is trivially true. If a' = a, then we have

$$d(m', k' \otimes \ell') = \max(0, l' - l'', r'' - r),$$

$$d(m', k) = \max(0, l - l'', r'' - r),$$

$$d(m', \ell) = \max(0, l' - l'', r'' - r').$$

Thus we need to show that

$$\max(0, l' - l'', r'' - r) \le \max(0, l - l'', r'' - r, l' - l'', r'' - r'),$$

which is clear as all left-hand terms also appear on the right-hand side.

In case r < l', we let k' = (a, [l, l']) and $\ell' = (a, [r, r'])$. Then $k \leq k'$, $\ell \leq \ell'$, and $k' \otimes \ell' = (a, [r, l'])$ is defined. Also, $d(k', k) = d(\ell', \ell) = l' - r \neq \infty$.

Let m' = (a', [l'', r'']) as before, then the case $a' \neq a$ is again trivial. We have

$$d(m', k' \otimes \ell') = \max(0, r - l'', r'' - l'),$$

so we need to show that

$$\max(0, r - l'', r'' - l') \le \max(0, l - l'', r'' - r, l' - l'', r'' - r')$$
$$= \max(0, r'' - r, l' - l''),$$

where the equality follows from $l \leq l'$, hence $l - l'' \leq l' - l''$, and $r \leq r'$, hence $r'' - r' \leq r'' - r$. But $0 \leq l' - r$, r - l'' < l' = l'', and r'' - l' < r'' - r because of r < l', so the inequality follows.

For the accumulating distance, it then follows that \otimes is relaxed conjunctively bounded by $C(\alpha, \alpha') = \alpha + \alpha'$: Using the notation from Lemma 7.37, we need to show (7.3), *i.e.*, that $d(m', l' \otimes l') + \lambda(\alpha + \alpha') \leq d(m', k) + \lambda\alpha + d(m', l) + \lambda\alpha'$, which however is clear by $d(m', k' \otimes l') \leq \max(d(m', k), d(m', l)) \leq d(m', k) + d(m', l)$.

- **7.38 Example.** For the pointwise distance, \otimes is relaxed conjunctively bounded by $C(\alpha, \alpha') = \max(\alpha, \alpha')$: (7.3) is then equivalent to $\max(d(m', k' \otimes \ell'), \alpha, \alpha') \leq \max(d(m', k), d(m', \ell), \alpha, \alpha')$, which follows from Lemma 7.37.
- **7.39 Example.** For the limit-average distance, \otimes is relaxed conjunctively bounded by $C(\alpha, \alpha') = \alpha \oplus_{\mathbb{L}} \alpha'$: Again using the notation from Lemma 7.37, we need to show (7.3), so we need to see that for all $j \in \mathbb{N}_+$,

$$\frac{1}{j+1}d(m',k'\otimes \ell') + \frac{j}{j+1}\alpha(j-1) + \frac{j}{j+1}\alpha'(j-1)
\leq \frac{1}{j+1}d(m',k) + \frac{j}{j+1}\alpha(j-1) + \frac{1}{j+1}d(m',\ell) + \frac{j}{j+1}\alpha'(j-1).$$

This follows again from $d(m', k' \otimes \ell') \leq \max(d(m', k), d(m', \ell)) \leq d(m', k) + d(m', \ell)$.

7.40 Example. Also for the maximum-lead distance, \otimes is relaxed conjunctively bounded by $C(\alpha, \alpha') = \alpha \oplus_{\mathbb{L}} \alpha'$. To see this, we again use the notation from Lemma 7.37. We need to show that for all $\alpha, \alpha' \in \mathbb{L}$ and all $d \in \mathbb{R}$,

$$\sup_{l'' \le z \le r''} \inf_{r \le w \le l'} \max(|d + z - w|, \alpha(d + z - w) + \alpha'(d + z - w)) \\
\le \sup_{l'' \le z \le r''} \inf_{l \le x \le r} \max(|d + z - x|, \alpha(d + z - x)) \\
+ \sup_{l'' \le z \le r''} \inf_{l' \le y \le r'} \max(|d + z - y|, \alpha'(d + z - y))$$
(7.4)

Now for all $z \in [l'', r'']$, we have

$$\inf_{r \leq w \leq l'} |d+z-w| \leq \inf_{l \leq x \leq r} |d+z-x| + \inf_{l' \leq y \leq r'} |d+z-y|$$

and

$$\inf_{r \le w \le l'} (\alpha(d+z-w) + \alpha'(d+z-w))$$

$$\leq \inf_{l \le x \le r} \alpha(d+z-x) + \inf_{l' \le y \le r'} \alpha'(d+z-y);$$

both can be shown by simply considering all cases of the placement of the infima. But then also

$$\begin{split} \max(\inf_{r \leq w \leq l'} |d+z-w|, \inf_{r \leq w \leq l'} (\alpha(d+z-w) + \alpha'(d+z-w))) \\ &\leq \max \begin{cases} \inf_{l \leq x \leq r} |d+z-x| + \inf_{l' \leq y \leq r'} |d+z-y| \\ \inf_{l \leq x \leq r} \alpha(d+z-x) + \inf_{l' \leq y \leq r'} \alpha'(d+z-y)) \\ &\leq \max(\inf_{l \leq x \leq r} |d+z-x|, \inf_{l \leq x \leq r} \alpha(d+z-x)) \\ &+ \max(\inf_{l' \leq y \leq r'} |d+z-y|, \inf_{l' \leq y \leq r'} \alpha'(d+z-y)) \,, \end{split}$$

the last inequality by distributivity of + over max. As this holds for all z, we have proven (7.4).

7.5 Logical Characterizations

We show that quantitative refinement admits a logical characterization. Our results extend the logical characterization of modal transition systems in [Lar89]. Our logic \mathcal{L} is the smallest set of expressions generated by the following abstract syntax:

$$\phi, \phi_1, \phi_2 := \mathbf{tt} \mid \mathbf{ff} \mid \langle \ell \rangle \phi \mid [\ell] \phi \mid \phi_1 \land \phi_2 \mid \phi_1 \lor \phi_2 \qquad (\ell \in \mathsf{Spec})$$

The semantics of a formula $\phi \in \mathcal{L}$ is a mapping $\llbracket \phi \rrbracket : S \to \mathbb{L}$ given inductively as follows:

$$\begin{split} \llbracket \mathbf{t} \mathbf{t} \rrbracket s &= \bot & \llbracket \mathbf{f} \mathbf{f} \rrbracket s &= \top \\ \llbracket (\phi_1 \wedge \phi_2) \rrbracket s &= \max(\llbracket \phi_1 \rrbracket s, \llbracket \phi \rrbracket_2 s) & \llbracket (\phi_1 \vee \phi_2) \rrbracket s &= \min(\llbracket \phi_1 \rrbracket s, \llbracket \phi_2 \rrbracket s) \\ \llbracket \langle \ell \rangle \phi \rrbracket s &= \inf \{ F(k, \ell, \llbracket \phi \rrbracket t) \mid s \xrightarrow{k} t, d(k, \ell) \neq \top_{\mathbb{L}} \} \\ \llbracket \llbracket \ell \rrbracket \phi \rrbracket s &= \sup \{ F(k, \ell, \llbracket \phi \rrbracket t) \mid s \xrightarrow{k} t, d(k, \ell) \neq \top_{\mathbb{L}} \} \end{split}$$

For a SMTS S we write $\llbracket \phi \rrbracket S = \llbracket \phi \rrbracket s_0$.

The below theorems express the fact that \mathcal{L} is quantitatively sound for refinement distance, i.e., the value of a formula in a specification is bounded by its value in any other specification together with their distance, and that the disjunction-free fragment of \mathcal{L} is quantitatively implementation complete, i.e., the value of any disjunction-free formula in a specification S is bounded above by its value in any implementation of S. Note that disjunction-freeness is a very common assumption in this context, cf. [Lar89, BCK11].

7.41 Theorem. For all $\phi \in \mathcal{L}$ and all SMTS $S, T, [\![\phi]\!]S \sqsubseteq_{\mathbb{L}} [\![\phi]\!]T \oplus_{\mathbb{L}} d_{\mathsf{m}}(S,T)$.

Proof: Structural induction. The claim obviously holds for $\phi = \mathbf{tt}$ and $\phi = \mathbf{ff}$; if $\phi = \phi_1 \wedge \phi_2$, then $\llbracket \phi_i \rrbracket s_1 \sqsubseteq_{\mathbb{L}} \llbracket \phi_i \rrbracket s_2 \oplus_{\mathbb{L}} d_{\mathsf{m}}(s_1, s_2)$ for i = 1, 2 imply that also $\max(\llbracket \phi_1 \rrbracket s_1, \llbracket \phi_2 \rrbracket s_1) \sqsubseteq_{\mathbb{L}} \max(\llbracket \phi_1 \rrbracket s_2, \llbracket \phi_2 \rrbracket s_2) \oplus_{\mathbb{L}} d_{\mathsf{m}}(s_1, s_2)$, and similarly for $\phi = \phi_1 \vee \phi_2$.

For the case $\phi = \langle \ell \rangle \phi'$, there is nothing to prove if there are no transitions $s_2 \longrightarrow_2$ or if $d_{\mathsf{m}}(s_1,s_2) = \top_{\mathbb{L}}$. Let thus $s_2 \xrightarrow{k_2} t_2$, then there exist $s_1 \xrightarrow{k_1} t_1$ with $F(k_1,k_2,d_{\mathsf{m}}(t_1,t_2)) \sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}(s_1,s_2)$. Now by induction hypothesis, $\llbracket \phi' \rrbracket t_1 \sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}(t_1,t_2) \oplus_{\mathbb{L}} \llbracket \phi' \rrbracket t_2$, and then, using the triangle inequality,

$$F(k_1, \ell, \llbracket \phi' \rrbracket t_1) \sqsubseteq_{\mathbb{L}} F(k_1, k_2, d_{\mathsf{m}}(t_1, t_2)) \oplus_{\mathbb{L}} F(k_2, \ell, \llbracket \phi' \rrbracket t_2)$$
$$\sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}(s_1, s_2) \oplus_{\mathbb{L}} F(k_2, \ell, \llbracket \phi' \rrbracket t_2).$$

As $s_2 \xrightarrow{k_2} t_2$ was arbitrary, this entails

$$\inf\{F(k_1,\ell,\llbracket\phi'\rrbracket t_1)\mid s_1\xrightarrow{k_1}t_1\}$$

$$\sqsubseteq_{\mathbb{L}}\inf\{F(k_2,\ell,\llbracket\phi'\rrbracket t_2)\mid s_1\xrightarrow{k_2}t_2\}\oplus_{\mathbb{L}}d_{\mathsf{m}}(s_1,s_2).$$

For the case $\phi = [\ell] \phi'$ the proof is similar: We have nothing to prove if $d_{\mathsf{m}}(s_1, s_2) = \top_{\mathbb{L}}$ or if there are no transitions $s_1 \stackrel{k_1}{---} t_1$ with $F(k_1, \ell, \llbracket \phi' \rrbracket t_1) \neq \top_{\mathbb{L}}$, so assume there is such a transition. Then we also have $s_2 \stackrel{k_2}{----} t_2$ with $F(k_1, k_2, d_{\mathsf{m}}(t_1, t_2)) \sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}(s_1, s_2)$, and

$$F(k_1, \ell, \llbracket \phi' \rrbracket t_1) \sqsubseteq_{\mathbb{L}} F(k_1, k_2, d_{\mathsf{m}}(t_1, t_2)) \oplus_{\mathbb{L}} F(k_2, \ell, \llbracket \phi' \rrbracket t_2)$$
$$\sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}(s_1, s_2) \oplus_{\mathbb{L}} F(k_2, \ell, \llbracket \phi' \rrbracket t_2). \quad \Box$$

7.42 Theorem. For all disjunction-free formulae $\phi \in \mathcal{L}$ and all SMTS S, $\llbracket \phi \rrbracket S = \sup_{I \in \llbracket S \rrbracket} \llbracket \phi \rrbracket I$.

Proof: Theorem 7.41 entails $\llbracket \phi \rrbracket I \sqsubseteq_{\mathbb{L}} \llbracket \phi \rrbracket S \oplus_{\mathbb{L}} d_{\mathsf{m}}(I,S) = \llbracket \phi \rrbracket S$ for all $I \in \llbracket S \rrbracket$, hence also $\sup_{I \in \llbracket S \rrbracket} \llbracket \phi \rrbracket I \sqsubseteq_{\mathbb{L}} \llbracket \phi \rrbracket S$. To show that $\llbracket \phi \rrbracket S \sqsubseteq_{\mathbb{L}} \sup_{I \in \llbracket S \rrbracket} \llbracket \phi \rrbracket I$ we use structural induction on ϕ . If $\phi = \mathbf{tt}$, both sides are $\bot_{\mathbb{L}}$, and if $\phi = \mathbf{ff}$, both sides are $\top_{\mathbb{L}}$, so the induction base is clear.

The case $\phi = \phi_1 \wedge \phi_2$ is also clear: By hypothesis, $\llbracket \phi_1 \rrbracket S \sqsubseteq_{\mathbb{L}} \sup_{I \in \llbracket S \rrbracket} \llbracket \phi_1 \rrbracket I$ and similarly for ϕ_2 , hence

$$\begin{split} \llbracket \phi \rrbracket S &= \max(\llbracket \phi_1 \rrbracket S, \llbracket \phi_2 \rrbracket S) \sqsubseteq_{\mathbb{L}} \max(\sup_{I \in \llbracket S \rrbracket} \llbracket \phi_1 \rrbracket I, \sup_{I \in \llbracket S \rrbracket} \llbracket \phi_2 \rrbracket I) \\ &= \sup_{I \in \llbracket S \rrbracket} \max(\llbracket \phi_1 \rrbracket I, \llbracket \phi_2 \rrbracket I) \,. \end{split}$$

For the case $\phi = \langle \ell \rangle \phi'$, we are done if $\llbracket \phi \rrbracket S = \bot_{\mathbb{L}}$. Otherwise, let $\alpha \sqsubseteq_{\mathbb{L}} \llbracket \phi \rrbracket S$; we want to expose $I \in \llbracket S \rrbracket$ for which $\alpha \sqsubseteq_{\mathbb{L}} \llbracket \phi \rrbracket I$. Start by letting $I = \{i_0\}$ and $\longrightarrow_I = \emptyset$.

Now for each transition $s_0 \xrightarrow{k}_S t$, we have $\alpha \sqsubseteq_{\mathbb{L}} F(k, \ell, \llbracket \phi' \rrbracket t)$, so (assuming for the moment that $\llbracket \phi' \rrbracket t \neq \bot_{\mathbb{L}}$) there is $\alpha'_k \sqsubseteq_{\mathbb{L}} \llbracket \phi' \rrbracket t$ for which $F(k, \ell, \alpha'_k) \sqsupseteq_{\mathbb{L}} \alpha$. By induction hypothesis, there is $J \in \llbracket t, S \rrbracket$ for which $\alpha'_k \sqsubseteq_{\mathbb{L}} \llbracket \phi' \rrbracket J$; let $n \in \llbracket k \rrbracket$ such that $F(n, \ell, \llbracket \phi' \rrbracket J) = F(k, \ell, \llbracket \phi' \rrbracket J)$, and add J together with a transition $i_0 \xrightarrow{n}_I j_0$ to I. In case $\llbracket \phi' \rrbracket t = \bot_{\mathbb{L}}$, we just take an arbitrary $J \in \llbracket t, S \rrbracket$.

For the so-constructed implementation I we have

$$\llbracket \phi \rrbracket I = \inf \{ F(m, \ell, \llbracket \phi' \rrbracket j \mid i_0 \xrightarrow{m}_I j \}
= \inf \{ F(k, \ell, \llbracket \phi' \rrbracket J) \mid s_0 \xrightarrow{k}_S t, J \in \llbracket t, S \rrbracket, \llbracket \phi' \rrbracket t = \bot_{\mathbb{L}} \text{ or } \alpha'_k \sqsubset_{\mathbb{L}} \llbracket \phi' \rrbracket J \}
\sqsupset_{\mathbb{L}} \inf \{ \{ F(k, \ell, \alpha'_k) \mid s_0 \xrightarrow{k}_S t \} \cup \{ F(k, \ell, \llbracket \phi' \rrbracket t) \}) \rightrightarrows_{\mathbb{L}} \alpha,$$
(7.5)

the strict inequality in (7.5) because S is compactly branching.

For the case $\phi = [\ell]\phi'$, let again $\alpha \sqsubseteq_{\mathbb{L}} \llbracket \phi \rrbracket S$, and let $I \in \llbracket S \rrbracket$ be any implementation (there exists one because of local consistency of S). If $F(k,\ell,\llbracket \phi' \rrbracket t) = \bigcup_{\mathbb{L}} f$ for all $s_0 \xrightarrow{k} t$, then $\llbracket \phi \rrbracket S = \sup \emptyset = \bot_{\mathbb{L}}$ and we are done. Otherwise let $s_0 \xrightarrow{k} t$ be such that $\llbracket \phi \rrbracket S = F(k,\ell,\llbracket \phi' \rrbracket t)$, which exists because S is compactly branching. Then $\alpha \sqsubseteq_{\mathbb{L}} F(k,\ell,\llbracket \phi' \rrbracket t)$, so (assuming that $\llbracket \phi' \rrbracket t \neq \bot_{\mathbb{L}}$) we have $\alpha'_k \sqsubseteq_{\mathbb{L}} \llbracket \phi' \rrbracket t$ with $F(k,\ell,\alpha'_k) \rightrightarrows_{\mathbb{L}} \alpha$.

Let $J \in [t, S]$ such that $\alpha'_k \sqsubseteq_{\mathbb{L}} [\phi']J$, let $n \in [k]$ such that $F(n, \ell, [\phi']J) = F(k, \ell, [\phi']J)$, and add J together with a transition $i_0 \xrightarrow{n}_I j_0$ to I. Then

$$\llbracket \phi \rrbracket I = \sup \{ F(m, \ell, \llbracket \phi' \rrbracket n) \mid i_0 \xrightarrow{m}_I j \}$$

$$\sqsubseteq_{\mathbb{L}} F(n, \ell, \llbracket \phi' \rrbracket J) = F(k, \ell, \llbracket \phi' \rrbracket J) \sqsupseteq_{\mathbb{L}} F(k, \ell, \alpha'_k) \sqsupset_{\mathbb{L}} \alpha.$$

In case $\llbracket \phi' \rrbracket t = \bot_{\mathbb{L}}$ instead, we again take an arbitrary $J \in \llbracket t, S \rrbracket$, and then $\llbracket \phi \rrbracket I \supseteq_{\mathbb{L}} F(k, \ell, \llbracket \phi' \rrbracket t) \supseteq_{\mathbb{L}} \alpha$.

8 Logical vs. Behavioral Specifications 1

In this chapter we depart from the quantitative setting of this thesis and introduce disjunctive modal transition systems. We show that this generalization of MTS is closely related to other specification formalisms, viz. acceptance automata and the modal ν -calculus and that it admits both disjunction and conjunction as well as a general notion of quotient which was unavailable for MTS.

8.1 Specification Formalisms

In this section we introduce the four specification formalisms with which this chapter is concerned. For the rest of the chapter, we fix a finite alphabet Σ . In each of the formalisms, the semantics of a specification is a set of implementations, in our case always a set of (finite) labeled transition systems (LTS) over Σ , i.e., structures $\mathcal{I} = (S, s^0, \longrightarrow)$ consisting of a finite set S of states, an initial state $s^0 \in S$, and a transition relation $\longrightarrow \subseteq S \times \Sigma \times S$.

8.1.1 Disjunctive Modal Transition Systems

8.1 Definition. A disjunctive modal transition system (DMTS) is a structure $\mathcal{D} = (S, S^0, \dashrightarrow, \longrightarrow)$ consisting of finite sets $S \supseteq S^0$ of states and initial states, a may-transition relation $\dashrightarrow \subseteq S \times \Sigma \times S$, and a disjunctive must-transition relation $\longrightarrow \subseteq S \times 2^{\Sigma \times S}$. It is assumed that for all $(s, N) \in \longrightarrow$ and all $(a, t) \in N$, $(s, a, t) \in \dashrightarrow$; furthermore, if $(s, \emptyset) \in \longrightarrow$ then there are no a and t such that $(s, a, t) \in \dashrightarrow$.

As customary, we write $s \stackrel{a}{\dashrightarrow} t$ instead of $(s, a, t) \in \dashrightarrow$, $s \longrightarrow N$ instead of $(s, N) \in \longrightarrow$, $s \stackrel{a}{\dashrightarrow}$ if there exists t for which $s \stackrel{a}{\dashrightarrow} t$, and $s \stackrel{a}{\leadsto}$ if there does not.

The intuition is that may-transitions $s \xrightarrow{a} t$ specify which transitions are permitted in an implementation, whereas a must-transition $s \longrightarrow N$ stipulates a disjunctive requirement: at least one of the choices $(a,t) \in N$ has to be

 $^{^1{}m This}$ chapter is based on the journal paper [BFK $^+20$] published in Information and Computation.

implemented. A DMTS $(S, S^0, \dashrightarrow, \longrightarrow)$ is an *implementation* if $S^0 = \{s^0\}$ is a singleton and $\longrightarrow = \{(s, \{(a,t)\}) \mid s \xrightarrow{a} t\}.$

DMTS were introduced in [LX90b] in the context of equation solving. They are a natural extension of the modal transition systems (MTS) of previous chapters. We say that a DMTS $(S, S^0, -----, ----)$ is a MTS if $S^0 = \{s^0\}$ is a singleton and for all $s \to N$ it holds that $N = \{(a, t)\}$ is also a singleton. When speaking about MTS, we usually write $s \xrightarrow{a} t$ instead of $s \to \{(a, t)\}$.

An LTS $(S, s^0, \longrightarrow)$ can be translated to a DMTS implementation $(S, S^0, \dashrightarrow, \longrightarrow')$ by setting $S^0 = \{s^0\}, \dashrightarrow = \longrightarrow \text{ and } \longrightarrow' = \{(s, \{(a, t)\}) \mid s \stackrel{a}{\longrightarrow} t\}$. This defines an embedding of LTS into DMTS whose image is precisely the set of DMTS implementations.

- **8.2 Definition.** Let $\mathcal{D}_1 = (S_1, S_1^0, -\rightarrow_1, \longrightarrow_1)$ and $\mathcal{D}_2 = (S_2, S_2^0, -\rightarrow_2, \longrightarrow_2)$ be DMTS. A relation $R \subseteq S_1 \times S_2$ is a *modal refinement* if for all $(s_1, s_2) \in R$ the following conditions hold:
 - for all $s_1 \xrightarrow{a}_{1} t_1$ there is $t_2 \in S_2$ with $s_2 \xrightarrow{a}_{2} t_2$ and $(t_1, t_2) \in R$, and
 - for all $s_2 \longrightarrow_2 N_2$ there is $s_1 \longrightarrow_1 N_1$ such that for each $(a, t_1) \in N_1$ there is $(a, t_2) \in N_2$ with $(t_1, t_2) \in R$.

We say that \mathcal{D}_1 modally refines \mathcal{D}_2 , denoted $\mathcal{D}_1 \leq_{\mathsf{m}} \mathcal{D}_2$, whenever there exists a modal refinement R such that for all $s_1^0 \in S_1^0$, there exists $s_2^0 \in S_2^0$ for which $(s_1^0, s_2^0) \in R$.

We write $\mathcal{D}_1 \equiv_{\mathsf{m}} \mathcal{D}_2$ if $\mathcal{D}_1 \leq_{\mathsf{m}} \mathcal{D}_2$ and $\mathcal{D}_2 \leq_{\mathsf{m}} \mathcal{D}_1$. For states $s_1 \in S_1$, $s_2 \in S_2$, we write $s_1 \leq_{\mathsf{m}} s_2$ if $(S_1, \{s_1\}, -\rightarrow_1, \longrightarrow_1) \leq_{\mathsf{m}} (S_2, \{s_2\}, -\rightarrow_2, \longrightarrow_2)$. Sometimes we will refer to the last property of a modal refinement relation, $\forall s_1^0 \in S_1^0 : \exists s_2^0 \in S_2^0 : (s_1^0, s_2^0) \in R$, as being *initialised*. Note that modal refinement is reflexive and transitive, *i.e.*, a preorder on DMTS.

The set of implementations of a DMTS \mathcal{D} is $\llbracket \mathcal{D} \rrbracket = \{ \mathcal{I} \leq_{\mathsf{m}} \mathcal{D} \mid \mathcal{I} \text{ implementation} \}$. This is, thus, the set of all LTS which satisfy the specification given by the DMTS \mathcal{D} . We say that \mathcal{D}_1 thoroughly refines \mathcal{D}_2 , and write $\mathcal{D}_1 \leq_{\mathsf{t}} \mathcal{D}_2$, if $\llbracket \mathcal{D}_1 \rrbracket \subseteq \llbracket \mathcal{D}_2 \rrbracket$. We write $\mathcal{D}_1 \equiv_{\mathsf{t}} \mathcal{D}_2$ if $\mathcal{D}_1 \leq_{\mathsf{t}} \mathcal{D}_2$ and $\mathcal{D}_2 \leq_{\mathsf{t}} \mathcal{D}_1$. For states $s_1 \in S_1, s_2 \in S_2$, we write $\llbracket s_1 \rrbracket = \llbracket (S_1, \{s_1\}, \dashrightarrow_1, \longrightarrow_1) \rrbracket$ and $s_1 \leq_{\mathsf{t}} s_2$ if $\llbracket s_1 \rrbracket \subseteq \llbracket s_2 \rrbracket$.

The proposition below, which follows directly from transitivity of modal refinement, shows that modal refinement is *sound* with respect to thorough refinement; in the context of specification theories, this is what one would expect, and we only include it for completeness of presentation. It can be shown that modal refinement is also *complete* for *deterministic* DMTS [BCK11], but we will not need this here.

8.3 Proposition. For all DMTS \mathcal{D}_1 , \mathcal{D}_2 , $\mathcal{D}_1 \leq_{\mathsf{m}} \mathcal{D}_2$ implies $\mathcal{D}_1 \leq_{\mathsf{t}} \mathcal{D}_2$.

8.1.2 The Modal ν -Calculus

We recall the syntax and semantics of the modal ν -calculus, the fragment of the modal μ -calculus [SdB69,Koz83] with only greatest fixed points. Instead of an explicit greatest fixed point operator, we use the representation by equation systems in Hennessy-Milner logic developed in [Lar90b].

For a finite set X of variables, let $\mathcal{H}(X)$ be the set of Hennessy-Milner formulae, generated by the abstract syntax

$$\mathcal{H}(X) \ni \phi ::= \mathbf{tt} \mid \mathbf{ff} \mid x \mid \langle a \rangle \phi \mid [a] \phi \mid \phi \land \phi \mid \phi \lor \phi$$

for $a \in \Sigma$ and $x \in X$.

A declaration is a mapping $\Delta: X \to \mathcal{H}(X)$; we recall the greatest fixed point semantics of declarations from [Lar90b]. For an LTS $(S, s^0, \longrightarrow)$, an assignment is a mapping $\sigma: X \to 2^S$. The set of assignments forms a complete lattice with order $\sigma_1 \sqsubseteq \sigma_2$ iff $\sigma_1(x) \subseteq \sigma_2(x)$ for all $x \in X$ and least upper bound $(\bigsqcup_{i \in I} \sigma_i)(x) = \bigcup_{i \in I} \sigma_i(x)$.

The semantics of a formula is a subset of S, given relative to an assignment σ , defined as follows: $\llbracket \mathbf{tt} \rrbracket \sigma = S$, $\llbracket \mathbf{ff} \rrbracket \sigma = \emptyset$, $\llbracket x \rrbracket \sigma = \sigma(x)$, $\llbracket \phi \wedge \psi \rrbracket \sigma = \llbracket \phi \rrbracket \sigma \cap \llbracket \psi \rrbracket \sigma$, $\llbracket \phi \vee \psi \rrbracket \sigma = \llbracket \phi \rrbracket \sigma \cup \llbracket \psi \rrbracket \sigma$, and

The semantics of a declaration Δ is then the assignment defined by

$$\llbracket \Delta \rrbracket = \bigsqcup \{\sigma: X \to 2^S \mid \forall x \in X: \sigma(x) \subseteq \llbracket \Delta(x) \rrbracket \sigma \};$$

the greatest (post) fixed point of Δ .

A ν -calculus expression is a structure $\mathcal{N}=(X,X^0,\Delta)$, with $X^0\subseteq X$ sets of variables and $\Delta:X\to\mathcal{H}(X)$ a declaration. We say that an LTS $\mathcal{I}=(S,s^0,\longrightarrow)$ implements (or models) the expression, and write $\mathcal{I}\models\mathcal{N}$, if there is $x^0\in X^0$ such that $s^0\in \llbracket\Delta\rrbracket(x^0)$. We write $\llbracket\mathcal{N}\rrbracket$ for the set of implementations (models) of a ν -calculus expression \mathcal{N} . As for DMTS, we write $\llbracket x\rrbracket=\llbracket(X,\{x\},\Delta)\rrbracket$ for $x\in X$, and thorough refinement of expressions and variables is defined accordingly.

We are now going to introduce a normal form for ν -calculus expressions. The purpose of this normal form is twofold. One is to allow us to define modal refinement for ν -calculus, an analogue to the DMTS modal refinement that can be seen as a sound approximation of the logical implication, cf. Proposition 8.3. The second purpose is to facilitate a simple translation between DMTS and ν -calculus expressions, see Section 8.2.1 below.

²Any ν -calculus expression is thoroughly equivalent to one with precisely one initial state, *i.e.*, X^0 a singleton, however this is not true for ν -calculus expressions in normal form as defined below. See also Section 8.2.5.

8.4 Lemma. For any ν -calculus expression $\mathcal{N}_1 = (X_1, X_1^0, \Delta_1)$, there exists another expression $\mathcal{N}_2 = (X_2, X_2^0, \Delta_2)$ with $[\![\mathcal{N}_1]\!] = [\![\mathcal{N}_2]\!]$ and such that for any $x \in X$, $\Delta_2(x)$ is of the form

$$\Delta_2(x) = \bigwedge_{i \in I} \left(\bigvee_{j \in J_i} \langle a_{ij} \rangle x_{ij} \right) \wedge \bigwedge_{a \in \Sigma} [a] \left(\bigvee_{j \in J_a} y_{a,j} \right)$$
(8.1)

for finite (possibly empty) index sets I, J_a , for $i \in I$ and $a \in \Sigma$, and all $x_{ij}, y_{a,j} \in X_2$. Additionally, for all $i \in I$ and $j \in J_i$, there exists $j' \in J_{a_{ij}}$ for which $x_{ij} \leq_t y_{a_{ij},j'}$. Also if at least one of the $J_i = \emptyset$ then $J_a = \emptyset$ for all a.

Remark that this normal form includes a semantic check $(x_{ij} \leq_{\mathsf{t}} y_{a_{ij},j'})$, so it is not entirely syntactic.

Proof: It is shown in [BL92] that any Hennessy-Milner formula is equivalent to one in so-called *strong normal form*, *i.e.*, of the form $\bigvee_{i \in I} (\bigwedge_{j \in J_i} \langle a_{ij} \rangle \phi_{ij} \wedge \bigwedge_{a \in \Sigma} [a] \psi_{i,a})$ for HML formulas ϕ_{ij} , $\psi_{i,a}$ which are also in strong normal form, and such that for all i, j, $\phi_{ij} \leq_{\mathsf{t}} \psi_{i,a_{ij}}$.

We can replace the ϕ_{ij} , $\psi_{i,a}$ by (new) variables x_{ij} , $y_{i,a}$ and add declarations $\Delta_2(x_{ij}) = \phi_{ij}$, $\Delta_2(y_{i,a}) = \psi_{i,a}$ to arrive at an expression in which all formulae are of the form $\Delta_2(x) = \bigvee_{i \in I} (\bigwedge_{j \in J_i} \langle a_{ij} \rangle x_{ij} \wedge \bigwedge_{a \in \Sigma} [a] y_{i,a})$ and such that for all $i, j, x_{ij} \leq_t y_{i,a_{ij}}$.

Now for each such formula, replace (recursively) x by new variables $\{\tilde{x}^i \mid i \in I\}$, similarly for y, and set $\Delta_2(\tilde{x}^i) = \bigwedge_{j \in J_i} \langle a_{ij} \rangle (\bigvee_k \tilde{x}^k_{ij}) \wedge \bigwedge_{a \in \Sigma} [a](\bigvee_k \tilde{y}^k_{i,a})$. Using initial variables $X_2^0 = \{\tilde{x}^i \mid x \in X_1^0\}$, the so-constructed ν -calculus expression is equivalent to the original one. We know that for all $i, j, \bigvee_k \tilde{x}^k_{ij} \leq_{\mathsf{t}} \bigvee_{k} \tilde{y}^k_{i,a_{ij}}$, hence for all i, j, k there exists k' such that $\tilde{x}^k_{ij} \leq_{\mathsf{t}} \tilde{y}^{k'}_{i,a_{ij}}$. We can thus rename variables and apply the distributivity of $\langle \cdot \rangle$ over \bigvee .

Finally, if at least one of the $J_i = \emptyset$ then Δ_2 is false and we can simply set $J_a = \emptyset$ for all a without changing the semantics of Δ_2 .

As this is a type of *conjunctive normal form*, it is clear that translating a ν -calculus expression into normal form may incur an exponential blow-up.

We introduce some notation for ν -calculus expressions in normal form which will make our life easier later. Let $\mathcal{N}=(X,X^0,\Delta)$ be such an expression and $x\in X$, with $\Delta(x)=\bigwedge_{i\in I}\left(\bigvee_{j\in J_i}\langle a_{ij}\rangle x_{ij}\right)\wedge\bigwedge_{a\in\Sigma}[a]\left(\bigvee_{j\in J_a}y_{a,j}\right)$ as in the lemma. Define $\Diamond(x)=\{\{(a_{ij},x_{ij})\mid j\in J_i\}\mid i\in I\}$ and, for each $a\in\Sigma$, $\Box^a(x)=\{y_{a,j}\mid j\in J_a\}$. Note that now,

$$\Delta(x) = \bigwedge_{N \in \Diamond(x)} \left(\bigvee_{(a,y) \in N} \langle a \rangle y \right) \wedge \bigwedge_{a \in \Sigma} [a] \left(\bigvee_{y \in \Box^a(x)} y \right).$$

8.5 Definition. Let $\mathcal{N}_1 = (X_1, X_1^0, \Delta_1)$, $\mathcal{N}_2 = (X_2, X_2^0, \Delta_2)$ be ν -calculus expressions in normal form and $R \subseteq X_1 \times X_2$. The relation R is a modal refinement if it holds for all $(x_1, x_2) \in R$ that

- for all $a \in \Sigma$ and every $y_1 \in \square_1^a(x_1)$, there is $y_2 \in \square_2^a(x_2)$ for which $(y_1, y_2) \in R$, and
- for all $N_2 \in \Diamond_2(x_2)$ there is $N_1 \in \Diamond_1(x_1)$ such that for each $(a, y_1) \in N_1$, there exists $(a, y_2) \in N_2$ with $(y_1, y_2) \in R$.

We say that \mathcal{N}_1 modally refines \mathcal{N}_2 , denoted $\mathcal{N}_1 \leq_{\mathsf{m}} \mathcal{N}_2$, whenever there exists a modal refinement R such that for every $x_1^0 \in X_1^0$ there exists $x_2^0 \in X_2^0$ for which $(x_1^0, x_2^0) \in R$.

We say that a ν -calculus expression (X, X^0, Δ) in normal form is an *implementation* if $X^0 = \{x^0\}$ is a singleton, $\Diamond(x) = \{\{(a,y)\} \mid y \in \Box^a(x), a \in \Sigma\}$ and $\Box^a(x) = \emptyset$ for all $a \notin \Sigma$, for all $x \in X$.

We can translate an LTS $(S, s^0, \longrightarrow)$ to a ν -calculus expression (S, S^0, Δ) in normal form by setting $S^0 = \{s^0\}$ and $\Diamond(s) = \{\{(a,t)\} \mid s \stackrel{a}{\longrightarrow} t\}$ and $\Box^a(s) = \{t \mid s \stackrel{a}{\longrightarrow} t\}$ for all $s \in S$, $a \in \Sigma$. Like for DMTS, this defines an embedding of LTS into the modal ν -calculus whose image are precisely the ν -calculus implementations.

We will show below in Theorem 8.12 that for any LTS \mathcal{I} and any ν -calculus expression \mathcal{N} in normal form, $\mathcal{I} \models \mathcal{N}$ iff $\mathcal{I} \leq_m \mathcal{N}$, hence the fixed-point semantics of [Lar90b] and our refinement semantics agree. As a corollary of this result, we get that modal refinement is a sound approximation to logical implication, *i.e.*, that $\mathcal{N}_1 \leq_m \mathcal{N}_2$ implies that for all implementations \mathcal{I} , $(\mathcal{I} \models \mathcal{N}_1) \Rightarrow (\mathcal{I} \models \mathcal{N}_2)$.

8.1.3 Nondeterministic Acceptance Automata

8.6 Definition. A nondeterministic acceptance automaton (AA) is a structure $\mathcal{A} = (S, S^0, \text{Tran})$, with $S \supseteq S^0$ finite sets of states and initial states and Tran : $S \to 2^{2^{\Sigma \times S}}$ an assignment of transition constraints.

Acceptance automata were first introduced in [Rac07] (see also [Rac08], where a slightly different language-based approach is taken), based on the notion of acceptance trees in [Hen85]; however, these are *deterministic*. We extend the formalism to a nondeterministic setting here. The following notion of modal refinement was introduced in [BKL⁺11].

8.7 Definition. Let $A_1 = (S_1, S_1^0, \operatorname{Tran}_1)$ and $A_2 = (S_2, S_2^0, \operatorname{Tran}_2)$ be AA. A relation $R \subseteq S_1 \times S_2$ is a *modal refinement* if it holds for all $(s_1, s_2) \in R$ and all $M_1 \in \operatorname{Tran}_1(s_1)$ that there exists $M_2 \in \operatorname{Tran}_2(s_2)$ such that

$$\forall (a, t_1) \in M_1 : \exists (a, t_2) \in M_2 : (t_1, t_2) \in R,
\forall (a, t_2) \in M_2 : \exists (a, t_1) \in M_1 : (t_1, t_2) \in R.$$
(8.2)

We say that \mathcal{A}_1 modally refines \mathcal{A}_2 , and write $\mathcal{A}_1 \leq_{\mathsf{m}} \mathcal{A}_2$, whenever there exists a modal refinement R such that for all $s_1^0 \in S_1^0$, there exists $s_2^0 \in S_2^0$ for which $(s_1^0, s_2^0) \in R$.

An AA is an *implementation* if $S^0 = \{s^0\}$ is a singleton and, for all $s \in S$, $\operatorname{Tran}(s) = \{M\}$ is a singleton. An LTS $(S, s^0, \longrightarrow)$ can be translated to an AA by setting $S^0 = \{s^0\}$ and $\operatorname{Tran}(s) = \{\{(a,t) \mid s \stackrel{a}{\longrightarrow} t\}\}$. This defines an embedding of LTS into AA whose image are precisely the AA implementations. As for DMTS, we write $[\![\mathcal{A}]\!]$ for the set of implementations of an AA \mathcal{A} , and through refinement and equivalence are defined accordingly.

8.1.4 Hybrid Modal Logic

As our fourth specification formalism, we introduce a hybrid modal logic, closely related to the Boolean modal transition systems of [BKL⁺11] and hybrid in the sense of [Pri68, Bla00]: it contains nominals, and the semantics of a nominal is given as all sets which contain the nominal.

For a finite set X of nominals, let $\mathcal{L}(X)$ be the set of formulae generated by the abstract syntax $\mathcal{L}(X) \ni \phi \coloneqq \mathbf{tt} \mid \mathbf{ff} \mid \langle a \rangle x \mid \neg \phi \mid \phi \land \phi$, for $a \in \Sigma$ and $x \in X$. The semantics of a formula is a set of subsets of $\Sigma \times X$, given as follows: $\llbracket \mathbf{tt} \rrbracket = 2^{\Sigma \times X}, \llbracket \mathbf{ff} \rrbracket = \emptyset, \llbracket \neg \phi \rrbracket = 2^{\Sigma \times X} \backslash \llbracket \phi \rrbracket, \llbracket \langle a \rangle x \rrbracket = \{ M \subseteq \Sigma \times X \mid (a, x) \in M \},$ and $\llbracket \phi \land \psi \rrbracket = \llbracket \phi \rrbracket \cap \llbracket \psi \rrbracket$. We also define disjunction $\phi_1 \lor \phi_2 = \neg (\neg \phi_1 \land \neg \phi_2)$. An \mathcal{L} -expression is a structure $\mathcal{E} = (X, X^0, \Phi)$ consisting of finite sets $X^0 \subseteq X$ of variables and a mapping $\Phi : X \to \mathcal{L}(X)$. Such an expression is an implementation if $\llbracket \Phi(x) \rrbracket = \{ M \}$ is a singleton for each $x \in X$.

- **8.8 Definition.** Let $\mathcal{E}_1 = (X_1, X_1^0, \Phi_1)$ and $\mathcal{E}_2 = (X_2, X_2^0, \Phi_2)$ be \mathcal{L} -expressions. A relation $R \subseteq X_1 \times X_2$ is a modal refinement if it holds for all $(x_1, x_2) \in R$ and all $M_1 \in \llbracket \Phi_1(x_1) \rrbracket$ that there exists $M_2 \in \llbracket \Phi_2(x_2) \rrbracket$ such that
 - $\forall (a, y_1) \in M_1 : \exists (a, y_2) \in M_2 : (y_1, y_2) \in R$,
 - $\forall (a, y_2) \in M_2 : \exists (a, y_1) \in M_1 : (y_1, y_2) \in R.$

We say that \mathcal{E}_1 modally refines \mathcal{E}_2 , denoted $\mathcal{E}_1 \leq_{\mathsf{m}} \mathcal{E}_2$, whenever there exists a modal refinement R such that for all $x_1^0 \in X_1^0$, there exists $x_2^0 \in X_2^0$ for which $(x_1^0, x_2^0) \in R$.

We can translate an LTS $(S, s^0, \longrightarrow)$ to an \mathcal{L} -expression (S, S^0, Φ) by setting $S^0 = \{s^0\}$ and $\Phi(s) = \bigwedge_{s \xrightarrow{a} t} \langle a \rangle t \wedge \bigwedge_{s \not \stackrel{b}{\longrightarrow} u} \neg \langle b \rangle u$. This defines an embedding of LTS into \mathcal{L} -expressions whose image are precisely the \mathcal{L} -implementations. As for DMTS, we write $[\![\mathcal{E}]\!]$ for the set of implementations of an \mathcal{L} -expression \mathcal{E} , and through refinement and equivalence are defined accordingly.

8.9 Remark. As all our four specification formalisms have the same type of implementations, labeled transition systems, we can use thorough refinement and equivalence cross-formalism. As an example example, for a given DMTS \mathcal{D} and an AA \mathcal{A} , the expression $\mathcal{D} \leq_t \mathcal{A}$ is valid. We will use these types of thorough refinement and equivalence in many places throughout the paper.

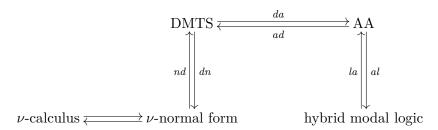


Figure 8.1: Six translations between specification formalisms

8.2 Structural Equivalence

We proceed to show that our four specification formalisms are structurally equivalent. To this end, we shall expose six translations between them, see Figure 8.1. Section 8.2.1 is concerned with dn and nd, Section 8.2.2 with al and la, and Section 8.2.3 with da and ad. We show in Theorems 8.11, 8.13 and 8.14 that all six translations preserve and reflect modal refinement.

8.2.1 DMTS vs. the Modal ν -Calculus

Our first two translations are rather straight-forward. For a DMTS $\mathcal{D} = (S, S^0, \dashrightarrow, \longrightarrow)$ and all $s \in S$, define $\Diamond(s) = \{N \mid s \longrightarrow N\}$ and, for each $a \in \Sigma$, $\Box^a(s) = \{t \mid s \stackrel{a}{\dashrightarrow} t\}$. Then, let

$$\Delta(s) = \bigwedge_{N \in \Diamond(s)} \left(\bigvee_{(a,t) \in N} \langle a \rangle t \right) \wedge \bigwedge_{a \in \Sigma} [a] \left(\bigvee_{t \in \Box^a(s)} t \right)$$
 (8.3)

and define the (normal-form) ν -calculus expression $dn(\mathcal{D}) = (S, S^0, \Delta)$.

Note how the formula precisely expresses that we demand at least one of every choice of disjunctive must-transitions (first part) and permit all may-transitions (second part); this is similar to the *characteristic formulae* of [Lar89].

Conversely, for a ν -calculus expression $\mathcal{N}=(X,X^0,\Delta)$ in normal form, let

and define the DMTS $nd(\mathcal{N}) = (X, X^0, \dashrightarrow, \longrightarrow)$. Note how this is a simple syntactic translation from diamonds to disjunctive must-transitions and from boxes to may-transitions. Also, the two translations are inverse to each other: $dn(nd(\mathcal{N})) = \mathcal{N}$ and $nd(dn(\mathcal{D})) = \mathcal{D}$.

8.10 Example. Consider the ν -calculus formula

$$X = (\langle a \rangle (\langle b \rangle X \wedge [a] \mathbf{ff}) \wedge [b] \mathbf{ff}) \vee [a] \mathbf{ff}.$$

$$X_1 = \langle a \rangle Y \qquad \wedge [a] \mathbf{t} \mathbf{t} \wedge [b] \mathbf{f} \mathbf{f}$$

$$X_2 = \qquad [a] \mathbf{f} \mathbf{f} \wedge [b] \mathbf{t} \mathbf{t}$$

$$Y = \langle b \rangle (X_1 \vee X_2) \wedge [a] \mathbf{f} \mathbf{f} \wedge [b] \mathbf{t} \mathbf{t}$$

$$\rightarrow 0$$

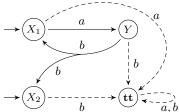


Figure 8.2: ν -calculus expression in normal form and its DMTS translation, cf. Example 8.10. The state corresponding to ff is inconsistent and not shown

Converting the formula into the normal form of Lemma 8.4 yields the result given in Figure 8.2 (left), where both X_1 and X_2 are initial variables. The resulting DMTS is illustrated in Figure 8.2 (right).

The following theorem follows easily:

8.11 Theorem. For all DMTS \mathcal{D}_1 , \mathcal{D}_2 , $\mathcal{D}_1 \leq_{\mathsf{m}} \mathcal{D}_2$ iff $dn(\mathcal{D}_1) \leq_{\mathsf{m}} dn(\mathcal{D}_2)$. For all ν -calculus expressions \mathcal{N}_1 , \mathcal{N}_2 in normal form, $\mathcal{N}_1 \leq_{\mathsf{m}} \mathcal{N}_2$ iff $nd(\mathcal{N}_1) \leq_{\mathsf{m}} nd(\mathcal{N}_2)$.

As a consequence, we can now show that the fixed-point semantics and our refinement semantics for the modal ν -calculus agree:

8.12 Theorem. For any LTS \mathcal{I} and any ν -calculus expression \mathcal{N} in normal form, $\mathcal{I} \models \mathcal{N}$ iff $\mathcal{I} \leq_m \mathcal{N}$.

Proof: We show that $\mathcal{I} \leq_m \mathcal{D}$ iff $\mathcal{I} \models \mathit{dn}(\mathcal{D})$ for any DMTS \mathcal{D} ; the claim then follows because $\mathcal{I} \leq_m \mathcal{N}$ iff $\mathcal{I} \leq_m \mathit{nd}(\mathcal{N})$ iff $\mathcal{I} \models \mathit{dn}(\mathit{nd}(\mathcal{N})) = \mathcal{N}$.

Write $\mathcal{I} = (I, i^0, \longrightarrow_I)$, $\mathcal{D} = (S, S^0, \dashrightarrow, \longrightarrow)$, and $dn(\mathcal{D}) = (S, S^0, \Delta)$.

We start with the only-if part. The proof is done by coinduction. We define the assignment $\sigma: S \to 2^I$ as follows: $\sigma(t) = \{j \in I \mid j \leq_{\mathsf{m}} t\}$. We need to show that for every $s \in S$, $\sigma(s) \subseteq [\![\Delta(s)]\!]\sigma$. Let $i \in \sigma(s)$.

As $i \leq_{\mathsf{m}} s$, we know that (1) $\forall s \longrightarrow N : \exists i \xrightarrow{a}_{I} j, (a,t) \in N : j \leq_{\mathsf{m}} t$ and (2) $\forall i \xrightarrow{a}_{I} j : \exists s \xrightarrow{a}_{i} t : j \leq_{\mathsf{m}} t$.

Due to (1), we see that for all $N \in \Diamond(s)$, there is $i \xrightarrow{a}_{I} j$ and $(a,t) \in N$ such that $j \in \sigma(t)$ and $i \in [\![\langle a \rangle t]\!] \sigma$. Hence $i \in \bigwedge_{N \in \Diamond(s)} \bigvee_{(a,t) \in N} [\![\langle a \rangle t]\!] \sigma = [\![\bigwedge_{N \in \Diamond(s)} \bigvee_{(a,t) \in N} \langle a \rangle t)]\!] \sigma$.

Due to (2), it holds that for every $a \in \Sigma$ and every $i \xrightarrow{a}_I j$, there is $t \in \Box^a(s)$ such that $j \in \sigma(t) \subseteq \llbracket \bigvee_{t \in \Box^a(s)} t \rrbracket \sigma$. Hence $i \in \bigwedge_{a \in \Sigma} \llbracket [a](\bigvee_{t \in \Box^a(s)} t) \rrbracket \sigma = \llbracket \bigwedge_{a \in \Sigma} [a](\bigvee_{t \in \Box^a(s)} t) \rrbracket \sigma$. Altogether, we have shown that $i \in \llbracket \Delta(s) \rrbracket \sigma$.

Clearly, there is $s^0 \in S^0$ such that $i^0 \in \sigma(s^0)$. Therefore, $\mathcal{I} \models dn(\mathcal{D})$.

For the other direction, define a relation $R \subseteq I \times S$ by $R = \{(j,t) \mid j \models t\}$. We show that R satisfies the conditions of modal refinement.

Let $(i, s) \in R$. As $i \models s$, we know that $(1) \forall N \in \Diamond(s) : \exists (a, t) \in N : i \models \langle a \rangle t$ and $(2) \forall a \in \Sigma : i \models [a](\bigvee_{t \in \Box^a(s)} t)$.

By (1), we know that for all $s \longrightarrow N$, there is $(a,t) \in N$ and $i \xrightarrow{a}_I j$ such that $j \models t$. By (2), it holds that for all $i \xrightarrow{a}_I j$, there is $s \xrightarrow{a}_I t$ so that $j \models t$. We have shown that $i \leq_m s$.

Clearly, there is $s^0 \in S^0$ for which $(i^0, s^0) \in R$, hence $\mathcal{I} \leq_{\mathsf{m}} \mathcal{D}$.

8.2.2 AA vs. Hybrid Modal Logic

Also the translations between AA and our hybrid modal logic are straightforward. For an AA $\mathcal{A} = (S, S^0, \text{Tran})$ and all $s \in S$, let

$$\Phi(s) = \bigvee_{M \in \text{Tran}(s)} \left(\bigwedge_{(a,t) \in M} \langle a \rangle t \wedge \bigwedge_{(b,u) \notin M} \neg \langle b \rangle u \right)$$

and define the \mathcal{L} -expression $al(\mathcal{A}) = (S, S^0, \Phi)$.

For an \mathcal{L} -expression $\mathcal{E} = (X, X^0, \Phi)$ and all $x \in X$, let $Tran(x) = \llbracket \Phi(x) \rrbracket$ and define the AA $la(\mathcal{E}) = (X, X^0, Tran)$.

8.13 Theorem. For all AA A_1 , A_2 , $A_1 \leq_m A_2$ iff $al(A_1) \leq_m al(A_2)$. For all \mathcal{L} -expressions \mathcal{E}_1 , \mathcal{E}_2 , $\mathcal{E}_1 \leq_m \mathcal{E}_2$ iff $la(\mathcal{E}_1) \leq_m la(\mathcal{E}_2)$.

Proof: We show that for any AA (S, S^0, Tran) and any \mathcal{L} -expression (S, S^0, Φ) , $\text{Tran}(s) = \llbracket \Phi(s) \rrbracket$ for every $s \in S$, for both translations. For the second one, la, this is clear by definition, and for the first,

$$\begin{split} \llbracket \Phi(s) \rrbracket &= \llbracket \bigvee_{M \in \operatorname{Tran}(s)} \big(\bigwedge_{(a,t) \in M} \langle a \rangle t \wedge \bigwedge_{(b,u) \notin M} \neg \langle b \rangle u \big) \rrbracket \\ &= \bigcup_{M \in \operatorname{Tran}(s)} \big(\bigcap_{(a,t) \in M} \{ M' \mid (a,t) \in M' \} \cap \bigcap_{(b,u) \notin M} \{ M' \mid (b,u) \notin M' \} \big) \\ &= \bigcup_{M \in \operatorname{Tran}(s)} \big(\{ M' \mid \forall (a,t) \in M : (a,t) \in M' \} \\ &\qquad \qquad \cap \{ M' \mid \forall (b,u) \notin M : (b,u) \notin M' \} \big) \\ &= \bigcup_{M \in \operatorname{Tran}(s)} \big(\{ M' \mid M \subseteq M' \} \cap \{ M' \mid M' \subseteq M \} \big) \\ &= \bigcup_{M \in \operatorname{Tran}(s)} \{ M \} = \operatorname{Tran}(s) \end{split}$$

as was to be shown.

8.2.3 DMTS vs. AA

The translations between DMTS and AA are somewhat more intricate. For a DMTS $\mathcal{D} = (S, S^0, \dashrightarrow, \longrightarrow)$ and all $s \in S$, let

$$\operatorname{Tran}(s) = \{ M \subseteq \Sigma \times S \mid \forall (a,t) \in M : s \xrightarrow{a} t, \forall s \longrightarrow N : N \cap M \neq \emptyset \}$$

and define the AA $da(\mathcal{D}) = (S, S^0, \text{Tran}).$

For an AA $\mathcal{A} = (S, S^0, \text{Tran})$, define the DMTS $ad(\mathcal{A}) = (D, D^0, \dashrightarrow, \longrightarrow)$ as follows:

$$D = \{M \in \operatorname{Tran}(s) \mid s \in S\} \cup \{ \sharp \}$$

$$D^{0} = \{M^{0} \in \operatorname{Tran}(s^{0}) \mid s^{0} \in S^{0}\} \cup \{ \sharp \mid \text{if } \exists s^{0} \in S^{0} : \operatorname{Tran}(s^{0}) = \emptyset \}$$

$$\longrightarrow = \{ (M, \{(a, M') \mid M' \in \operatorname{Tran}(t)\}) \mid (a, t) \in M, \operatorname{Tran}(t) \neq \emptyset \} \cup$$

$$\{ (M, \{(a, \sharp)\}) \mid (a, t) \in M, \operatorname{Tran}(t) = \emptyset \} \cup \{ (\sharp, \emptyset) \}$$

$$\longrightarrow = \{ (M, a, M') \mid \exists M \longrightarrow N : (a, M') \in N \}$$

Note that the state spaces of \mathcal{A} and $ad(\mathcal{A})$ are not the same; the one of $ad(\mathcal{A})$ may be exponentially larger.

8.14 Theorem. For all DMTS \mathcal{D}_1 , \mathcal{D}_2 , $\mathcal{D}_1 \leq_{\mathsf{m}} \mathcal{D}_2$ iff $da(\mathcal{D}_1) \leq_{\mathsf{m}} da(\mathcal{D}_2)$. For all $AA \mathcal{A}_1$, \mathcal{A}_2 , $\mathcal{A}_1 \leq_{\mathsf{m}} \mathcal{A}_2$ iff $ad(\mathcal{A}_1) \leq_{\mathsf{m}} ad(\mathcal{A}_2)$.

Proof: There are four parts to this proof, two implications to show the first claim of the theorem and two implications for the second claim.

• $\mathcal{D}_1 \leq_{\mathsf{m}} \mathcal{D}_2 \text{ implies } da(\mathcal{D}_1) \leq_{\mathsf{m}} da(\mathcal{D}_2)$:

Write $\mathcal{D}_1 = (S_1, S_1^0, -- \rightarrow_1, \longrightarrow_1)$, $\mathcal{D}_2 = (S_2, S_2^0, -- \rightarrow_2, \longrightarrow_2)$. We have a modal refinement relation (in the DMTS sense) $R \subseteq S_1 \times S_2$. Now let $(s_1, s_2) \in R$ and $M_1 \in \text{Tran}_1(s_1)$, and define

$$M_2 = \{(a, t_2) \mid s_2 \xrightarrow{a}_{-} t_2, \exists (a, t_1) \in M_1 : (t_1, t_2) \in R\}.$$

We prove that $M_2 \in \operatorname{Tran}_2(s_2)$. First we notice that by construction, indeed $s_2 \xrightarrow{a}_2 t_2$ for all $(a, t_2) \in M_2$. Now let $s_2 \longrightarrow_2 N_2$; we need to show that $N_2 \cap M_2 \neq \emptyset$.

By DMTS refinement, we have $s_1 \longrightarrow_1 N_1$ such that $\forall (a,t_1) \in N_1 : \exists (a,t_2) \in N_2 : (t_1,t_2) \in R$. We know that $N_1 \cap M_1 \neq \emptyset$, so let $(a,t_1) \in N_1 \cap M_1$. Then there also is $(a,t_2) \in N_2$ with $(t_1,t_2) \in R$. But $(a,t_2) \in N_2$ implies $s_2 \xrightarrow{a} t_2$, hence $(a,t_2) \in M_2$.

Now the condition

$$\forall (a, t_2) \in M_2 : \exists (a, t_1) \in M_1 : (t_1, t_2) \in R$$

in the definition of AA refinement is satisfied by construction. For the inverse condition, let $(a, t_1) \in M_1$, then $s_1 \stackrel{a}{\dashrightarrow}_1 t_1$, so by DMTS refinement, there is $t_2 \in S_2$ with $s_2 \stackrel{a}{\dashrightarrow}_2 t_2$ and $(t_1, t_2) \in R$, whence $(a, t_2) \in M_2$ by construction.

• $da(\mathcal{D}_1) \leq_{\mathsf{m}} da(\mathcal{D}_2)$ implies $\mathcal{D}_1 \leq_{\mathsf{m}} \mathcal{D}_2$:

Let $R \subseteq S_1 \times S_2$ be a modal refinement relation in the AA sense and $(s_1, s_2) \in R$. Let $s_1 \stackrel{a}{\dashrightarrow}_1 t_1$ and $M_1 = \{(a, t_1)\} \cup \bigcup_{s_1 \longrightarrow_1 N_1} N_1$, then $M_1 \in \operatorname{Tran}_1(s_1)$ by construction. As R is a modal refinement, this implies that there is $M_2 \in \operatorname{Tran}_2(s_2)$ and $(a, t_2) \in M_2$ with $(t_1, t_2) \in R$, but then also $s_2 \stackrel{a}{\dashrightarrow}_2 t_2$ as was to be shown.

Let $s_2 \longrightarrow_2 N_2$ and assume, for the sake of contradiction, that there is no $s_1 \longrightarrow_1 N_1$ for which $\forall (a,t_1) \in N_1 : \exists (a,t_2) \in N_2 : (t_1,t_2) \in R$ holds. Then for each $s_1 \longrightarrow_1 N_1$, there is an element $(a_{N_1},t_{N_1}) \in N_1$ for which there is no $(a_{N_1},t_2) \in N_2$ with $(t_{N_1},t_2) \in R$.

Let $M_1 = \{(a_{N_1}, t_{N_1}) \mid s_1 \longrightarrow_1 N_1\}$, then $M_1 \in \text{Tran}_1(s_1)$ by construction. Hence we have $M_2 \in \text{Tran}_2(s_2)$ satisfying the conditions in the definition of AA refinement. By construction of $\text{Tran}_2(s_2)$, $s_2 \longrightarrow_2 N_2$ and $N_2 \cap M_2 \neq \emptyset$, so let $(a, t_2) \in N_2 \cap M_2$. Then there exists $(a, t_1) \in M_1$ for which $(t_1, t_2) \in R$, in contradiction to the definition of M_1 .

• $A_1 \leq_{\mathsf{m}} A_2 \text{ implies } ad(A_1) \leq_{\mathsf{m}} ad(A_2)$:

Write $A_1 = (S_1, S_1^0, \text{Tran}_1)$, $A_2 = (S_2, S_2^0, \text{Tran}_2)$, with DMTS translations $(D_1, D_1^0, \longrightarrow_1, -\rightarrow_1)$, $(D_2, D_2^0, \longrightarrow_2, -\rightarrow_2)$. We have a modal refinement relation (in the AA sense) $R \subseteq S_1 \times S_2$. Define $R' \subseteq D_1 \times D_2$ by

```
R' = \{ (\nleq, M_2) \mid M_2 \in D_2 \}
\cup \{ (M_1, M_2) \mid \exists (s_1, s_2) \in R : M_1 \in \operatorname{Tran}_1(s_1), M_2 \in \operatorname{Tran}_1(s_2),
\forall (a, t_1) \in M_1 : \exists (a, t_2) \in M_2 : (t_1, t_2) \in R,
\forall (a, t_2) \in M_2 : \exists (a, t_1) \in M_1 : (t_1, t_2) \in R \}.
```

We show that R' is a modal refinement in the DMTS sense. Let $(M_1, M_2) \in R'$. If $M_1 = \mnote{\mnote$

Let $M_2 \longrightarrow_2 \{(a, \normalfoldsymbol{\not})\}$. By construction of \longrightarrow_2 , there is $(a, t_2) \in M_2$ with $\operatorname{Tran}_2(t_2) = \emptyset$. Then $(M_1, M_2) \in R'$ implies that there must be $(a, t_1) \in M_1$ for which $(t_1, t_2) \in R$. As R is a AA refinement, this means that $\operatorname{Tran}_1(t_1) = \emptyset$ and thus $M_1 \longrightarrow_1 \{(a, \normalfoldsymbol{\not})\}$. Clearly, $(a, \normalfoldsymbol{\not})$ is matched by $(a, \normalfoldsymbol{\not})$ and $(\normalfoldsymbol{\not}) \in R'$.

Let $M_2 \longrightarrow_2 N_2 \neq \{(a, \xi)\}$. By construction of \longrightarrow_2 , there is $(a, t_2) \in M_2$ such that $N_2 = \{(a, M_2') \mid M_2' \in \operatorname{Tran}_2(t_2)\} \neq \emptyset$. Then $(M_1, M_2) \in R'$ implies that there must be $(a, t_1) \in M_1$ for which $(t_1, t_2) \in R$.

If $\operatorname{Tran}_1(t_1) = \emptyset$, we know that $M_1 \longrightarrow_1 \{(a, \xi)\}$. We can then match (a, ξ) with arbitrary $(a, M_2') \in N_2$ as $(\xi, M_2') \in R'$.

If $\operatorname{Tran}_1(t_1) \neq \emptyset$, we can define $N_1 = \{(a, M_1') \mid M_1' \in \operatorname{Tran}_1(t_1)\}$, whence $M_1 \longrightarrow_1 N_1$. We show that $\forall (a, M_1') \in N_1 : \exists (a, M_2') \in N_2 : (M_1', M_2') \in R'$. Let $(a, M_1') \in N_1$, then $M_1' \in \operatorname{Tran}_1(t_1)$. From $(t_1, t_2) \in R$ we hence get $M_2' \in \operatorname{Tran}_2(t_2)$, and then $(a, M_2') \in N_2$ by construction of N_2 and $(M_1', M_2') \in R'$ due to the conditions of AA refinement (applied to $(t_1, t_2) \in R$).

Let $M_1 \stackrel{a}{\dashrightarrow}_1 \not \xi$. We then have $M_1 \longrightarrow \{(a, \xi)\}$ which means that there is $(a, t_1) \in M_1$ with $\operatorname{Tran}_1(t_1) = \emptyset$. By $(M_1, M_2) \in R'$ we get $(a, t_2) \in M_2$, hence $M_2 \longrightarrow_2 N_2$ with some $(a, M_2') \in N_2$ and $M_2 \stackrel{a}{\dashrightarrow} M_2'$. By definition of R', $(\xi, M_2') \in R'$.

Let $M_1 \xrightarrow{a}_1 M'_1 \neq \xi$, then we have $M_1 \longrightarrow_1 N_1$ for which $(a, M'_1) \in N_1$ and by construction of $-\rightarrow_1$. This in turn implies that there must be $(a, t_1) \in$

 M_1 such that $N_1 = \{(a, M_1'') \mid M_1'' \in \operatorname{Tran}_1(t_1)\} \neq \emptyset$, and then by $(M_1, M_2) \in R'$, we get $(a, t_2) \in M_2$ for which $(t_1, t_2) \in R$. Due to the conditions of AA refinement, $\operatorname{Tran}_2(t_2) \neq \emptyset$. Let $N_2 = \{(a, M_2') \mid M_2' \in \operatorname{Tran}_2(t_2)\}$, then $M_2 \longrightarrow_2 N_2$ and hence $M_2 \xrightarrow{a} M_2'$ for all $(a, M_2') \in N_2$. Furthermore, the argument above shows that there is $(a, M_2') \in N_2$ for which $(M_1', M_2') \in R'$.

We miss to show that R' is initialised. If $\xi \in D_1^0$, then $(\xi, M_2^0) \in R'$ for any $M_2^0 \in D_2^0$. If $\xi \neq M_1^0 \in D_1^0$, then we have $s_1^0 \in S_1^0$ with $M_1^0 \in \operatorname{Tran}_1(s_1^0)$. As R is initialised, this entails that there is $s_2^0 \in S_2^0$ with $(s_1^0, s_2^0) \in R$, which gives us $M_2^0 \in \operatorname{Tran}_2(s_2^0)$ which satisfies the AA refinement conditions, whence $(M_1^0, M_2^0) \in R'$.

• $ad(A_1) \leq_{\mathsf{m}} ad(A_2)$ implies $A_1 \leq_{\mathsf{m}} A_2$:

Let $R \subseteq D_1 \times D_2$ be a modal refinement relation in the DMTS sense and define $R' \subseteq S_1 \times S_2$ by

$$R' = \{(s_1, s_2) \mid \forall M_1 \in \text{Tran}_1(s_1) : \exists M_2 \in \text{Tran}_2(s_2) : (M_1, M_2) \in R\};$$

we will show that R' is an AA modal refinement.

Observe first that $(M_1, \normalfoldsymbol{\not}) \in R$ implies $M_1 = \normalfoldsymbol{\not}$ as $\normalfoldsymbol{\not}$ is the only state in D_1 that has a must transition $\longrightarrow_1 \emptyset$. We shall occasionally refer to this observation in the following.

Let $(s_1, s_2) \in R'$ and $M_1 \in \text{Tran}_1(s_1)$, then by construction of R', we have $M_2 \in \text{Tran}_2(s_2)$ with $(M_1, M_2) \in R$.

Let $(a, t_2) \in M_2$; we need to find $(a, t_1) \in M_1$ such that $(t_1, t_2) \in R'$.

If $\operatorname{Tran}_2(t_2) = \emptyset$, then $M_2 \longrightarrow_2 \{(a, \normalfont)\}$. From $(M_1, M_2) \in R$ we get $M_1 \longrightarrow_1 N_1$ such that all $(a, M_1') \in N_1$ are matched by (a, \normalfont) with $(M_1', \normalfont) \in R$. By the observation above, this means that $M_1' = \normalfont$ and thus $N_1 = \{(a, \normalfont)\}$ due to the construction. This means that there exists $(a, t_1) \in M_1$ with $\operatorname{Tran}_1(t_1) = \emptyset$. Hence $(t_1, t_2) \in R'$.

If $\operatorname{Tran}_2(t_2) \neq \emptyset$, define $N_2 = \{(a, M_2') \mid M_2' \in \operatorname{Tran}_2(t_2)\}$, then $M_2 \longrightarrow_2 N_2$. Now $(M_1, M_2) \in R$ implies that there must be $M_1 \longrightarrow_1 N_1$ satisfying $\forall (a, M_1') \in N_1 : \exists (a, M_2') \in N_2 : (M_1', M_2') \in R$. If $N_1 = \{(a, \xi)\}$, we have $(a, t_1) \in M_1$ with $\operatorname{Tran}_1(t_1) = \emptyset$ and thus trivially $(t_1, t_2) \in R'$. Otherwise, we have $(a, t_1) \in M_1$ such that $N_1 = \{(a, M_1') \mid M_1' \in \operatorname{Tran}_1(t_1)\}$; we only miss to show that $(t_1, t_2) \in R'$.

Let $M'_1 \in \text{Tran}_1(t_1)$, then $(a, M'_1) \in N_1$, hence there is $(a, M'_2) \in N_2$ with $(M'_1, M'_2) \in R$, but $(a, M'_2) \in N_2$ also entails $M'_2 \in \text{Tran}_2(t_2)$; thus $(t_1, t_2) \in R'$.

Let now $(a, t_1) \in M_1$; we need to find $(a, t_2) \in M_2$ such that $(t_1, t_2) \in R'$.

If $\operatorname{Tran}_1(t_1) = \emptyset$, then $M_1 \longrightarrow_1 \{(a, \xi)\}$ and $M_1 \stackrel{a}{\dashrightarrow} \xi$. By modal refinement, we have $M_2 \stackrel{a}{\dashrightarrow} M'_2$ with $(\xi, M'_2) \in R$. In any case (whether $M'_2 = \xi$ or not), this means that there exists some $(a, t_2) \in M_2$ and trivially $(t_1, t_2) \in R'$.

In case $\operatorname{Tran}_1(t_1) \neq \emptyset$, define $N_1 = \{(a, M_1') \mid M_1' \in \operatorname{Tran}_1(t_1)\}$, then $N_1 \neq \emptyset$ and $M_1 \longrightarrow_1 N_1$. Now let $(a, M_1') \in N_1$, then $M_1 \stackrel{a}{\dashrightarrow}_1 M_1'$, hence we have $M_2 \stackrel{a}{\dashrightarrow}_2 M_2'$ for some $(M_1', M_2') \in R$ by modal refinement. Note that

 $M_1' \neq f$ implies $M_2' \neq f$ due to the observation above. By construction of $-- \rightarrow_2$, this implies that there is $M_2 \longrightarrow_2 N_2$ with $(a, M_2') \in N_2$, and we have $(a, t_2) \in M_2$ for which $N_2 = \{(a, M_2'') \mid M_2'' \in \operatorname{Tran}_2(t_2)\}$. We show that $(t_1, t_2) \in R'$.

Let $M_1'' \in \operatorname{Tran}_1(t_1)$, then $(a, M_1'') \in N_1$, thus $M_1 \xrightarrow{a} M_1''$, so that there is $M_2 \xrightarrow{a} M_2''$ with $(M_1'', M_2'') \in R$. By construction of $-- \rightarrow_2$, there is $M_2 \xrightarrow{a} N_2'$ with $(a, M_2'') \in N_2'$, hence also $M_2'' \in \operatorname{Tran}_2(t_2)$.

We miss to show that R' is initialised. Let $s_1^0 \in S_1^0$; if $\operatorname{Tran}_1(s_1^0) = \emptyset$, then trivially $(s_1^0, s_2^0) \in R'$ for any $s_2^0 \in S_2^0$. If $\operatorname{Tran}_1(s_1^0) \neq \emptyset$, then there is $M_1^0 \in \operatorname{Tran}_1(s_1^0)$. As R is initialised, this gets us $M_2^0 \in D_2$ with $(M_1^0, M_2^0) \in R$, but $M_2^0 \in \operatorname{Tran}_2(s_2^0)$ for some $s_2^0 \in S_2^0$, and then $(s_1^0, s_2^0) \in R'$.

8.15 Corollary. For all DMTS \mathcal{D} , ν -calculus expressions \mathcal{N} , AA \mathcal{A} , and \mathcal{L} -expressions \mathcal{E} , $dn(\mathcal{D}) \equiv_{\mathsf{t}} da(\mathcal{D}) \equiv_{\mathsf{t}} \mathcal{D}$, $nd(\mathcal{N}) \equiv_{\mathsf{t}} \mathcal{N}$, $ad(\mathcal{A}) \equiv_{\mathsf{t}} al(\mathcal{A}) \equiv_{\mathsf{t}} \mathcal{A}$, and $la(\mathcal{E}) \equiv_{\mathsf{t}} \mathcal{E}$.

8.2.4 Translation Complexity

We have shown that our four specification formalisms are structurally equivalent, which will be useful for us from a theoretical point of view. From a practical point of view however, some of the translations may incur exponential blow-ups, hence care has to be taken. On the other hand, all our translations can be implemented in an on-the-fly manner, only creating states when necessary.

We already noticed that the translation of ν -calculus expressions into normal form may incur an exponential blow-up, so this also affects our translation from the modal ν -calculus to DMTS. When considering only normal-form expressions, the translations to and from DMTS incur no blow-ups.

When translating from AA to our hybrid modal logic, we see that, due to the complementation $(b, u) \notin M$, the length of a formula $\Phi(s)$ is quadratic in the representation of $\operatorname{Tran}(s)$. For the reverse translation, the number of Tran constraints can be exponential in the number of states. The worst case is $\Phi(s) = \mathbf{tt}$, which gets translated to $\operatorname{Tran}(s) = 2^{2^{\Sigma \times S}}$.

8.16 Remark. There is a direct translation from DMTS to hybrid modal logic: for a DMTS $\mathcal{D} = (S, S^0, \dashrightarrow, \longrightarrow)$, define $dl(\mathcal{D}) = (S, S^0, \Phi)$ with

$$\Phi(s) = \bigwedge_{s \longrightarrow N} \bigvee_{(a,t) \in N} \langle a \rangle t \wedge \bigwedge_{s \not \stackrel{\rho}{\leftarrow} \iota u} \neg \langle a \rangle u$$

for all $s \in S$. This translation is again quadratic, and $da(\mathcal{D}) = la(dl(\mathcal{D}))$.

The translations between DMTS and AA may involve exponential blow-ups both ways. For the first translation, we can see this by considering the one-state DMTS ($\{s\}, \{s\}, \dashrightarrow, \longrightarrow$) with $\dashrightarrow = \{(s, a, s) \mid a \in \Sigma\}$ and $\longrightarrow = \emptyset$. Then $\text{Tran}(s) = 2^{2^{\Sigma \times S}}$.

The fact that also the translation from AA to DMTS may be exponential in space is evident from the definition. To see that this blow-up is unavoidable, we expose a special property of the Tran-sets arising in the DMTS-to-AA translation.

8.17 Lemma. Let $\mathcal{D} = (S, S^0, \dashrightarrow, \longrightarrow)$ be a DMTS and $s \in S$. For all $M_1, M_2 \in \operatorname{Tran}(s)$ and all $M \subseteq \Sigma \times S$ with $M_1 \subseteq M \subseteq M_1 \cup M_2$, also $M \in \operatorname{Tran}(s)$.

Proof: For i = 1, 2, since $M_i \in \text{Tran}(s)$, we know that

- for all $(a,t) \in M_i$, $s \xrightarrow{a} t$, and
- for all $s \longrightarrow N$, there is $(a, t) \in M_i \cap N$.

Now as $M \subseteq M_1 \cup M_2$, it directly follows that for all $(a,t) \in M$, we have $s \xrightarrow{a} t$. Moreover, since $M_1 \subseteq M$, we also have that for all $s \longrightarrow N$, there exists $(a,t) \in M \cap N$. As a consequence, $M \in \text{Tran}(s)$.

Using this, we can show the following.

8.18 Proposition. There exists a one-state AA \mathcal{A} for which any DMTS $\mathcal{D} \equiv_{\mathsf{t}} \mathcal{A}$ has at least 2^{n-1} states, where n is the size of the alphabet Σ .

Proof: Let $\Sigma = \{a_1, \ldots, a_n\}$ and $\mathcal{A} = (\{s^0\}, \{s^0\}, \text{Tran})$ the AA with $\text{Tran}(s^0) = \{M \subseteq \Sigma \times \{s^0\} \mid \exists k : |M| = 2k\}$ the transition constraint containing all disjunctive choices of even cardinality. Let $\mathcal{D} = (T, T^0, \dashrightarrow, \longrightarrow)$ be a DMTS with $\mathcal{D} \equiv_{\mathsf{t}} \mathcal{A}$; we claim that \mathcal{D} must have at least 2^{n-1} initial states.

Assume, for the purpose of contradiction, that $T^0 = \{t_1^0, \dots, t_m^0\}$ with $m < 2^{n-1}$. As $\mathcal{D} \equiv_{\mathsf{t}} \mathcal{A}$, we must have $\bigcup_{i=1}^m \operatorname{Tran}(t_i^0) = \{M \subseteq \Sigma \times T \mid \exists k : |M| = 2k\}$, so that there is an index $j \in \{1, \dots, m\}$ for which $\operatorname{Tran}(t_j^0) = \{M_1, M_2\}$ contains two different disjunctive choices from $\operatorname{Tran}(s^0)$. By Lemma 8.17, also $M \in \operatorname{Tran}(t_j^0)$ for any M with $M_1 \subseteq M \subseteq M_1 \cup M_2$. But $M_1 \cup M_2$ has greater cardinality than M_1 , so that there will be an $M \in \operatorname{Tran}(t_j^0)$ with odd cardinality. \square

Figure 8.3 sums up the translation complexities.

8.2.5 Initial States

We finish this section with a justification for why we allow our specifications to have several (or possibly zero) initial states. The first lemma shows that for AA, and up to *thorough* refinement, this is inessential; due to their close relationship, this also holds for \mathcal{L} -expressions.

8.19 Lemma. For any AA A_1 , there is an AA $A_2 = (S_2, S_2^0, \text{Tran}_2)$ with $S_2^0 = \{s_2^0\}$ a singleton and $A_1 \equiv_t A_2$.

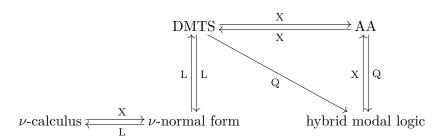


Figure 8.3: Complexity of the translations between specification formalisms. "L" stands for linear (no blow-up), "Q" for quadratic blow-up, and "X" for exponential blow-up

Proof: Write $\mathcal{A}_1 = (S_1, S_1^0, \operatorname{Tran}_1)$. If $S_1^0 = \emptyset$, we can let $S_2 = S_2^0 = \{s_2^0\}$ and $\operatorname{Tran}_2(s_2^0) = \emptyset$. Otherwise, we let $S_2 = S_1 \cup \{s_2^0\}$, where s_2^0 is a new state, and $\operatorname{Tran}_2(s) = \operatorname{Tran}_1(s)$ for $s \in S_1$, $\operatorname{Tran}_2(s_2^0) = \bigcup_{s_1^0 \in S_1^0} \operatorname{Tran}_1(s_1^0)$. Let $R = \operatorname{id}_{S_1} \cup \{(s_1^0, s_2^0) \mid s_1^0 \in S_1^0\}$, then R is easily seen to be a modal refinement showing $\mathcal{A}_1 \leq_{\mathsf{m}} \mathcal{A}_2$.

We show that $A_2 \leq_t A_1$. Let $\mathcal{I} = (S, s^0, \longrightarrow) \in [A_2]$, then we have a modal refinement $R_2 \subseteq S \times S_2$, *i.e.*, such that for all $(s, s_2) \in R_2$, there exists $M_2 \in \text{Tran}_2(s_2)$ for which

$$\forall s \xrightarrow{a} t : \exists (a, t_2) \in M_2 : (t, t_2) \in R_2, \forall (a, t_2) \in M_2 : \exists s \xrightarrow{a} t : (t, t_2) \in R_2.$$

$$(8.4)$$

Now $(s^0, s_2^0) \in R_2$ implies that there must be $M_2 \in \text{Tran}_2(s_2^0)$ for which (8.4) holds, but by definition of $\text{Tran}_2(s_2^0)$, this entails that there is $s_1^0 \in S_1^0$ for which $M_2 \in \text{Tran}_1(s_1^0)$. Define $R_1 \subseteq S \times S_1$ by

$$R_1 = \{(s, s_2) \mid (s, s_2) \in R_2, s_2 \neq s_2^0\} \cup \{(s^0, s_1^0)\},\$$

then R_1 is a modal refinement showing $\mathcal{I} \leq_{\mathsf{m}} \mathcal{A}_1$.

In order to show that the above statement does *not* hold for DMTS, we expose a special property of DMTS with single initial states, *cf.* [BKLS09, Example 7.8]. Recall that for LTS $\mathcal{I}_1 = (S_1, s_1^0, \longrightarrow_1)$, $\mathcal{I}_2 = (S_2, s_2^0, \longrightarrow_2)$, their *nondeterministic sum* is given by $\mathcal{I}_1 + \mathcal{I}_2 = (S, s^0, \longrightarrow)$ with $S = S_1 \cup S_2 \cup \{s^0\}$ (with the unions disjoint), where s^0 is a new state, and transitions $s \xrightarrow{a} t$ iff $s \xrightarrow{a}_1 t$ or $s \xrightarrow{a}_2 t$ together with $s^0 \xrightarrow{a} t$ for all t with $s_1^0 \xrightarrow{a}_1 t$ or $s_2^0 \xrightarrow{a}_2 t$.

8.20 Lemma. If $\mathcal{D} = (S, \{s^0\}, --\rightarrow, \longrightarrow)$ is a DMTS with a single initial state and $\mathcal{I}_1, \mathcal{I}_2 \in [\![\mathcal{D}]\!]$, then also $\mathcal{I}_1 + \mathcal{I}_2 \in [\![\mathcal{D}]\!]$.

Proof: Let i_1^0 and i_2^0 be the initial states of \mathcal{I}_1 and \mathcal{I}_2 , respectively. Let further i^0 be the initial state of $\mathcal{I}_1 + \mathcal{I}_2$. Assume that we have modal refinements R_1

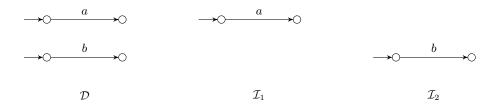


Figure 8.4: DMTS \mathcal{D} with two initial states and its two implementations \mathcal{I}_1 , \mathcal{I}_2

and R_2 such that $(i_1^0, s^0) \in R_1$ and $(i_2^0, s^0) \in R_2$. Let $R = R_1 \cup R_2 \cup \{(i^0, s^0)\}$. Clearly, R is a modal refinement witnessing $\mathcal{I}_1 + \mathcal{I}_2 \leq_{\mathsf{m}} \mathcal{D}$.

Now let \mathcal{D} be the DMTS, with two initial states, depicted in Figure 8.4, then $\llbracket \mathcal{D} \rrbracket = \{\mathcal{I}_1, \mathcal{I}_2\}$ as also seen in Figure 8.4, but $\mathcal{I}_1 + \mathcal{I}_2 \notin \llbracket \mathcal{D} \rrbracket$. Hence \mathcal{D} is not thoroughly equivalent to any DMTS with a single initial state.

Applying the construction from the proof of Lemma 8.19 to the AA generated by the DMTS in Figure 8.4 gives an AA $\mathcal{A}_2 = (S_2, \{s_2^0\}, \text{Tran}_2)$ with $\text{Tran}_2(s_2^0) = \{\{(a, s_1)\}, \{(b, t_1)\}\}$ (where s_1 and t_1 are the target states of the a and b transitions in \mathcal{D} , respectively). This specifies an exclusive disjunction: one of a and b has to be implemented, but not both. This also serves to show that Lemma 8.20 does not hold for AA.

8.21 Corollary. There is a DMTS
$$\mathcal{D}_1$$
 for which there is no DMTS $\mathcal{D}_2 = (S_2, S_2^0, \longrightarrow)$ with $S_2^0 = \{s_2^0\}$ a singleton and $\mathcal{D}_1 \equiv_{\mathsf{t}} \mathcal{D}_2$.

Due to their close relationship with DMTS, this property also holds for ν -calculus expressions in normal form: there exist ν -calculus expressions which are not equivalent to any normal-form ν -calculus expression with a single initial variable. (Of course, omitting "normal form" would make this statement invalid; as disjunction is part of the syntax, any ν -calculus expression is thoroughly equivalent to one with only one initial variable.)

We also remark that the above argument can easily be extended to show that for any $k \in \mathbb{N}$, there exists a DMTS with k+1 initial states which is not thoroughly equivalent to any DMTS with at most k initial states.

Using again the example in Figure 8.4, we can also show that the statement in Lemma 8.19 does not hold when thorough equivalence is replaced by modal equivalence. Let $\mathcal{A}_1 = da(\mathcal{D})$, with initial states s^0 and t^0 , be the AA translation of the DMTS in Figure 8.4 and assume that there exists an AA \mathcal{A}_2 with single initial state s_2^0 for which $\mathcal{A}_2 \leq_{\mathsf{m}} \mathcal{A}_1$. Then there is a modal refinement R with $(s_2^0, s^0), (s_2^0, t^0) \in R$. Let $M_2 \in \mathrm{Tran}_2(s_2^0)$, then by $(s_2^0, s^0) \in R$, there must be some $(a, t_2) \in M_2$ with $(t_2, s_1) \in R$. By $(s_2^0, t^0) \in R$, this implies that there must be $(a, t_1) \in \mathrm{Tran}_1(t^0)$, a contradiction.

8.3 Specification Theory

Behavioral specifications typically come equipped with operations which allow for *compositional reasoning*, viz. conjunction, composition and quotient, cf. [BDH $^+$ 12]. On deterministic MTS, these operations can be given easily using simple structural operational rules. For non-deterministic systems this is significantly harder.

We remark that composition and quotient operators are well-known from some logics, such as, e.g., linear [Gir87] or spatial logic [CC03], and were extended to quite general contexts [CLM11]. However, whereas these operators are part of the formal syntax in those logics, for us they are simply operations on logical expressions (or DMTS, or AA). Consequently, composition is generally only a sound over-approximation of the semantic composition.

Given the structural equivalence of DMTS, the modal ν -calculus, AA, and our hybrid modal logic exposed in the previous section, it suffices to introduce the operations for *one* of the four types of specifications. On the other hand, we will often state properties for *all* four types of specifications at the same time, letting S stand for a specification of any type.

8.3.1 Disjunction and Conjunction

Disjunction of specifications is easily defined as we allow multiple initial states. For DMTS $\mathcal{D}_1 = (S_1, S_1^0, -\rightarrow_1, \longrightarrow_1)$, $\mathcal{D}_2 = (S_2, S_2^0, -\rightarrow_2, \longrightarrow_2)$, we can hence define $\mathcal{D}_1 \vee \mathcal{D}_2 = (S_1 \cup S_2, S_1^0 \cup S_2^0, -\rightarrow_1 \cup -\rightarrow_2, \longrightarrow_1 \cup \longrightarrow_2)$ (with all unions disjoint). Similar definitions are available for the other types of specifications, and disjunction commutes with the translations.

Conjunction for DMTS is an extension of the construction from [BCK11] for multiple initial states. Given two DMTS $\mathcal{D}_1 = (S_1, S_1^0, -- \rightarrow_1, \longrightarrow_1), \mathcal{D}_2 = (S_2, S_2^0, -- \rightarrow_2, \longrightarrow_2)$, we define $\mathcal{D}_1 \wedge \mathcal{D}_2 = (S, S^0, -- \rightarrow, \longrightarrow)$ with $S = S_1 \times S_2$, $S^0 = S_1^0 \times S_2^0$, and

- $(s_1, s_2) \xrightarrow{a} (t_1, t_2)$ iff $s_1 \xrightarrow{a} t_1$ and $s_2 \xrightarrow{a} t_2$,
- for all $s_1 \longrightarrow_1 N_1$, $(s_1, s_2) \longrightarrow \{(a, (t_1, t_2)) \mid (a, t_1) \in N_1, s_2 \xrightarrow{a}_{-} t_2\}$,
- for all $s_2 \longrightarrow_2 N_2$, $(s_1, s_2) \longrightarrow \{(a, (t_1, t_2)) \mid (a, t_2) \in N_2, s_1 \xrightarrow{a}_{-} t_1\}$.

For AA, conjunction can be defined using auxiliary projection functions $\pi_i: 2^{\Sigma \times S_1 \times S_2} \to 2^{\Sigma \times S_i}$ given by

$$\pi_1(M) = \{(a, s_1) \mid \exists s_2 \in S_2 : (a, s_1, s_2) \in M\}, \pi_2(M) = \{(a, s_2) \mid \exists s_1 \in S_1 : (a, s_1, s_2) \in M\}.$$

Then for AA $\mathcal{A}_1 = (S_1, S_1^0, \operatorname{Tran}_1), \ \mathcal{A}_2 = (S_2, S_2^0, \operatorname{Tran}_2), \ \text{we let } \mathcal{A}_1 \wedge \mathcal{A}_2 = (S, S^0, \operatorname{Tran}), \ \text{with } S = S_1 \times S_2, \ S^0 = S_1^0 \times S_2^0 \ \text{and} \ \operatorname{Tran}_1(s_1, s_2)) = \{M \subseteq \Sigma \times S \mid \pi_1(M) \in \operatorname{Tran}_1(s_1), \pi_2(M) \in \operatorname{Tran}_2(s_2)\}.$

We can also define conjunction for \mathcal{L} -expressions, using similar auxiliary mappings on formulae. For sets X_1 , X_2 and $i \in \{1, 2\}$, we define $\rho_i : \mathcal{L}(X_i) \to \mathcal{L}(X_1 \times X_2)$ inductively, by

- $\rho_i(\mathbf{tt}) = \mathbf{tt}$, $\rho_i(\mathbf{ff}) = \mathbf{ff}$, $\rho_i(\neg \phi) = \neg \rho_i(\phi)$, $\rho_i(\phi_i \land \phi_2) = \rho_i(\phi_i) \land \rho_i(\phi_2)$,
- $\rho_1(\langle a \rangle x_1) = \bigvee_{x_2 \in X_2} \langle a \rangle (x_1, x_2),$
- $\rho_2(\langle a \rangle x_2) = \bigvee_{x_1 \in X_1} \langle a \rangle (x_1, x_2).$

Then, for \mathcal{L} -expressions $\mathcal{E}_1 = (X_1, X_1^0, \Phi_1), \mathcal{E}_2 = (X_2, X_2^0, \Phi_2), \text{ we let } \mathcal{E}_1 \wedge \mathcal{E}_2 = (X_1 \times X_2, X_1^0 \times X_2^0, \Phi) \text{ with } \Phi((x_1, x_2)) = \rho_1(\Phi_1(x_1)) \wedge \rho_2(\Phi_2(x_2)).$

8.22 Lemma. For all DMTS \mathcal{D}_1 , \mathcal{D}_2 , AA \mathcal{A}_1 , \mathcal{A}_2 , and \mathcal{L} -expressions \mathcal{E}_1 , \mathcal{E}_2 , $da(\mathcal{D}_1 \wedge \mathcal{D}_2) = da(\mathcal{D}_1) \wedge da(\mathcal{D}_2)$, $al(\mathcal{A}_1 \wedge \mathcal{A}_2) = al(\mathcal{A}_1) \wedge al(\mathcal{A}_2)$, and $la(\mathcal{E}_1 \wedge \mathcal{E}_2) = la(\mathcal{E}_1) \wedge la(\mathcal{E}_2)$.

Note that above makes no statement about the *ad* translation; due to the change of state space during the translation, equality does not hold here.

Proof: The last two claims follow easily once one notices that for $i \in \{1, 2\}$ and all M, $M \models \rho_i(\phi)$ iff $\pi_i(M) \models \phi$. To show the first claim, let $\mathcal{D}_1 = (S_1, S_1^0, -\rightarrow_1, \longrightarrow_1)$ and $\mathcal{D}_2 = (S_2, S_2^0, -\rightarrow_2, \longrightarrow_2)$ be DMTS, with AA translations $da(\mathcal{D}_1) = (S_1, S_1^0, \operatorname{Tran}_1)$ and $da(\mathcal{D}_2) = (S_2, S_2^0, \operatorname{Tran}_2)$. Write $\mathcal{A}^{\wedge} = da(\mathcal{D}_1 \wedge \mathcal{D}_2)$ and $\mathcal{A}_{\wedge} = da(\mathcal{D}_1) \wedge da(\mathcal{D}_2)$; we show that $\mathcal{A}^{\wedge} = \mathcal{A}_{\wedge}$.

First, remark that \mathcal{A}^{\wedge} and \mathcal{A}_{\wedge} have precisely the same state space $S_1 \times S_2$ and initial states $S_1^0 \times S_2^0$. We now show that they have the same transition constraints. Let $\operatorname{Tran}_{\wedge}$ (resp. $\operatorname{Tran}^{\wedge}$) be the transition constraints mapping of \mathcal{A}_{\wedge} (resp. \mathcal{A}^{\wedge}). Let $(s_1, s_2) \in S_1 \times S_2$ and $M \in \operatorname{Tran}_{\wedge}(s_1, s_2)$.

By construction of $\operatorname{Tran}_{\wedge}$, there must be $M_1 \in \operatorname{Tran}_1(s_1)$ and $M_2 \in \operatorname{Tran}_2(s_2)$ such that $\pi_1(M) = M_1$ and $\pi_2(M) = M_2$. We show that $M \in \operatorname{Tran}^{\wedge}(s_1, s_2)$. Let $(a, (t_1, t_2)) \in M$. Since $\pi_1(M) = M_1$ and $\pi_2(M) = M_2$, we have $(a, t_1) \in M_1$ and $(a, t_2) \in M_2$. As a consequence, there are transitions $s_1 \stackrel{a}{\dashrightarrow} t_1$ and $s_2 \stackrel{a}{\dashrightarrow} t_2$ in \mathcal{D}_1 and \mathcal{D}_2 , respectively. Thus, by construction, there is a transition $(s_1, s_2) \stackrel{a}{\dashrightarrow} (t_1, t_2)$ in $\mathcal{D}_1 \wedge \mathcal{D}_2$.

Let $(s_1, s_2) \longrightarrow N$ in $\mathcal{D}_1 \wedge \mathcal{D}_2$. By construction, N is such that either (1) there exists N_1 such that $s_1 \longrightarrow N_1$ in \mathcal{D}_1 and $N = \{(a, (t_1, t_2)) \mid (a, t_1) \in N_1, (s_1, s_2) \stackrel{a}{\dashrightarrow} (t_1, t_2)\}$, or (2) there exists N_2 such that $s_2 \longrightarrow N_2$ in \mathcal{D}_2 and $N = \{(a, (t_1, t_2)) \mid (a, t_2) \in N_2, (s_1, s_2) \stackrel{a}{\dashrightarrow} (t_1, t_2)\}$. Assume that (1) holds (case (2) being symmetric). Since $M_1 \in \text{Tran}_1(s_1)$, there must be $(a, t_1) \in N_1 \cap M_1$. As $\pi_1(M) = M_1$, there must be $t_2 \in S_2$ such that $(a, (t_1, t_2)) \in M$. As a consequence, there is $(a, (t_1, t_2)) \in M \cap N$.

We have shown that $M \in \operatorname{Tran}^{\wedge}(s_1, s_2)$. Similarly, we can show that for all $M \in \operatorname{Tran}^{\wedge}(s_1, s_2)$, we also have $M \in \operatorname{Tran}_{\wedge}(s_1, s_2)$. We can thus conclude that $\operatorname{Tran}^{\wedge} = \operatorname{Tran}_{\wedge}$, hence $\mathcal{A}^{\wedge} = \mathcal{A}_{\wedge}$.

8.23 Theorem. For all specifications S_1 , S_2 , S_3 ,

- $S_1 \vee S_2 \leq_m S_3$ iff $S_1 \leq_m S_3$ and $S_2 \leq_m S_3$,
- $S_1 \leq_{\mathsf{m}} S_2 \wedge S_3$ iff $S_1 \leq_{\mathsf{m}} S_2$ and $S_1 \leq_{\mathsf{m}} S_3$,
- $[S_1 \vee S_2] = [S_1] \cup [S_2]$, and $[S_1 \wedge S_2] = [S_1] \cap [S_2]$.

Proof: The proof falls into six parts. We show the second statement of the theorem separately for DMTS and for AA. For the other formalisms, the statement then follows by structural equivalence.

- $S_1 \vee S_2 \leq_{\mathsf{m}} S_3$ iff $S_1 \leq_{\mathsf{m}} S_3$ and $S_2 \leq_{\mathsf{m}} S_3$: Any modal refinement $R \subseteq (S_1 \cup S_2) \times S_3$ splits into two refinements $R_1 \subseteq S_1 \times S_3$, $R_2 \subseteq S_2 \times S_3$ and vice versa.
- $\bullet \ \mathcal{S}_1 \leq_{\mathsf{m}} \mathcal{S}_2 \wedge \mathcal{S}_3 \ \textit{is implied by} \ \mathcal{S}_1 \leq_{\mathsf{m}} \mathcal{S}_2 \ \textit{and} \ \mathcal{S}_1 \leq_{\mathsf{m}} \mathcal{S}_3 \text{:}$

We first show this proof for DMTS. Let $S_i = (S_i, S_i^0, -- \cdot_i, \longrightarrow_i)$, for i = 1, 2, 3, be DMTS and $R_2 \subseteq S_1 \times S_2$, $R_3 \subseteq S_1 \times S_3$ modal refinements and define $R = \{(s_1, (s_2, s_3)) \mid (s_1, s_2) \in R_1, (s_1, s_3) \in R_3\} \subseteq S_1 \times (S_2 \times S_3)$. Then R is initialised.

Now let $(s_1, (s_2, s_3)) \in R$, then $(s_1, s_2) \in R_2$ and $(s_1, s_3) \in R_3$. Assume that $s_1 \stackrel{a}{\dashrightarrow} t_1$, then by $S_1 \leq_{\mathsf{m}} S_2$, we have $s_2 \stackrel{a}{\dashrightarrow} t_2$ with $(t_1, t_2) \in R_2$. Similarly, by $S_1 \leq_{\mathsf{m}} S_3$, we have $s_3 \stackrel{a}{\dashrightarrow} t_3$ with $(t_1, t_3) \in R_3$. But then also $(t_1, (t_2, t_3)) \in R$, and $(s_2, s_3) \stackrel{a}{\dashrightarrow} (t_2, t_3)$ by definition.

Assume that $(s_2, s_3) \longrightarrow N$. Without loss of generality we can assume that there is $s_2 \longrightarrow_2 N_2$ such that $N = \{(a, (t_2, t_3)) \mid (a, t_2) \in N_2, s_3 \stackrel{a}{\dashrightarrow}_3 t_3\}$. By $S_1 \leq_{\mathsf{m}} S_2$, we have $s_1 \longrightarrow_1 N_1$ such that

$$\forall (a, t_1) \in N_1 : \exists (a, t_2) \in N_2 : (t_1, t_2) \in R_2 \tag{8.5}$$

Let $(a, t_1) \in N_1$, then also $s_1 \stackrel{a}{\dashrightarrow}_1 t_1$, so by $S_1 \leq_{\mathsf{m}} S_3$, there is $s_3 \stackrel{a}{\dashrightarrow}_3 t_3$ with $(t_1, t_3) \in R_3$. By (8.5), we also have $(a, t_2) \in N_2$ such that $(t_1, t_2) \in R_2$, but then $(a, (t_2, t_3)) \in N$ and $(t_1, (t_2, t_3)) \in R$.

• $S_1 \leq_{\mathsf{m}} S_2 \wedge S_3$ implies $S_1 \leq_{\mathsf{m}} S_2$ and $S_1 \leq_{\mathsf{m}} S_3$:

Let $R \subseteq S_1 \times (S_2 \times S_3)$ be a (DMTS) modal refinement. We show that $S_1 \leq_{\mathsf{m}} S_2$, the proof of $S_1 \leq_{\mathsf{m}} S_3$ being entirely analogous. Define $R_2 = \{(s_1, s_2) \mid \exists s_3 \in S_3 : (s_1, (s_2, s_3)) \in R\} \subseteq S_1 \times S_2$, then R_2 is initialised.

Let $(s_1, s_2) \in R_2$, then we must have $s_3 \in S_3$ such that $(s_1, (s_2, s_3)) \in R$. Assume that $s_1 \stackrel{a}{\dashrightarrow} t_1$, then also $(s_2, s_3) \stackrel{a}{\dashrightarrow} (t_2, t_3)$ and $(t_1, (t_2, t_3)) \in R$. By construction we have $s_2 \stackrel{a}{\dashrightarrow} t_2$ and $s_3 \stackrel{a}{\dashrightarrow} t_3$. Moreover, by definition of R_2 , $(t_1, t_2) \in R_2$.

Assume that $s_2 \longrightarrow_2 N_2$, then by construction of $\mathcal{S}_2 \wedge \mathcal{S}_3$, $(s_2, s_3) \longrightarrow N = \{(a, (t_2, t_3)) \mid (a, t_2) \in N_2, s_3 \stackrel{a}{\dashrightarrow}_3 t_3\}$. By $\mathcal{S}_1 \leq_{\mathsf{m}} \mathcal{S}_2 \wedge \mathcal{S}_3$, we have $s_1 \longrightarrow_1 N_1$ such that $\forall (a, t_1) \in N_1 : \exists (a, (t_2, t_3)) \in N : (t_1, (t_2, t_3)) \in R$.

Let $(a, t_1) \in N_1$, then we have $(a, (t_2, t_3)) \in N$ for which $(t_1, (t_2, t_3)) \in R$. By construction of N, this implies that there are $(a, t_2) \in N_2$ and $s_3 \stackrel{a}{-} \rightarrow_3 t_3$. Moreover, by definition of R_2 , $(t_1, t_2) \in R$.

• $A_1 \leq_{\mathsf{m}} A_2 \wedge A_3$ is implied by $A_1 \leq_{\mathsf{m}} A_2$ and $A_1 \leq_{\mathsf{m}} A_3$:

Now we show the second part of the lemma for AA. Let $\mathcal{A}_i = (\mathcal{S}_i, \mathcal{S}_i^0, \operatorname{Tran}_i)$, for i = 1, 2, 3, be AA and $R_2 \subseteq S_1 \times S_2$, $R_3 \subseteq S_1 \times S_3$ modal refinements and define $R = \{(s_1, (s_2, s_3)) \mid (s_1, s_2) \in R_1, (s_1, s_3) \in R_3\} \subseteq S_1 \times (S_2 \times S_3)$. Then R is initialised.

Let $(s_1, (s_2, s_3)) \in R$, then $(s_1, s_2) \in R_2$ and $(s_1, s_3) \in R_3$. Let $M_1 \in \operatorname{Tran}_1(s_1)$, then we have $M_2 \in \operatorname{Tran}_2(s_2)$ and $M_3 \in \operatorname{Tran}_3(s_3)$ such that the pairs M_1, M_2 and M_1, M_3 verify the conditions (8.2) in Definition 8.7. Let $M = \{(a, (t_2, t_3)) \mid (a, t_2) \in M_2, (a, t_3) \in M_3\} \subseteq \Sigma \times S$.

We show that $\pi_2(M) = M_2$ and, similarly, $\pi_3(M) = M_3$: It is clear that $\pi_2(M) \subseteq M_2$, so let $(a, t_2) \in M_2$. By the refinement R_2 , there is $(a, t_1) \in M_1$, so by the refinement R_3 , there is $(a, t_3) \in M_3$, but then $(a, t_2) \in \pi_2(M)$. Using $\pi_2(M) = M_2$ and $\pi_3(M) = M_3$, we can now conclude that $M \in \text{Tran}((s_2, s_3))$.

Let $(a, t_1) \in M_1$, then we have $(a, t_2) \in M_2$ and $(a, t_3) \in M_3$ such that $(t_1, t_2) \in R_2$ and $(t_1, t_3) \in R_3$. But then $(t_1, (t_2, t_3)) \in R$ and $(a, (t_2, t_3)) \in M$.

Let $(a,(t_2,t_3)) \in M$, then $(a,t_2) \in M_2$ and $(a,t_3) \in M_3$. By the refinements R_2 and R_3 , we have $(a,t_1) \in M_1$ such that $(t_1,t_2) \in R_2$ and $(t_1,t_3) \in R_3$, but then also $(t_1,(t_2,t_3)) \in R$.

• $A_1 \leq_{\mathsf{m}} A_2 \wedge A_3$ implies $A_1 \leq_{\mathsf{m}} A_2$ and $A_1 \leq_{\mathsf{m}} A_3$:

Let $R \subseteq S_1 \times (S_2 \times S_3)$ be a (AA) modal refinement. We show that $\mathcal{A}_1 \leq_{\mathsf{m}} \mathcal{A}_2$; the proof of $\mathcal{A}_1 \leq_{\mathsf{m}} \mathcal{A}_3$ is similar. Define $R_2 = \{(s_1, s_2) \mid \exists s_3 \in S_3 : (s_1, (s_2, s_3)) \in R\} \subseteq S_1 \times S_2$, then R_2 is initialised.

Let $(s_1, s_2) \in R_2$, then there is $t_3 \in S_3$ with $(t_1, (t_2, t_3)) \in R$. Let $M_1 \in \text{Tran}_1(s_1)$, then we have $M \in \text{Tran}_1(s_2, s_3)$ such that the pair M_1, M satisfies conditions (8.2). Let $M_2 = \pi_2(M) \in \text{Tran}_2(s_2)$.

Let $(a, t_1) \in M_1$, then there is $(a, (t_2, t_3)) \in M$ such that $(t_1, (t_2, t_3)) \in R$; hence $(t_1, t_2) \in R$ and $(a, t_2) \in M_2$.

Let $(a, t_2) \in M_2$, then there is $t_3 \in S_3$ such that $(a, (t_2, t_3)) \in M$. But then we also have $(a, t_1) \in M_1$ such that $(t_1, (t_2, t_3)) \in R$, thus $(t_1, t_2) \in R_2$.

• $\llbracket \mathcal{S}_1 \vee \mathcal{S}_2 \rrbracket = \llbracket \mathcal{S}_1 \rrbracket \cup \llbracket \mathcal{S}_2 \rrbracket$ and $\llbracket \mathcal{S}_1 \wedge \mathcal{S}_2 \rrbracket = \llbracket \mathcal{S}_1 \rrbracket \cap \llbracket \mathcal{S}_2 \rrbracket$:

 $[\![\mathcal{S}_1 \wedge \mathcal{S}_2]\!] = [\![\mathcal{S}_1]\!] \cap [\![\mathcal{S}_2]\!]$ is clear from what we just proved: for all implementations \mathcal{I} , $\mathcal{I} \leq_m \mathcal{S}_1 \wedge \mathcal{S}_2$ iff $\mathcal{I} \leq_m \mathcal{S}_1$ and $\mathcal{I} \leq_m \mathcal{S}_2$. For the other part, it is clear by construction that for any implementation \mathcal{I} , any witness R for $\mathcal{I} \leq_m \mathcal{S}_1$ is also a witness for $\mathcal{I} \leq_m \mathcal{S}_1 \vee \mathcal{S}_2$, and similarly for \mathcal{S}_2 , hence $[\![\mathcal{S}_1]\!] \cup [\![\mathcal{S}_2]\!] \subseteq [\![\mathcal{S}_1 \vee \mathcal{S}_2]\!]$.

To show the other inclusion, we note that an initialised refinement R witnessing $\mathcal{I} \leq_{\mathsf{m}} \mathcal{S}_1 \vee \mathcal{S}_2$ must relate the initial state of \mathcal{I} either to an initial state of \mathcal{S}_1 or to an initial state of \mathcal{S}_2 . In the first case, and by disjointness, R witnesses $\mathcal{I} \leq_{\mathsf{m}} \mathcal{S}_1$, in the second, $\mathcal{I} \leq_{\mathsf{m}} \mathcal{S}_2$. Note how it is essential here

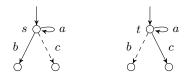


Figure 8.5: DMTS S and T whose composition cannot be captured precisely

that implementations have but one initial state; this part of the proof would break down if we were to allow several initial states for implementations. \Box

8.24 Corollary. With operations \vee and \wedge , each of our four classes of specifications forms a bounded distributive lattice up to $\equiv_{\mathfrak{m}}$.

Proof: The bottom elements (up to \equiv_{m}) in the lattices are given by specifications with empty initial state sets. The top elements are the DMTS $(\{s^0\}, \{s^0\}, \{(s^0, a, s^0) \mid a \in \Sigma\}, \emptyset)$ and its respective translations. The other lattice properties follow from Theorem 8.23.

We miss to verify distributivity. Let $\mathcal{A}_i = (S_i, S_i^0, \operatorname{Tran}_i)$, for i = 1, 2, 3, be AA. The set of states of both $\mathcal{A}_1 \wedge (\mathcal{A}_2 \vee \mathcal{A}_3)$ and $(\mathcal{A}_1 \wedge \mathcal{A}_2) \vee (\mathcal{A}_1 \wedge \mathcal{A}_3)$ is $S_1 \times (S_2 \cup S_3) = S_1 \times S_2 \cup S_1 \times S_3$, and one easily sees that the identity relation is a two-sided modal refinement. Things are similar for the other distributive law.

8.3.2 Composition

The composition operator for a specification theory is to mimic, at specification level, the parallel composition of implementations. That is to say, if \parallel is a composition operator for implementations (LTS), then the goal is to extend \parallel to specifications such that for all specifications S_1 , S_2 ,

$$[S_1 | S_2] = \{ \mathcal{I}_1 | \mathcal{I}_2 | \mathcal{I}_1 \in [S_1], \mathcal{I}_2 \in [S_2] \}.$$
 (8.6)

For simplicity, we use CSP-style synchronisation for parallel composition of LTS, however, our results readily carry over to other types of composition. Analogously to the situation for MTS [BKLS09], we have the following negative result:

8.25 Theorem. There is no operator \parallel , for any of our specification formalisms, which satisfies (8.6).

Proof: We show that there exist DMTS S and T such that there is no DMTS D with $[\![D]\!] = [\![S]\!] | [\![T]\!] := \{ \mathcal{I} | \mathcal{J} \mid \mathcal{I} \in [\![S]\!], \mathcal{J} \in [\![T]\!] \}$. They are given in Figure 8.5; S has initial state s, while T has initial state t. Note that in fact, S and T are MTS, i.e., no disjunctive must transitions are used.

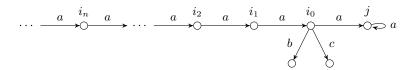


Figure 8.6: Implementation state space in the proof of Theorem 8.25

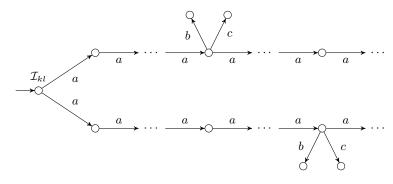


Figure 8.7: The nondeterministic sum of \mathcal{I}_k and \mathcal{I}_l , unfolded

We make the following observations about implementations of S and T. They always admit one or more infinite runs labeled with a's with one-step b or c branches. Moreover, all infinite runs in these implementations are of this form. To each infinite a-run of an implementation we assign its signature, that is a word over $2^{\{b,c\}}$ that describes which one-step branches are available at each step. This means that every implementation of S has runs with signatures from $\{\{b\},\{b,c\}\}^{\omega}$, while every implementation of T has runs with signatures from $\{\{c\},\{b,c\}\}^{\omega}$.

We now construct an implementation state space as illustrated in Figure 8.6. Consider the implementations $\mathcal{I}_1, \mathcal{I}_2, \ldots$ that share the same state space and have the initial state i_1, i_2, \ldots , respectively. The implementation \mathcal{I}_n has only one a-run with the signature $\emptyset^n\{b,c\}\emptyset^\omega$. Note that \mathcal{I}_n is the composition of an implementation of \mathcal{S} that has only one a-run with the signature $\{b\}^n\{b,c\}\{b\}^\omega$ and an implementation of \mathcal{T} that has only one a-run with the signature $\{c\}^n\{b,c\}\{c\}^\omega$.

Assume now that there exists a DMTS \mathcal{D} with $\llbracket \mathcal{D} \rrbracket = \llbracket \mathcal{S} \rrbracket \Vert \llbracket \mathcal{T} \rrbracket$. As all \mathcal{I}_n belong to $\llbracket \mathcal{D} \rrbracket$ and there is only a finite number of initial states of \mathcal{D} , there has to be at least one initial state of \mathcal{D} , say d, such that there exists a modal refinement R containing both (i_k, d) and (i_l, d) for some numbers k < l. Let \mathcal{D}_d be created from \mathcal{D} by changing the set of initial states to the singleton $\{d\}$. As both $\mathcal{I}_k \leq_{\mathsf{m}} \mathcal{D}_d$ and $\mathcal{I}_l \leq_{\mathsf{m}} \mathcal{D}_d$ and \mathcal{D}_d has only one initial state, we know by Lemma 8.20 that also $\mathcal{I}_{kl} = \mathcal{I}_k + \mathcal{I}_l \leq_{\mathsf{m}} \mathcal{D}_d$. The unfolding of this implementation is illustrated in Figure 8.7.

We now argue that $\mathcal{I}_{kl} \notin [S] ||[T]|$. We actually show that it cannot even

be bisimilar to any $\mathcal{I} \parallel \mathcal{J}$ with $\mathcal{I} \in \llbracket \mathcal{S} \rrbracket$ and $\mathcal{J} \in \llbracket \mathcal{T} \rrbracket$. Let us assume that there exist such \mathcal{I} and \mathcal{J} . We make the following observations:

- \mathcal{I} has to contain at least one a-run with signature $\{b\}^k \{b,c\} \{b\}^{\omega}$. Otherwise, it would be impossible to create the \mathcal{I}_k part of \mathcal{I}_{kl} .
- \mathcal{J} has to contain at least one a-run with signature $\{c\}^l\{b,c\}\{c\}^{\omega}$. Otherwise, it would be impossible to create the \mathcal{I}_l part of \mathcal{I}_{kl} .

However, these observations mean that $\mathcal{I} \parallel \mathcal{J}$ contains at least one a-run with signature $\emptyset^k \{c\} \emptyset^{l-k-1} \{b\} \emptyset^{\omega}$. It is thus not bisimilar to \mathcal{I}_{kl} .

Given that we cannot have (8.6), the revised goal is to have a sound composition operator for which the right-to-left inclusion holds in (8.6). For AA $A_1 = (S_1, S_1^0, \operatorname{Tran}_1)$, $A_2 = (S_2, S_2^0, \operatorname{Tran}_2)$, we define $A_1 \| A_2 = (S, S^0, \operatorname{Tran})$ with $S = S_1 \times S_2$, $S^0 = S_1^0 \times S_2^0$, and for all $(s_1, s_2) \in S$, $\operatorname{Tran}(s_1, s_2) = \{M_1 \| M_2 \mid M_1 \in \operatorname{Tran}_1(s_1), M_2 \in \operatorname{Tran}_2(s_2)\}$, where $M_1 \| M_2 = \{(a, (t_1, t_2)) \mid (a, t_1) \in M_1, (a, t_2) \in M_2\}$. Composition for DMTS is defined using the translations to and from AA; note that this may incur an exponential blow-up.

8.26 Lemma. Up to \equiv_{m} , the operator \parallel on AA is associative and commutative, distributes over \vee , and has unit U , where U is the LTS $(\{s\}, s, \longrightarrow)$ with $s \xrightarrow{a} s$ for all $a \in \Sigma$.

Proof: Associativity and commutativity are clear. To show distributivity over \vee , let $\mathcal{A}_i = (S_i, S_i^0, \operatorname{Tran}_i)$, for i = 1, 2, 3, be AA. We prove that $\mathcal{A}_1 \| (\mathcal{A}_1 \vee \mathcal{A}_3) \equiv_{\mathsf{m}} \mathcal{A}_1 \| \mathcal{A}_2 \vee \mathcal{A}_1 \| \mathcal{A}_3$; right-distributivity will follow by commutativity. The state spaces of both sides are $S_1 \times S_2 \cup S_1 \times S_3$, and it is easily verified that the identity relation is a two-sided modal refinement.

For the claim that $\mathcal{A}\|\mathsf{U} \equiv_{\mathsf{m}} \mathcal{A}$ for all AA $\mathcal{A} = (S, S^0, \operatorname{Tran})$, let u be the unique state of U and define $R = \{((s, u), s) \mid s \in S\} \subseteq S \times \mathsf{U} \times S$. We show that R is a two-sided modal refinement. Let $((s, u), s) \in R$ and $M \in \operatorname{Tran}(s, u)$, then there must be $M_1 \in \operatorname{Tran}(s)$ for which $M = M_1\|(\Sigma \times \{u\})$. Thus $M_1 = \{(a, t) \mid (a, (t, u)) \in M\}$. Then any element of M has a corresponding one in M_1 , and vice versa, and their states are related by R. For the other direction, let $M_1 \in \operatorname{Tran}(s)$, then $M = M_1\|(\Sigma \times \{u\}) = \{(a, (t, u)) \mid (a, t) \in M_1\} \in \operatorname{Tran}(s, u)$, and the same argument applies. \square

The next theorem is one of *independent implementability*, as it ensures that a composition of refinements is a refinement of compositions:

8.27 Theorem. For all specifications S_1 , S_2 , S_3 , S_4 , $S_1 \leq_m S_3$ and $S_2 \leq_m S_4$ imply $S_1 \| S_2 \leq_m S_3 \| S_4$.

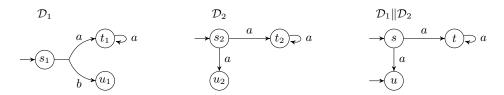


Figure 8.8: Two DMTS and the reachable parts of the DMTS translation of their composition. Here, $s = \{(a, (t_1, t_2)), (a, (t_1, u_2))\}, t = \{(a, (t_1, t_2))\}$ and $u = \emptyset$

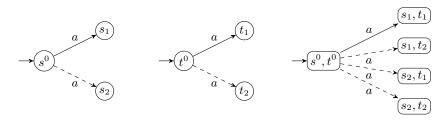


Figure 8.9: Two MTS and their MTS composition according to [Lar89]

Proof: Let $S_1 \leq_m S_3$ and $S_2 \leq_m S_4$, then $S_1 \vee S_3 \equiv_m S_3$ and $S_2 \vee S_4 \equiv_m S_4$. By distributivity,

$$S_3 \| S_4 \equiv_{\mathsf{m}} (S_1 \vee S_3) \| (S_2 \vee S_4)$$

$$\equiv_{\mathsf{m}} S_1 \| S_2 \vee S_1 \| S_4 \vee S_3 \| S_2 \vee S_3 \| S_4 \rangle$$

thus

$$|\mathcal{S}_1||\mathcal{S}_2 \vee \mathcal{S}_1||\mathcal{S}_4 \vee \mathcal{S}_3||\mathcal{S}_2 \leq_{\mathsf{m}} \mathcal{S}_3||\mathcal{S}_4|.$$

But

$$\mathcal{S}_1 \| \mathcal{S}_2 \leq_{\mathsf{m}} \mathcal{S}_1 \| \mathcal{S}_2 \vee \mathcal{S}_1 \| \mathcal{S}_4 \vee \mathcal{S}_3 \| \mathcal{S}_2 ,$$

finishing the argument.

8.28 Example. An example of composition is shown in Figure 8.8. Here the DMTS translation of $\mathcal{D}_1 \| \mathcal{D}_2$ has two initial states; it can be shown that no DMTS with a single initial state is thoroughly equivalent.

Remark that AA composition is more precise than the composition for MTS introduced in [Lar89]. The MTS composition is given by the following rules: $(s_1, s_2) \stackrel{a}{\longrightarrow} (t_1, t_2)$ whenever $s_1 \stackrel{a}{\longrightarrow} t_1$ and $s_2 \stackrel{a}{\longrightarrow} t_2$, $(s_1, s_2) \stackrel{a}{\longrightarrow} (t_1, t_2)$ whenever $s_1 \stackrel{a}{\longrightarrow} t_1$ and $s_2 \stackrel{a}{\longrightarrow} t_2$. The difference between the two compositions is illustrated in Figure 8.9. The figure shows two MTS and their

MTS composition; for their AA composition,

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\operatorname{Tran}(s^{0}, t^{0}) = \{\{(a, (s_{1}, t_{1}))\}, \{(a, (s_{1}, t_{1})), (a, (s_{1}, t_{2}))\}, \{(a, (s_{1}, t_{1})), (a, (s_{2}, t_{1}))\}, \{(a, (s_{1}, t_{1})), (a, (s_{1}, t_{2})), (a, (s_{2}, t_{1})), (a, (s_{2}, t_{2}))\}\}. (8.7)
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The AA translation of their MTS composition has eight transition constraints instead of four; note how the four constraints in (8.7) precisely correspond to the four implementation choices for s^0 and t^0 .

It can easily be shown that generally, AA composition is a refinement of MTS composition. The following lemma shows a stronger relationship, namely that the MTS composition is a conservative approximation of the AA composition.

8.29 Lemma. Let \mathcal{M}_1 , \mathcal{M}_2 , \mathcal{M}_3 be MTS and let $\|_{M}$ and $\|_{A}$ be the MTS and AA composition, respectively. It holds that $\mathcal{M}_1\|_{M}\mathcal{M}_2 \leq_{\mathsf{m}} \mathcal{M}_3$ iff $\mathcal{M}_1\|_{A}\mathcal{M}_2 \leq_{\mathsf{m}} \mathcal{M}_3$.

Proof: Let $\mathcal{M}_i = (S_i^0, \{s_i^0\}, -\rightarrow_i, \longrightarrow_i)$ be MTS for i = 1, 2, 3. In the following, we use the notation $s_1 \parallel_A s_2$ to denote the states (s_1, s_2) of $\mathcal{M}_1 \parallel_A \mathcal{M}_2$ and similarly for \parallel_M .

For an MTS translated into AA, the Tran sets have a special structure, namely, for all states s, $\operatorname{Tran}(s)$ always has a maximal element $\{(a,t) \mid s \stackrel{a}{\longrightarrow} t\}$ and a minimal element $\{(a,t) \mid s \stackrel{a}{\longrightarrow} t\}$ (with respect to set inclusion; cf. Lemma 8.17 for the similar property for DMTS). Furthermore, we note that $\operatorname{Tran}(s_1||_{A}s_2) \subseteq \operatorname{Tran}(s_1||_{M}s_2)$ and, moreover, $\operatorname{Tran}(s_1||_{A}s_2)$ also has a minimal and a maximal element and these elements correspond to the minimal and maximal element of $\operatorname{Tran}(s_1||_{M}s_2)$.

The fact that $\mathcal{M}_1\|_{A}\mathcal{M}_2 \leq_{\mathsf{m}} \mathcal{M}_1\|_{M}\mathcal{M}_2$ follows from the observation that $\operatorname{Tran}(s_1\|_{A}s_2) \subseteq \operatorname{Tran}(s_1\|_{M}s_2)$ for all (s_1, s_2) . This proves the 'only-if' part of the lemma.

To prove the 'if' part of the lemma, we let $R = \{(s_1|_{M}s_2, s_3) \mid s_1|_{A}s_2 \leq_{\mathsf{m}} s_3\}$ and show that it is a modal refinement relation witnessing $\mathcal{M}_1|_{M}\mathcal{M}_2 \leq_{\mathsf{m}} \mathcal{M}_3$. Let $(s_1|_{M}s_2, s_3) \in R$.

- Let $s_1|_{\mathbf{M}}s_2 \xrightarrow{a} t_1|_{\mathbf{M}}t_2$. Then $(a, t_1|_{\mathbf{M}}t_2)$ belongs to the maximal element of $\operatorname{Tran}(s_1|_{\mathbf{M}}s_2)$, which is also in $\operatorname{Tran}(s_1|_{\mathbf{A}}s_2)$. Due to $s_1|_{\mathbf{A}}s_2 \leq_{\mathsf{m}} s_3$ we have some $N \in \operatorname{Tran}(s_3)$ with $(a, t_3) \in N$ such that $t_1|_{\mathbf{A}}t_2 \leq_{\mathsf{m}} t_3$. Thus $s_3 \xrightarrow{a} t_3$ and $(t_1|_{\mathbf{M}}t_2, t_3) \in R$.
- Let $s_3 \stackrel{a}{\longrightarrow} t_3$. Then all elements of $\operatorname{Tran}(s_3)$ contain (a, t_3) . If we now chose the minimal element $M \in \operatorname{Tran}(s_1||_{A}s_2)$ then it has to contain $(a, t_1||_{A}t_2)$ such that $(t_1||_{A}t_2 \leq_{\mathsf{m}} t_3)$. This means that $s_1||_{M}s_2 \stackrel{a}{\longrightarrow} t_1||_{M}t_2$ and $(t_1||_{M}t_2, t_3) \in R$.

8.3.3 Quotient

The quotient operator for a specification theory is used to synthesise specifications for components of a composition. Hence it is to have the property, for all specifications \mathcal{S} , \mathcal{S}_1 and all implementations \mathcal{I}_1 , \mathcal{I}_2 , that

$$\mathcal{I}_1 \in \llbracket \mathcal{S}_1 \rrbracket \text{ and } \mathcal{I}_2 \in \llbracket \mathcal{S}/\mathcal{S}_1 \rrbracket \text{ imply } \mathcal{I}_1 \Vert \mathcal{I}_2 \in \llbracket \mathcal{S} \rrbracket.$$
 (8.8)

Furthermore, $\mathcal{S}/\mathcal{S}_1$ is to be as permissive as possible.

Quotient for MTS

Before we describe the general construction of the quotient, we start with a simpler construction that works for the important special case of MTS. However, MTS are not closed under quotient, *cf.* [Lar90a, Thm. 5.5]; we show that the quotient of two MTS will generally be a DMTS.

Recall that MTS have only one initial state and all their must transitions are singletons. Let $\mathcal{M}_1 = (S_1, s_1^0, -\rightarrow_1, \longrightarrow_1)$ and $\mathcal{M}_2 = (S_2, s_2^0, -\rightarrow_2, \longrightarrow_2)$ be MTS. We define $\mathcal{M}_1/\mathcal{M}_2 = (S, s^0, -\rightarrow, \longrightarrow)$ with $S = 2^{S_1 \times S_2}$, $s^0 = \{(s_1^0, s_2^0)\}$, and the transition relations given as follows.

For $s = \{(s_1^1, s_2^1), \dots, (s_1^n, s_2^n)\} \in S$ we say that $a \in \Sigma$ is permissible from s if for all $i = 1, \dots, n$ either $s_1^i \stackrel{a}{\dashrightarrow}$ or $s_2^i \stackrel{a}{\leadsto}$.

For a permissible from s and $i \in \{1, \ldots, n\}$, let $\{t_2^{i,1}, \ldots, t_2^{i,m_i}\} = \{t_2 \in S_2 \mid s_2^i \xrightarrow{a} t_2\}$ be an enumeration of the possible states in S_2 after an a-transition from s_2^i . We then define the set of possible transitions from s under a as $pt_a(s) = \{\{(t_1^{i,j}, t_2^{i,j}) \mid i = 1, \ldots, n, j = 1, \ldots, m_i\} \mid \forall i, j : s_1^i \xrightarrow{a} t_1^{i,j}\}$. The transitions of s are now given as follows: for every a permissible from

The transitions of s are now given as follows: for every a permissible from s and every $t \in pt_a(s)$, let $s \stackrel{a}{\dashrightarrow} t$. Furthermore, for every $s_1^i \stackrel{a}{\longrightarrow} t_1$ let $s \longrightarrow \{(a, M) \in \{a\} \times pt_a(s) \mid \exists t_2 : (t_1, t_2) \in M, s_2^i \stackrel{a}{\longrightarrow} t_2\}.$

Note that as a special case we obtain $\emptyset \xrightarrow{a} \emptyset$ for all $a \in \Sigma$ and there are no must transitions from \emptyset .

8.30 Example. We illustrate the construction on an example. Let S and T be the MTS on the top of Figure 8.10. We construct S/T, displayed below; this can be further simplified into the system on the bottom.

First we construct the may-successors of s^0/t^0 . Both b and c are admissible due to $t^0 \stackrel{b}{\leadsto}$ and $t^0 \stackrel{c}{\leadsto}$ and thus with $pt(_)b(s^0/t^0) = pt(_)c(s^0/t^0) = \{\emptyset\}$. Consequently, the only successor here is \emptyset . Further, a is also admissible due to $s^0 \stackrel{a}{\dashrightarrow}$. For may-transitions under a, we have to consider all mappings of successors of t^0 to successors of s^0 , namely $\{s_1/t_1, s_1/t_2\}$, $\{s_1/t_1, s_2/t_2\}$, $\{s_2/t_1, s_1/t_2\}$, and $\{s_2/t_1, s_2/t_2\}$. Besides, since there is a must-transition from s^0 (to s_1), we create a disjunctive must-transition to all successors that can be used to yield this must-transition $s^0 \stackrel{a}{\longrightarrow} s_1$ when composed with the must-transition $t^0 \stackrel{a}{\longrightarrow} t_1$. These are all successors where t_1 is mapped to s_1 , hence the first two.

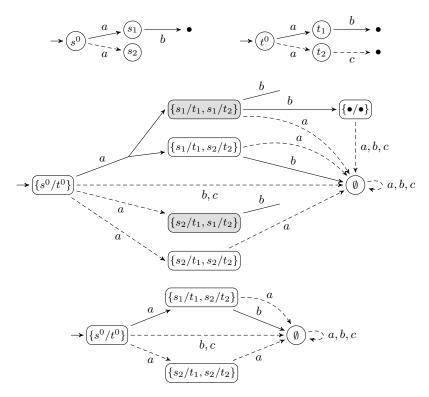


Figure 8.10: Two nondeterministic MTS, their quotient, and its simplification by pruning

Further, $\{s_1/t_1, s_2/t_2\}$ is obliged to have a must under b so that it refines s_1 when composed with t_1 , but cannot have any c in order to match s_2 when composed with t_2 . Similarly, $\{s_2/t_1, s_2/t_2\}$ has neither c nor b.

The first and third successor of s^0/t^0 deserve special attention. Firstly, $\{s_1/t_1, s_1/t_2\}$ has (apart from a may-transition under a) transitions under b: may to $\{\bullet/\bullet\}$ and two musts. Both musts are due to s_1 , but the first one because of s_1 in s_1/t_1 (leading to $\{\bullet/\bullet\}$) and the second one because of s_1 in s_1/t_2 (leading to an empty disjunction because $t_2 \not\longrightarrow$). The empty disjunction is drawn as a line not branching anywhere. Note that it is very different from a may-transition to \emptyset and cannot be implemented. States with such a transition are drawn in gray here and called *inconsistent*.

Secondly, $\{s_2/t_1, s_1/t_2\}$ is inconsistent for the same reason: it requires to refine s_1 by a composition with t_2 . As t_2 has no must under b, the composition has none either, hence the must of s_1 can never be matched.

We have seen that the construction may produce empty must-disjunctions and thus also inconsistent states, i.e., states s such that $s \longrightarrow \emptyset$. Since inconsistent states have no implementations, their presence in the system is useless and we can remove them from the system using the procedure of *prun*-

ing. This procedure produces a more readable system that has the same set of implementations and, moreover, is modally refining the original system. The procedure is standard for MTS, see e.g. [BJL+12a]; here we describe its straightforward adaptation for DMTS. The procedure exhaustively repeats the following: if there is an inconsistent state s, then remove it together with all its outgoing transitions (both may and must) and incoming may-transitions, and each remaining must-transition $t \longrightarrow D$ is modified into $t \longrightarrow D \setminus (\Sigma \times \{s\})$; intuitively, we are removing the incoming must-branches (not the whole transitions). This may of course turn other states inconsistent and thus the procedure is repeated until there are no more inconsistent states.

8.31 Example. When we apply pruning to the quotient in the previous example, we obtain the system on the bottom of Figure 8.10. Here the gray inconsistent states are removed and the disjunctive must from $\{s^0/t^0\}$ leads only to $\{s_1/t_1, s_2/t_2\}$.

Now it is easy to see that $T||(S/T) \equiv_{\mathsf{m}} S$ in this case.

Recall from Lemma 8.29 that the MTS composition is a conservative approximation to the AA composition. This means that the following theorem holds regardless of which of the two compositions is used.

8.32 Theorem. For all MTS specifications \mathcal{M}_1 , \mathcal{M}_2 and \mathcal{M}_3 , $\mathcal{M}_1 \| \mathcal{M}_2 \leq_m \mathcal{M}_3$ iff $\mathcal{M}_2 \leq_m \mathcal{M}_3 / \mathcal{M}_1$.

Proof: In this proof only, let \parallel denote MTS composition.

Write $\mathcal{M}_i = (S_i, s_i^0, -\rightarrow_i, \longrightarrow_i)$ for i = 1, 2, 3. We use the following notation to help distinguish states of $\mathcal{M}_1 \| \mathcal{M}_2$ and $\mathcal{M}_3 / \mathcal{M}_1$. The states of $\mathcal{M}_1 \| \mathcal{M}_2$ are denoted by $s_1 \| s_2$ instead of (s_1, s_2) while the states of $\mathcal{M}_3 / \mathcal{M}_1$ are denoted by $\{s_3/s_1, \ldots\}$ instead of $\{(s_3, s_1), \ldots\}$. We also note that for states of $\mathcal{M}_3 / \mathcal{M}_1$, $s \supseteq t$ implies $s \leq_{\mathsf{m}} t$ due to the construction.

Now assume that $\mathcal{M}_2 \leq_{\mathsf{m}} \mathcal{M}_3/\mathcal{M}_1$ and let $R = \{(s_1 \| s_2, s_3) \mid s_2 \leq_{\mathsf{m}} \{s_3/s_1\}\}$. We show that R is a witness for $\mathcal{M}_1 \| \mathcal{M}_2 \leq_{\mathsf{m}} \mathcal{M}_3$, *i.e.*, that it satisfies the conditions of Definition 8.2. Let $(s_1 \| s_2, s_3) \in R$.

- Let $s_1 \| s_2 \xrightarrow{a} t_1 \| t_2$. As $s_2 \leq_{\mathsf{m}} \{s_3/s_1\}$ this means that $\{s_3/s_1\} \xrightarrow{a} \{t_3^1/t_1^1, \ldots, t_3^k/t_1^k\} = t$ and $t_2 \leq_{\mathsf{m}} t$. Due to the construction of $\{s_3/s_1\}$, we know that there is an index j for which $t_1^j = t_1$ and $s_3 \xrightarrow{a} t_3^j$. Let $t_3 = t_3^j$. As $t \supseteq \{t_3/t_1\}$, $t \leq_{\mathsf{m}} \{t_3/t_1\}$. Therefore, since $t_2 \leq_{\mathsf{m}} t$, we have $t_2 \leq_{\mathsf{m}} \{t_3/t_1\}$ and thus $(t_1 \| t_2, t_3) \in R$.
- Let $s_3 \stackrel{a}{\longrightarrow} t_3$. This means that $\{s_3/s_1\} \longrightarrow U = \{(a,u) \in \{a\} \times pt_a(\{s_3/s_1\}) \mid \exists t_1 : t_3/t_1 \in u, s_1 \stackrel{a}{\longrightarrow} t_1\}$. As $s_2 \leq_{\mathsf{m}} \{s_3/s_1\}$, we know that $s_2 \stackrel{a}{\longrightarrow} t_2$ and $t_2 \leq_{\mathsf{m}} u$ for some $(a,u) \in U$. Due to the construction of U we know that there exists t_1 such that $t_3/t_1 \in u$ with $s_1 \stackrel{a}{\longrightarrow} t_1$. Thus $s_1||s_2 \stackrel{a}{\longrightarrow} t_1||t_2$. Again, as $u \supseteq \{t_3/t_1\}$, $t_2 \leq_{\mathsf{m}} \{t_3/t_1\}$. Therefore, $(t_1 || t_2, t_3) \in R$.

Assume, for the other direction of the proof, that $\mathcal{M}_1 \parallel \mathcal{M}_2 \leq_{\mathsf{m}} \mathcal{M}_3$. Define

$$R = \{(s_2, \{s_3^1/s_1^1, \dots, s_3^n/s_1^n\}) \mid \forall i = 1, \dots, n : s_1^i \parallel s_2 \leq_{\mathsf{m}} s_3^i\};$$

note that $(s_2,\emptyset) \in R$ for all $s_2 \in S_2$. We show that R is a witness for $\mathcal{M}_2 \leq_{\mathsf{m}} \mathcal{M}_3/\mathcal{M}_1$. Let $(s_2,s) \in R$ with $s = \{s_3^1/s_1^1, \ldots, s_3^n/s_1^n\}$.

- Let $s_2 \stackrel{a}{\longrightarrow} t_2$. If there is no i such that $s_1^i \stackrel{a}{\longrightarrow}$ then $s \stackrel{a}{\longrightarrow} \emptyset$ and $(t_2, \emptyset) \in R$. Otherwise, for each $i \in \{1, \ldots, n\}$ and each $j \in \{1, \ldots, m_i\}$ such that $s_1^i \stackrel{a}{\longrightarrow} t_1^{i,j}$ consider that we have $s_1^i \| s_2 \stackrel{a}{\longrightarrow} t_1^{i,j} \| t_2$ and as $s_1^i \| s_2 \leq_{\mathsf{m}} s_3^i$ we also have a corresponding $s_3^i \stackrel{a}{\longrightarrow} t_3^{i,j}$ with $t_1^{i,j} \| t_2 \leq_{\mathsf{m}} t_3^{i,j}$. We fix these $t_3^{i,j}$ for each i and j. Let $t = \{t_3^{i,j}/t_1^{i,j} \mid i \in \{1, \ldots, n\}, j \in \{1, \ldots, m_i\}\}$. Clearly, $s \stackrel{a}{\longrightarrow} t$ and $(t_2, t) \in R$.
- Let s oup U (note that this means that $s \neq \emptyset$) and let $s_3^i \overset{a}{\to} t_3^i$ be the corresponding must transition in the construction. As $s_1^i \| s_2 \leq_{\mathsf{m}} s_3^i$, this means that $s_2 \overset{a}{\to} t_2$ and $s_1^i \overset{a}{\to} t_1^i$ such that $t_1^i \| t_2 \leq_{\mathsf{m}} t_3^i$. This also means that $s_2 \overset{a}{\to} t_2$. We thus build t as we did in the previous case where for i, j such that $t_1^{i,j} = t_1^i$ we choose the corresponding $t_3^{i,j}$ to be t_3^i . Clearly $(t_2, t) \in R$.

Quotient for AA

We now introduce the general quotient operator for AA. The construction is similar to the previous one, with the notions of permissibility and $pt_a(s)$ adapted to the more general setting.

Let $\mathcal{A}_1 = (S_1, S_1^0, \operatorname{Tran}_1)$, $\mathcal{A}_2 = (S_2, S_2^0, \operatorname{Tran}_2)$ be AA and define $\mathcal{A}_1/\mathcal{A}_2 = (S, S^0, \operatorname{Tran})$, with $S = 2^{S_1 \times S_2}$. To define the set of initial states, let us first enumerate the initial states of \mathcal{A}_2 as follows: $S_2^0 = \{s_2^{0,1}, \dots, s_2^{0,p}\}$. The set of initial states is given by all possible assignments of states from S_1^0 to states of S_2^0 , formally: $S^0 = \{\{(s_1^{0,q}, s_2^{0,q}) \mid q \in \{1, \dots, p\}\} \mid \forall q : s_1^{0,q} \in S_1^0\}$. The assignment of transition constraints Tran is given as follows. Let

The assignment of transition constraints Tran is given as follows. Let $\operatorname{Tran}(\emptyset) = 2^{\Sigma \times \{\emptyset\}}$. For $s = \{(s_1^1, s_2^1), \dots, (s_1^n, s_2^n)\} \in S$, say that $a \in \Sigma$ is permissible from s if it holds for all $i = 1, \dots, n$ that there is $M_1 \in \operatorname{Tran}_1(s_1^i)$ and $t_1 \in S_1$ for which $(a, t_1) \in M_1$, or else there is no $M_2 \in \operatorname{Tran}_2(s_2^i)$ and $t_2 \in S_2$ with $(a, t_2) \in M_2$.

Let us now fix a nonempty $s = \{(s_1^1, s_2^1), \dots, (s_1^n, s_2^n)\} \in S$. We introduce some notation that we are going to use throughout this construction and the following proof to denote the successor states of s_2^i for all i. For each s_2^i let $\operatorname{Tran}_2(s_2^i) = \{M_2^{i,1}, \dots, M_2^{i,m_i}\}$ be a fixed enumeration of $\operatorname{Tran}_2(s_2^i)$. For each a permissible from s and for each $M_2^{i,j}$ we further fix an enumeration of all states in $M_2^{i,j}$ after an a-transition: $\{t_2 \in S_2 \mid (a,t_2) \in M_2^{i,j}\} = \{t^{i,j,a,1}, \dots, t^{i,j,a,r_{i,j,a}}\}$. This means that $t^{i,j,a,k}$ is the kth state (out of $r_{i,j,a}$)

with an a-transition in $M_2^{i,j}$, which is the jth member of ${\rm Tran}_2(s_2^i)$, where s_2^i is the state in the jth pair in s.

For a permissible from s, we define

$$pt_a(s) = \{ \{ (t_1^{i,j,a,k}, i, j, t_2^{i,j,a,k}) \mid i = 1, \dots, n, j = 1, \dots, m_i, k = 1, \dots, r_{i,j,a} \}$$
$$\mid \forall i, j, k : \exists M_1 \in \text{Tran}_1(s_1^i) : (a, t_1^{i,j,a,k}) \in M_1 \},$$

the set of all sets of possible assignments of next-a states from s_1^i to next-a states from s_2^i . Note that unlike the case of MTS quotient, we also keep the indices i, j in the assignments. We further define $pt(s) = \{(a, x) \mid a \text{ permissible from } s, x \in pt_a(s)\}$.

To deal with the elements of $pt_a(s)$ we define the following auxiliary operations. The first operation ℓ allows us to "forget" the indices i, j and is defined as follows: $\ell(x) = \{(t_1, t_2) \mid \exists i, j : (t_1, i, j, t_2) \in x\}$. The operation can be naturally lifted to subsets of pt(s) as follows: $\ell(N) = \{(a, \ell(x)) \mid (a, x) \in N\}$ where $N \subseteq pt(s)$.

The second operation is a type of projection that given the two indices $i,\ j$ and the state $t_2^{i,j,a,k}$ produces the next-a state of s_1 assigned in the given element x of $pt_a(s)$. Note that the projection is defined uniquely: $x \upharpoonright (i,j,t_2^{i,j,a,k}) = t_1^{i,j,a,k}$ where $x = \{\dots,(t_1^{i,j,a,k},i,j,t_2^{i,j,a,k}),\dots\}$. The projection operation can also be lifted to subsets N of pt(s) and sets $M_2 \in \operatorname{Tran}_2(s_i^2)$ as follows: $N \upharpoonright (i,j,M_2) = \{(a,x \upharpoonright (i,j,t_2)) \mid (a,t_2) \in M_2\}$. Note that the result of this operation is then a set of elements of the form (a,t_3) where t_3 is an a-successor of s_3 .

Having the two auxiliary operations, we can then finally define

$$\operatorname{Tran}(s) = \{ \ell(N) \mid N \subseteq pt(s), \forall i \in \{1, \dots, n\}, j \in \{1, \dots, m_j\} : \\ N \upharpoonright (i, j, M_2^{i,j}) \in \operatorname{Tran}_3(s_3^i) \}. \quad (8.9)$$

8.33 Theorem. For all specifications S_1 , S_2 , S_3 , $S_1 \| S_2 \leq_m S_3 \| S_2 \leq_m S_3 / S_1 \| S_2 \leq_m S_3 \| S_3 \|$

Proof: We show the proof for AA. Let $\mathcal{A}_1 = (S_1, S_1^0, \operatorname{Tran}_1)$ and $\mathcal{A}_2 = (S_2, S_2^0, \operatorname{Tran}_2)$, $\mathcal{A}_3 = (S_3, S_3^0, \operatorname{Tran}_3)$; we show that $\mathcal{A}_1 \| \mathcal{A}_2 \leq_{\mathsf{m}} \mathcal{A}_3$ iff $\mathcal{A}_2 \leq_{\mathsf{m}} \mathcal{A}_3 / \mathcal{A}_1$.

We use the notation introduced in the proof of Theorem 8.32, *i.e.*, $s_1||s_2$ instead of (s_1, s_2) when speaking about states of $\mathcal{A}_1||\mathcal{A}_2$ and $\{s_3/s_1, \ldots\}$ instead of $\{(s_3, s_1), \ldots\}$ when speaking about states of $\mathcal{A}_3/\mathcal{A}_1$. We further note that by construction, $s \supseteq t$ implies $s \subseteq_{\mathsf{m}} t$ for all $s, t \in 2^{S_3 \times S_1}$.

Now assume that $\mathcal{A}_2 \leq_{\mathsf{m}} \mathcal{A}_3/\mathcal{A}_1$ and let $R = \{(s_1 || s_2, s_3) \mid s_2 \leq_{\mathsf{m}} \{s_3/s_1\}\};$ we show that R is a witness for $\mathcal{A}_1 || \mathcal{A}_2 \leq_{\mathsf{m}} \mathcal{A}_3$.

Let $(s_1||s_2, s_3) \in R$ and $M_{\parallel} \in \operatorname{Tran}_{\parallel}(s_1||s_2)$. Then $M_{\parallel} = M_1||M_2$ with $M_1 \in \operatorname{Tran}_1(s_1)$ and $M_2 \in \operatorname{Tran}_2(s_2)$. As $s_2 \leq_{\mathsf{m}} \{s_3/s_1\}$, we can pair M_2 with an $M_{/} \in \operatorname{Tran}_{/}(\{s_3/s_1\})$, such that the conditions in (8.2) are satisfied (see Definition 8.7).

Note that, as $M_1 \in \text{Tran}_1(s_1)$, there exists some j such that $M_1 = M_1^{1,j}$, i.e., M_1 is the jth element of $\text{Tran}(s_1)$ in the enumeration as described in the construction of the quotient. Further note that $M_{\parallel} = \ell(N)$ for some $N \subseteq pt(\{s_3/s_1\})$ satisfying the conditions in (8.9).

We now define $M_3 = N \upharpoonright (1, j, M_1)$ and show that (8.2) holds for the pair M_{\parallel}, M_3 :

- Let $(a, t_1 || t_2) \in M_{\parallel}$, then there are $(a, t_1) \in M_1$ and $(a, t_2) \in M_2$. By (8.2) applied to the pair M_2 , $M_{/}$, there is $(a, t) \in M_{/}$ such that $t_2 \leq_{\mathsf{m}} t$. This means that there is $(a, x) \in N$ such that $t = \ell(x)$. Due to the construction of $pt(\{s_3/s_1\})$ there has to be some t_3 such that $(t_3, 1, j, t_1) \in x$. Due to the definition of M_3 , this means that $(a, t_3) \in M_3$. We also know that $t \supseteq \{t_3/t_1\}$, hence $t \leq_{\mathsf{m}} \{t_3/t_1\}$, and together with $t_2 \leq_{\mathsf{m}} t$ we get $t_2 \leq_{\mathsf{m}} \{t_3/t_1\}$. Thus $(t_1 || t_2, t_3) \in R$.
- Let $(a,t_3) \in M_3$. This means that there is some $(a,x) \in N$ with $(t_3,1,j,t_1) \in x$ and $(a,t_1) \in M_1$. Therefore, $\ell(x) = t \in M_{/}$ with $t_3/t_1 \in t$. Due to (8.2) applied to M_2 , $M_{/}$, there has to be a corresponding $(a,t_2) \in M_2$ such that $t_2 \leq_{\mathsf{m}} t$. Thus $(a,t_1||t_2) \in M_{||}$ and again by $t \supseteq \{t_3/t_1\}$ we have $t_2 \leq_{\mathsf{m}} \{t_3/t_1\}$ and hence $(t_1||t_2,t_3) \in R$.

It remains to show that R is initialised. Let $\hat{s}_1^0 \| \hat{s}_2^0$ be an initial state of $\mathcal{A}_1 \| \mathcal{A}_2$. By $\mathcal{A}_2 \leq_{\mathsf{m}} \mathcal{A}_3 / \mathcal{A}_1$ we know that $\hat{s}_2^0 \leq_{\mathsf{m}} s^0$ for some $s^0 \in S^0$. We then take $\hat{s}_3^0 \in S_3^0$ such that $\hat{s}_3^0 / \hat{s}_1^0 \in s^0$ (there has to be exactly one due to the definition of s^0). We then have $s^0 \supseteq \{\hat{s}_3^0 / \hat{s}_1^0\}$ and thus $s^0 \leq_{\mathsf{m}} \{\hat{s}_3^0 / \hat{s}_1^0\}$. This means that $\hat{s}_2^0 \leq_{\mathsf{m}} \{\hat{s}_3^0 / \hat{s}_1^0\}$ and hence $(\hat{s}_1^0 \| \hat{s}_2^0, \hat{s}_3^0) \in R$.

Assume, for the other direction of the proof, that $A_1 \| A_2 \leq_{\mathsf{m}} A_3$. Define $R \subseteq S_2 \times 2^{S_3 \times S_1}$ by

$$R = \{(s_2, \{s_3^1/s_1^1, \dots, s_3^n/s_1^n\}) \mid \forall i = 1, \dots, n: s_1^i \| s_2 \leq_{\mathsf{m}} s_3^i\} \,;$$

we show that R is a witness for $A_2 \leq_{\mathsf{m}} A_3/A_1$. We first note that $(s_2,\emptyset) \in R$ for all s_2 . Let now $(s_2,s) \in R$, with nonempty $s = \{s_3^1/s_1^1, \ldots, s_3^n/s_1^n\}$, and $M_2 \in \operatorname{Tran}_2(s_2)$.

Note that for every $M_1^{i,j} \in \operatorname{Tran}(s_1^i)$ we can build $M_{\parallel}^{i,j} = M_1^{i,j} \| M_2$, and as $s_1^i \| s_2 \leq_{\mathsf{m}} s_3^i$, there has to be a corresponding $M_3 \in \operatorname{Tran}(s_3^i)$ satisfying the conditions of (8.2). We fix such M_3 for every i, j and denote it by $M_3^{i,j}$.

We are going to build a subset N of pt(s). To that end, we first define an auxiliary notion of an adequate element of pt(s) with respect to $(a, t_2) \in M_2$ as follows. Let $(a, x) \in pt(s)$. We say that (a, x) is adequate w.r.t. (a, t_2) if for every $i \in \{1, \ldots, n\}, j \in \{1, \ldots, m_i\}$, and $k \in \{1, \ldots, r_{i,j,a}\}$, the projection $t_3 = x \upharpoonright (i, j, t_1^{i,j,k})$ satisfies $(a, t_3) \in M_3^{i,j}$ and $t_1^{i,j,k} \parallel t_2 \leq_m t_3$.

Clearly, if we have (a, x) adequate w.r.t. (a, t_2) , then $(t_2, \ell(x)) \in R$. We can now define

$$N = \{(a, x) \in pt(s) \mid \exists (a, t_2) \in M_2 : (a, x) \text{ is adequate w.r.t. } (a, t_2)\}.$$

We first need to show that $\ell(N) \in \operatorname{Tran}_{/}(s)$. Let $i \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,m_j\}$. We want to show that $N \upharpoonright (i,j,M_1^{i,j}) = M_3^{i,j}$. Let first $(a,t_3) \in N \upharpoonright (i,j,M_1^{i,j})$. This means that there is some $(a,x) \in N$ with $x = (t_3,i,j,t_1^{i,j,k})$. Due to the definition of $N, (a,t_3) \in M_3^{i,j}$. Let now $(a,t_3) \in M_3^{i,j}$. Recall that the pair $M_1^{i,j} \| M_2, M_3^{i,j}$ satisfies (8.2). This means that for $(a,t_3) \in M_3^{i,j}$ there exists $(a,t_1^{i,j,k} \| t_2) \in M_1^{i,j} \| M_2$ such that $t_1^{i,j,k} \| t_2 \leq_{\mathsf{m}} t_3$. Hence $(a,(t_3,i,j,t_1^{i,j,k}))$ is adequate w.r.t. (a,t_2) and thus $(a,t_3) \in N \upharpoonright (i,j,M_1^{i,j})$.

We now show that the pair M_2 , $M = \ell(N)$ satisfies the conditions of (8.2).

- Let $(a,t_2) \in M_2$. We need to show that there exists $(a,x) \in N$ adequate w.r.t. (a,t_2) . Recall that the pair $M_1^{i,j} \| M_2, M_3^{i,j}$ satisfies (8.2) for every i, j. For every $(a,t_1^{i,j,k} \| t_2)$ there thus has to be $(a,t_3) \in M_3^{i,j}$ with $t_1^{i,j,k} \| t_2 \leq_{\mathsf{m}} t_3$. We fix such t_3 for every i, j, k and denote it by $t_3^{i,j,k}$. We then set $x = \{(t_3^{i,j,k}, i, j, t_1^{i,j,k} \mid i \in \{1, \dots, n\}, j \in \{1, \dots, m_i\}, k \in \{1, \dots, r_{i,j,a}\}$. Clearly $(a,x) \in N$ and (a,x) is adequate w.r.t. (a,t_2) . As noted above, we have $(t_2, \ell(x)) \in R$.
- Let $(a,t) \in M$. This means that there is some $(a,x) \in N$ such that $t = \ell(x)$. Due to the definition of N, there exists $(a,t_2) \in M_2$ such that (a,x) is adequate w.r.t. (a,t_2) . Again, as noted above, we have $(t_2,t)=(t_2,\ell(x))\in R$.

It remains to show that R is initialised. Let s_2^0 be an initial state of \mathcal{A}_2 . By $\mathcal{A}_1 \| \mathcal{A}_2 \leq_{\mathsf{m}} \mathcal{A}_3$ we know that for every $s_1^{0,q} \in S_1^0$ (recall the enumeration of initial states in the construction of the quotient) there exists $s_3^0 \in S_3^0$ such that $s_1^{0,q} \| s_2^0 \leq_{\mathsf{m}} s_3^0$. Let us fix for every q such s_3^0 and denote it by $s_3^{0,q}$. Let then $s^0 = \{s_3^{0,q}/s_1^{0,q} \mid q \in \{1,\ldots,p\}\}$. Clearly, s^0 is an initial state of $\mathcal{A}_3/\mathcal{A}_1$ and $(s_2^0,s^0) \in R$.

As a corollary, we get (8.8): If $\mathcal{I}_2 \in [\![\mathcal{S}/\mathcal{S}_1]\!]$, *i.e.*, $\mathcal{I}_2 \leq_m \mathcal{S}/\mathcal{S}_1$, then $\mathcal{S}_1 \| \mathcal{I}_2 \leq_m \mathcal{S}$, which using $\mathcal{I}_1 \leq_m \mathcal{S}_1$ and Theorem 8.27 implies $\mathcal{I}_1 \| \mathcal{I}_2 \leq_m \mathcal{S}_1 \| \mathcal{I}_2 \leq_m \mathcal{S}$. The reverse implication in Theorem 8.33 implies that $\mathcal{S}/\mathcal{S}_1$ is as permissive as possible.

8.34 Corollary. With operations \land , \lor , \parallel and /, each of our four classes of specifications forms a commutative residuated lattice up to \equiv_m .

Proof: We have already seen in Corollary 8.24 that the class of AA forms a lattice, up to \equiv_m , under \land and \lor , and by Theorem 8.33, / is the residual, up to \equiv_m , of \parallel . All other properties (such as distributivity of \parallel over \lor or $\mathcal{N}\parallel\perp\equiv_m\perp$) follow.

8.4 Related Work

The modal ν -calculus is equivalent to the Hennessy-Milner logic with greatest fixed points, which arises from Hennessy-Milner logic (HML) [HM85] by introducing variables and greatest fixed points. If also least fixed points are allowed, one arrives at the full modal μ -calculus [SdB69, Pra81, Koz83]. Janin and Walukiewicz have in [JW95] introduced an automata-like representation for the modal μ -calculus which seems related to our AA.

DMTS have been proposed as solutions to algebraic process equations in Larsen and Xinxin's [LX90b] and further investigated also as a specification formalism [Lar90a, BCK11]. The DMTS formalism is a member of the modal transition systems (MTS) family and as such has also received attention recently. The MTS formalisms have proven to be useful in practice. Industrial applications started with Bruns' [Bru97] where MTS have been used for an air-traffic system at Heathrow airport. Besides, MTS classes are advocated as an appropriate base for interface theories by Raclet et.al. in [RBB+09] and for product line theories in Nyman's [Nym08]. Further, an MTS based software engineering methodology for design via merging partial descriptions of behavior has been established by Uchitel and Chechik in [UC04] and methods for supervisory control of MTS shown by Darondeau et.al. in [DDM10]. Tool support is quite extensive, e.g., [BLS95, DFFU07, BML11, KS13b].

Over the years, many extensions of MTS have been proposed, surveyed in more detail in [Kře17, FLLT14, FLT14a]. While MTS can only specify whether or not a particular transition is required, some extensions equip MTS with more general abilities to describe what *combinations* of transitions are possible. These include DMTS [LX90b], Fecher and Schmidt's 1-MTS [FS08] allowing to express exclusive disjunction, OTS [BK10] capable of expressing positive Boolean combinations, and Boolean MTS [BKL+11] covering all Boolean combinations. The last one is closely related to our AA as well as hybrid modal logic [Pri68, Bla00]. Our results show that all these formalisms are at most as expressive as DMTS.

Larsen has shown in [Lar89] that any finite acyclic MTS is equivalent to a HML formula (without recursion or fixed points), the *characteristic* formula of the given MTS, cf. (8.3). Conversely, Boudol and Larsen show in [BL92] that any consistent and prime HML formula is equivalent to a MTS.³ Here we extend these results to ν -calculus formulae, and show that any such formula is equivalent to a DMTS, solving a problem left open in [LX90b]. Hence the modal ν -calculus supports full compositionality and decomposition in the sense of [Lar90a]. This finishes some of the work started in [Lar89, BL92, Lar90a]. Recently, the graphical representability of a variant of alternating simulation called covariant-contravariant simulation has been studied

 $^{^3\}mathrm{A}$ HML formula is prime if implying a disjunction means implying one of the alternatives.

in $[AFdFE^+13]$.

Quotients are related to decomposition of processes and properties, an issue which has received considerable attention through the years. In [LX90b], a solution to bisimulation $C(X) \sim P$ for a given process P and context C is provided (as a DMTS). This solves the quotienting problem P/C for the special case where both P and C are processes. This is extended in [LX90a] to the setting where the context C can have several holes and $C(X_1, \ldots, X_n)$ must satisfy a ν -calculus property Q. However, C remains to be a process context, not a specification context. Our specification context allows for arbitrary specifications, representing infinite sets of processes and process equations. Other extensions use infinite conjunctions [FvGdW06], probabilistic processes [GF12] or processes with continuous time and space [CLM11].

Quotient operators, or guarantee or multiplicative implication as they are called there, are also well-known from various logical formalisms. Indeed, the algebraic properties of our parallel composition | and quotient / resemble closely those of multiplicative conjunction & and implication \multimap in linear logic [Gir87], and of spatial conjunction and implication in spatial logic [CC03] and separation logic [Rey02, ORY01]. For these and other logics, proof systems have been developed which allow one to reason about expressions containing these operators. In these logics, & and → are first-class operators on par with the other logical operators, and their semantics are defined as certain sets of processes. In contrast, for AA and hence, via the translations, also for ν -calculus, || and / are derived operators, and we provide constructions to reduce any expression which contains them, to one which does not. This is important from the perspective of reuse of components and useful in industrial applications. To the best of our knowledge, there are no other such reductions of quotient for the synchronisation type of composition in the context of specifications.

8.5 Conclusion

In this chapter we have introduced a general specification framework whose basis consists of four different but equally expressive formalisms: one of a graphical behavioral kind (DMTS), one logic-based (ν -calculus) and two intermediate languages between the former two (AA and hybrid modal logic). We have shown their structural equivalence.

The established connection implies several consequences. On the one hand, it allows for a graphical representation of ν -calculus. Further, composition on DMTS can be transferred to the modal ν -calculus, hence turning it into a modal process algebra. On the other hand, such a correspondence identifies a class of modal transition systems with a natural expressive power and provides another justification of this formalism. Further, this class is closed under both conjunction and disjunction, a requirement raised by component-based

design methods. However, it is not closed under complement and difference.⁴ Nevertheless, since DMTS are closed under conjunction, disjunction and composition, we still have a positive Boolean process algebra.

Altogether, we have shown that the framework possesses a rich algebraic structure that includes logical (conjunction, disjunction) and behavioral operations (parallel composition and quotient) and forms a complete specification theory in the sense of [Lar90a, BDH⁺12].

Moreover, the construction of the quotient solves an open problem in the area of MTS. All attempts to find the quotient for variants of MTS so far have been limited to the much simpler deterministic case [Rac08]. Here we have given the first solution to the quotient on nondeterministic specifications: first, a quotient construction for MTS, and then a quotient for general DMTS. Due to the established correspondence, the quotient can be applied also to ν -calculus formulae. We remark that all our translations and constructions are based on a new normal form for ν -calculus expressions, and that turning a ν -calculus expression into normal form may incur an exponential blow-up. However, the translations and constructions preserve the normal form, so that this translation only need be applied once in the beginning.

 $^{^4{\}rm Previous}$ results on difference [SCU11] are incorrect due to a mistake in [FU08] on conjunction of MTS, see [Kře14, p. 36].

9 Compositionality for Quantitative Specifications¹

This chapter continues and finishes the work of Chapter 7. It extends the quantitative theory of that chapter to the disjunctive modal transition systems (DMTS) of Chapter 8 and shows that also in the quantitative setting, DMTS are closely related to acceptance automata and the modal ν -calculus. The quantitative theory of DMTS is shown to be rather pleasant, with better properties than for pure MTS.

9.1 Structured Labels

Let Σ be a poset with partial order \leq . We think of \leq as *label refinement*, so that if $a \leq b$, then a is less permissive (more restricted) than b.

9.1 Definition. A label $a \in \Sigma$ is an *implementation label* if $b \leq a$ implies b = a for all $b \in \Sigma$. The set of implementation labels is denoted Γ , and for $a \in \Sigma$, we let $[\![a]\!] = \{b \in \Gamma \mid b \leq a\}$ denote the set of its implementations.

Hence a is an implementation label iff a cannot be further refined. Note that $a \preccurlyeq b$ implies $[\![a]\!] \subseteq [\![b]\!]$ for all $a,b \in \Sigma$.

9.2 Example. A trivial but important example of our label structure is the *discrete* one in which label refinement \leq is equality (and $\Gamma = \Sigma$). This is equivalent to the "standard" case of *unstructured* labels.

A typical label set in quantitative applications consists of a discrete component and real-valued weights. For specifications, weights are replaced by (closed) weight intervals, so that $\Sigma = U \times \{[l,r] \mid l \in \mathbb{R} \cup \{-\infty\}, r \in \mathbb{R} \cup \{\infty\}, l \leq r\}$ for a finite set U, cf. [BFJ⁺13, BJL⁺12a]. Label refinement is given by $(u_1, [l_1, r_1]) \leq (u_2, [l_2, r_2])$ iff $u_1 = u_2$ and $[l_1, r_1] \subseteq [l_2, r_2]$, so that labels are more refined if they specify smaller intervals; thus, $\Gamma = U \times \{[x, x] \mid x \in \mathbb{R}\} \approx U \times \mathbb{R}$.

For a quite general setting, we can instead start with an arbitrary set Γ of implementation labels, let $\Sigma = 2^{\Gamma}$, the powerset, and $\leq \subseteq$ be subset

¹This chapter is based on the journal paper [FKLT18] published in Soft Computing.

inclusion. Then [a] = a for all $a \in \Sigma$. (Hence we identify implementation labels with one-element subsets of Σ .)

9.1.1 Label operations

Specification theories come equipped with several standard operations that make compositional software design possible [BDH⁺12]: conjunction for merging viewpoints covering different system's aspects [UC04,BDCU13], structural composition for running components in parallel, and quotient to synthesize missing parts of systems [LX90b]. In order to provide them for DMTS, we first need the respective atomic operations on their action labels.

We hence assume that Σ comes equipped with a partial conjunction, i.e., an operator $\emptyset : \Sigma \times \Sigma \rightharpoonup \Sigma$ for which it holds that

- (1) if $a_1 \otimes a_2$ is defined, then $a_1 \otimes a_2 \leq a_1$ and $a_1 \otimes a_2 \leq a_2$, and
- (2) if $a_3 \leq a_1$ and $a_3 \leq a_2$, then $a_1 \otimes a_2$ is defined and $a_3 \leq a_1 \otimes a_2$.

Note that by these properties, any two partial conjunctions on Σ have to agree on elements for which they are both defined.

9.3 Example. For discrete labels, the unique conjunction operator is given by

$$a_1 \otimes a_2 = \begin{cases} a_1 & \text{if } a_1 = a_2 \,, \\ \text{undef.} & \text{otherwise} \,. \end{cases}$$

Indeed, by property (2), $a_1 \otimes a_2$ must be defined for $a_1 = a_2$, and by (1), if $a_1 \otimes a_2 = a_3$ is defined, then $a_3 = a_1$ and $a_3 = a_2$.

For labels in $U \times \{[l,r] \mid l,r \in \mathbb{R}, l \leq r\}$, the unique conjunction is

$$(u_1, [l_1, r_1]) \otimes (u_2, [l_2, r_2]) = \begin{cases} \text{undef.} & \text{if } u_1 \neq u_2 \text{ or } [l_1, r_1] \cap [l_2, r_2] = \emptyset, \\ (u_1, [l_1, r_1] \cap [l_2, r_2]) & \text{otherwise.} \end{cases}$$

To see uniqueness, let $a_i = (u_i, [l_i, r_i])$ for i = 1, 2, 3. Using property (2), we see that $a_1 \otimes a_2$ must be defined when $u_1 = u_2$ and $[l_1, r_1] \cap [l_2, r_2] \neq \emptyset$, and by (2), if $a_1 \otimes a_2 = a_3$ is defined, then $u_3 = u_1$ and $u_3 = u_2$, and $[l_3, r_3] \subseteq [l_1, r_1]$, $[l_3, r_3] \subseteq [l_2, r_2]$ imply $[l_1, r_1] \cap [l_2, r_2] \neq \emptyset$.

Finally, for the case of specification labels as sets of implementation labels, the unique conjunction is $a_1 \otimes a_2 = a_1 \cap a_2$.

For structural composition and quotient of specifications, we assume a partial label synchronization operator $\oplus: \Sigma \times \Sigma \to \Sigma$ which specifies how to compose labels. We assume \oplus to be associative and commutative, with the following technical property which we shall need later: For all $a_1, a_2, b_1, b_2 \in \Sigma$ with $a_1 \preccurlyeq a_2$ and $b_1 \preccurlyeq b_2$, $a_1 \oplus b_1$ is defined iff $a_2 \oplus b_2$ is, and if both are defined, then $a_1 \oplus b_1 \preccurlyeq a_2 \oplus b_2$.

9.4 Example. For discrete labels, the conjunction of Example 9.3 is the same as CSP-style composition, *i.e.*, $a \oplus b = a$ if a = b and undefined otherwise, but other compositions can easily be defined.

For labels in $U \times \{[l, r] \mid l, r \in \mathbb{R}, l \leq r\}$, several useful label synchronization operators may be defined for different applications. One is given by *addition* of intervals, *i.e.*,

$$(u_1, [l_1, r_1]) \stackrel{\perp}{\oplus} (u_2, [l_2, r_2]) = \begin{cases} \text{undef.} & \text{if } u_1 \neq u_2, \\ (u_1, [l_1 + l_2, r_1 + r_2]) & \text{otherwise,} \end{cases}$$

for example modeling computation time of actions on a single processor. Another operator, useful in scheduling, uses maximum instead of addition:

$$(u_1, [l_1, r_1]) \stackrel{\vee}{\oplus} (u_2, [l_2, r_2]) = \begin{cases} \text{undef.} & \text{if } u_1 \neq u_2, \\ (u_1, [\max(l_1, l_2), \max(r_1, r_2)]) & \text{otherwise.} \end{cases}$$

For set-valued specification labels, we may take any synchronization operator \oplus given on implementation labels Γ and lift it to one on Σ by $a_1 \oplus a_2 = \{b_1 \oplus b_2 \mid b_1 \in [a_1], b_2 \in [a_2]\}$.

9.2 Specification Formalisms

In this section we introduce the specification formalisms which we use in the rest of the paper. The universe of models for our specifications is the one of standard *labeled transition systems*. For simplicity of exposition, we work only with *finite* specifications and implementations, but most of our results extend to the infinite (but finitely branching) case.

A labeled transition system (LTS) is a structure $\mathcal{I} = (S, s^0, \longrightarrow)$ consisting of a finite set S of states, an initial state $s^0 \in S$, and a transition relation $\longrightarrow \subseteq S \times \Gamma \times S$. We usually write $s \xrightarrow{a} t$ instead of $(s, a, t) \in \longrightarrow$. Note that transitions are labeled with *implementation* labels.

9.2.1 Disjunctive Modal Transition Systems

A disjunctive modal transition system (DMTS) is a structure $\mathcal{D}=(S,S^0,--\to,\longrightarrow)$ consisting of finite sets $S\supseteq S^0$ of states and initial states, respectively, may-transitions $--\to\subseteq S\times\Sigma\times S$, and disjunctive must-transitions $\longrightarrow\subseteq S\times2^{\Sigma\times S}$. It is assumed that for all $(s,N)\in\longrightarrow$ and $(a,t)\in N$ there is $(s,b,t)\in --\to$ with $a\preccurlyeq b$.

Note that we allow multiple (or zero) initial states. We write $s \xrightarrow{a} t$ instead of $(s, a, t) \in --+$ and $s \longrightarrow N$ instead of $(s, N) \in --+$.

A DMTS $(S, S^0, \dashrightarrow, \longrightarrow)$ is an *implementation* if $\dashrightarrow \subseteq S \times \Gamma \times S, \longrightarrow = \{(s, \{(a,t)\}) \mid s \xrightarrow{a} t\}$, and $S^0 = \{s^0\}$ is a singleton; DMTS implementations are hence isomorphic to LTS.

DMTS were introduced in [LX90b] in the context of equation solving, or quotient of specifications by processes and are used e.g., in [BCK11] for LTL model checking. They are a natural extension of modal transition systems [LT88], which are DMTS in which all disjunctive must-transitions $s \rightarrow N$ lead to singletons $N = \{(a,t)\}$; in fact, DMTS are the closure of MTS under quotient [LX90b].

We introduce a notion of modal refinement of DMTS with structured labels. For discrete labels, it coincides with the classical definition [LX90b].

- **9.5 Definition.** Let $\mathcal{D}_1 = (S_1, S_1^0, \rightarrow_1, \longrightarrow_1)$ and $\mathcal{D}_2 = (S_2, S_2^0, \rightarrow_2, \longrightarrow_2)$ be DMTS. A relation $R \subseteq S_1 \times S_2$ is a *modal refinement* if it holds for all $(s_1, s_2) \in R$ that
 - for all $s_1 \xrightarrow{a_1} t_1$ there is $s_2 \xrightarrow{a_2} t_2$ such that $a_1 \preccurlyeq a_2$ and $(t_1, t_2) \in R$, and
 - for all $s_2 \longrightarrow_2 N_2$ there is $s_1 \longrightarrow_1 N_1$ such that for all $(a_1, t_1) \in N_1$ there is $(a_2, t_2) \in N_2$ with $a_1 \leq a_2$ and $(t_1, t_2) \in R$.

 \mathcal{D}_1 refines \mathcal{D}_2 , denoted $\mathcal{D}_1 \leq_{\mathsf{m}} \mathcal{D}_2$, if there exists an *initialized* modal refinement R, *i.e.*, one for which it holds that for every $s_1^0 \in S_1^0$ there is $s_2^0 \in S_2^0$ for which $(s_1^0, s_2^0) \in R$.

Note that this definition reduces to the one of [LX90b,BCK11] for discrete labels (*cf.* Example 9.2).

We write $\mathcal{D}_1 \equiv_{\mathsf{m}} \mathcal{D}_2$ if $\mathcal{D}_1 \leq_{\mathsf{m}} \mathcal{D}_2$ and $\mathcal{D}_2 \leq_{\mathsf{m}} \mathcal{D}_1$. The implementation semantics of a DMTS \mathcal{D} is $[\![\mathcal{D}]\!] = \{\mathcal{I} \leq_{\mathsf{m}} \mathcal{D} \mid \mathcal{I} \text{ implementation}\}$. This is, thus, the set of all LTS which satisfy the specification given by the DMTS \mathcal{D} . We say that \mathcal{D}_1 thoroughly refines \mathcal{D}_2 , and write $\mathcal{D}_1 \leq_{\mathsf{t}} \mathcal{D}_2$, if $[\![\mathcal{D}_1]\!] \subseteq [\![\mathcal{D}_2]\!]$.

The below proposition, which follows directly from transitivity of modal refinement, shows that modal refinement is *sound* with respect to thorough refinement; in the context of specification theories, this is what one would expect. It can be shown that modal refinement is also *complete* for *deterministic* DMTS [BKLS09], but we will not need this here.

9.6 Proposition. For all DMTS
$$\mathcal{D}_1$$
, \mathcal{D}_2 , $\mathcal{D}_1 \leq_{\mathsf{m}} \mathcal{D}_2$ implies $\mathcal{D}_1 \leq_{\mathsf{t}} \mathcal{D}_2$.

9.2.2 Acceptance automata

An acceptance automaton (AA) is a structure $\mathcal{A} = (S, S^0, \text{Tran})$, with $S \supseteq S^0$ finite sets of states and initial states and Tran : $S \to 2^{2^{\Sigma \times S}}$ an assignment of transition constraints. The intuition is that a transition constraint $\text{Tran}(s) = \{M_1, \ldots, M_n\}$ specifies a disjunction of n choices M_1, \ldots, M_n as to which transitions from s have to be implemented.

An AA is an *implementation* if $S^0 = \{s^0\}$ is a singleton and it holds for all $s \in S$ that $Tran(s) = \{M\} \subseteq 2^{\Gamma \times S}$ is a singleton; hence AA implementations

are isomorphic to LTS. Acceptance automata were first introduced in [Rac07], based on the notion of acceptance trees in [Hen85]; however, there they are restricted to be *deterministic*. We employ no such restriction here.

Let $\mathcal{A}_1 = (S_1, S_1^0, \operatorname{Tran}_1)$ and $\mathcal{A}_2 = (S_2, S_2^0, \operatorname{Tran}_2)$ be AA. A relation $R \subseteq S_1 \times S_2$ is a modal refinement if it holds for all $(s_1, s_2) \in R$ and all $M_1 \in \operatorname{Tran}_1(s_1)$ that there exists $M_2 \in \operatorname{Tran}_2(s_2)$ such that

$$\forall (a_1, t_1) \in M_1 : \exists (a_2, t_2) \in M_2 : a_1 \leq a_2, (t_1, t_2) \in R,
\forall (a_2, t_2) \in M_2 : \exists (a_1, t_1) \in M_1 : a_1 \leq a_2, (t_1, t_2) \in R.$$
(9.1)

The definition reduces to the one of [Rac07] in case labels are discrete. We will write $M_1 \leq_R M_2$ if M_1 , M_2 , R satisfy (9.1).

$$D = \{M \in \operatorname{Tran}(s) \mid s \in S\},$$

$$D^{0} = \{M^{0} \in \operatorname{Tran}(s^{0}) \mid s^{0} \in S^{0}\},$$

$$\longrightarrow = \{(M, \{(a, M') \mid M' \in \operatorname{Tran}(t)\}) \mid (a, t) \in M\},$$

$$- \rightarrow = \{(M, a, M') \mid \exists M \longrightarrow N : (a, M') \in N\}.$$

Theorem 8.14 is easily extended to our case of structured labels:

9.7 Theorem. For all DMTS
$$\mathcal{D}_1$$
, \mathcal{D}_2 and AA \mathcal{A}_1 , \mathcal{A}_2 , $\mathcal{D}_1 \leq_{\mathsf{m}} \mathcal{D}_2$ iff $da(\mathcal{D}_1) \leq_{\mathsf{m}} da(\mathcal{D}_2)$ and $\mathcal{A}_1 \leq_{\mathsf{m}} \mathcal{A}_2$ iff $ad(\mathcal{A}_1) \leq_{\mathsf{m}} ad(\mathcal{A}_2)$.

This structural equivalence will allow us to freely translate forth and back between DMTS and AA in the rest of the paper. Note, however, that the state spaces of \mathcal{A} and $ad(\mathcal{A})$ are not the same; the one of $ad(\mathcal{A})$ may be exponentially larger. Proposition 8.18 shows that this blow-up is unavoidable.

From a practical point of view, DMTS are a somewhat more useful specification formalism than AA. This is because they are usually more compact and easily drawn and due to their close relation to the modal ν -calculus, see below.

9.2.3 The Modal ν -Calculus

The modal ν -calculus [FP07] is the maximal-fixed point fragment of the modal μ -calculus [Koz83], *i.e.*, the modal μ -calculus without negation and without the minimal fixed point operator. This is also sometimes called *Hennessy-Milner logic with maximal fixed points* and represented using equation systems in Hennessy-Milner logic with variables, see [Lar90b, AILS07]. We will use this representation below. In Chapter 8 we have introduced translations

between DMTS and the modal ν -calculus, showing that for discrete labels, these formalisms are *structurally equivalent*.

For a finite set X of variables, let $\mathcal{H}(X)$ be the set of Hennessy-Milner formulae, generated by the abstract syntax $\mathcal{H}(X) \ni \phi := \mathbf{tt} \mid \mathbf{ff} \mid x \mid \langle a \rangle \phi \mid$ $[a]\phi \mid \phi \land \phi \mid \phi \lor \phi$, for $a \in \Sigma$ and $x \in X$. A ν -calculus expression is a structure $\mathcal{N} = (X, X^0, \Delta)$, with $X^0 \subseteq X$ sets of variables and $\Delta : X \to \mathcal{H}(X)$ a declaration.

We recall the greatest fixed point semantics of ν -calculus expressions from [Lar90b], but extend it to structured labels. Let $(S, S^0, \longrightarrow)$ be an LTS, then an assignment is a mapping $\sigma: X \to 2^S$. The set of assignments forms a complete lattice with order $\sigma_1 \sqsubseteq \sigma_2$ iff $\sigma_1(x) \subseteq \sigma_2(x)$ for all $x \in X$ and lowest upper bound $(\bigsqcup_{i \in I} \sigma_i)(x) = \bigcup_{i \in I} \sigma_i(x)$.

The semantics of a formula in $\mathcal{H}(X)$ is a function from assignments to subsets of S defined as follows: $\llbracket \mathbf{tt} \rrbracket \sigma = S$, $\llbracket \mathbf{ff} \rrbracket \sigma = \emptyset$, $\llbracket x \rrbracket \sigma = \sigma(x)$, $\llbracket \phi \wedge \psi \rrbracket \sigma = \llbracket \phi \rrbracket \sigma \cap \llbracket \psi \rrbracket \sigma$, $\llbracket \phi \vee \psi \rrbracket \sigma = \llbracket \phi \rrbracket \sigma \cup \llbracket \psi \rrbracket \sigma$, and

The semantics of a declaration Δ is then the assignment defined by

$$\llbracket \Delta \rrbracket = \big| \ \big| \{\sigma: X \to 2^S \mid \forall x \in X: \sigma(x) \subseteq \llbracket \Delta(x) \rrbracket \sigma \};$$

the greatest (pre)fixed point of Δ .

An LTS $\mathcal{I} = (S, s^0, \longrightarrow)$ implements (or models) the expression \mathcal{N} , denoted $\mathcal{I} \models \mathcal{N}$, if there is $x^0 \in X^0$ such that $s^0 \in [\![\Delta]\!](x^0)$.

In Chapter 8 we have introduced another semantics for ν -calculus expressions, which is given by a notion of refinement, like for DMTS and AA. For this we recall the normal form for ν -calculus expressions:

9.8 Lemma. For any ν -calculus expression $\mathcal{N}_1 = (X_1, X_1^0, \Delta_1)$, there exists another $\mathcal{N}_2 = (X_2, X_2^0, \Delta_2)$ with $[\![\mathcal{N}_1]\!] = [\![\mathcal{N}_2]\!]$ and such that for any $x \in X$, $\Delta_2(x)$ is of the form

$$\Delta_2(x) = \bigwedge_{i \in I} \left(\bigvee_{j \in J_i} \langle a_{ij} \rangle x_{ij} \right) \wedge \bigwedge_{a \in \Sigma} [a] \left(\bigvee_{j \in J_a} y_{a,j} \right)$$

for finite (possibly empty) index sets I, J_i , J_a and all $x_{ij}, y_{a,j} \in X_2$.

As this is a type of *conjunctive normal form*, it is clear that translating a ν -calculus expression into normal form may incur an exponential blow-up.

We introduce some notation for ν -calculus expressions in normal form. Let $\mathcal{N} = (X, X^0, \Delta)$ be such an expression and $x \in X$, with

$$\Delta(x) = \bigwedge_{i \in I} \left(\bigvee_{j \in J_i} \langle a_{ij} \rangle x_{ij} \right) \wedge \bigwedge_{a \in \Sigma} [a] \left(\bigvee_{j \in J_a} y_{a,j} \right)$$

as in the lemma. Define $\Diamond(x) = \{\{(a_{ij}, x_{ij}) \mid j \in J_i\} \mid i \in I\}$ and, for each $a \in \Sigma$, $\Box^a(x) = \{y_{a,j} \mid j \in J_a\}$. Intuitively, $\Diamond(x)$ collects all $\langle a \rangle$ -requirements from x, whereas $\Box^a(x)$ specifies the disjunction of [a]-properties which must hold from x. Note that now,

$$\Delta(x) = \bigwedge_{N \in \Diamond(x)} \left(\bigvee_{(a,y) \in N} \langle a \rangle y \right) \wedge \bigwedge_{a \in \Sigma} [a] \left(\bigvee_{y \in \Box^a(x)} y \right). \tag{9.2}$$

Let $\mathcal{N}_1 = (X_1, X_1^0, \Delta_1)$, $\mathcal{N}_2 = (X_2, X_2^0, \Delta_2)$ be ν -calculus expressions in normal form and $R \subseteq X_1 \times X_2$. The relation R is a *modal refinement* if it holds for all $(x_1, x_2) \in R$ that

- for all $a_1 \in \Sigma$ and $y_1 \in \square_1^{a_1}(x_1)$ there is $a_2 \in \Sigma$ and $y_2 \in \square_2^{a_2}(x_2)$ with $a_1 \leq a_2$ and $(y_1, y_2) \in R$, and
- for all $N_2 \in \Diamond_2(x_2)$ there is $N_1 \in \Diamond_1(x_1)$ such that for all $(a_1, y_1) \in N_1$ there exists $(a_2, y_2) \in N_2$ with $a_1 \preccurlyeq a_2$ and $(y_1, y_2) \in R$.

We say that a ν -calculus expression (X, X^0, Δ) in normal form is an *implementation* if $X^0 = \{x^0\}$ is a singleton, $\Diamond(x) = \{\{(a,y)\} \mid y \in \Box^a(x), a \in \Sigma\}$ and $\Box^a(x) = \emptyset$ for all $a \notin \Gamma$, for all $x \in X$.

We can translate a LTS $(S, S^0, \longrightarrow)$ to a ν -calculus expression (S, S^0, Δ) in normal form by setting $\Diamond(s) = \{\{(a,t)\} \mid s \stackrel{a}{\longrightarrow} t\}$ and $\Box^a(s) = \{t \mid s \stackrel{a}{\longrightarrow} t\}$ for all $s \in S$, $a \in \Sigma$. This defines a bijection between LTS and ν -calculus implementations, hence, like for DMTS and AA, an embedding of LTS into the modal ν -calculus.

We have shown in Chapter 8 that for discrete labels, the refinement semantics and the fixed point semantics of the modal ν -calculus agree; the proof can easily be extended to our case of structured labels:

9.9 Theorem. For any LTS \mathcal{I} and any ν -calculus expression \mathcal{N} in normal form, $\mathcal{I} \models \mathcal{N}$ iff $\mathcal{I} \leq_m \mathcal{N}$.

For a DMTS $\mathcal{D}=(S,S^0, \dashrightarrow, \longrightarrow)$ and all $s\in S$, let $\Diamond(s)=\{N\mid s\longrightarrow N\}$ and, for each $a\in \Sigma$, $\square^a(s)=\{t\mid s\stackrel{a}{\dashrightarrow}t\}$. Define the (normal-form) ν -calculus expression $dn(\mathcal{D})=(S,S^0,\Delta)$, with Δ given as in (9.2). For a ν -calculus expression $\mathcal{N}=(X,X^0,\Delta)$ in normal form, let $\dashrightarrow=\{(x,a,y)\in X\times \Sigma\times X\mid y\in \square^a(x)\}, \longrightarrow=\{(x,N)\mid x\in X,N\in \Diamond(x)\}$ and define the DMTS $nd(\mathcal{N})=(X,X^0,\dashrightarrow,\longrightarrow)$. Given that these translations are entirely syntactic, the following theorem is not a surprise:

9.10 Theorem. For DMTS \mathcal{D}_1 , \mathcal{D}_2 and ν -calculus expressions \mathcal{N}_1 , \mathcal{N}_2 , $\mathcal{D}_1 \leq_{\mathsf{m}} \mathcal{D}_2$ iff $dn(\mathcal{D}_1) \leq_{\mathsf{m}} dn(\mathcal{D}_2)$ and $\mathcal{N}_1 \leq_{\mathsf{m}} \mathcal{N}_2$ iff $nd(\mathcal{N}_1) \leq_{\mathsf{m}} nd(\mathcal{N}_2)$.

9.3 Specification theory

Structural specifications typically come equipped with operations which permit compositional reasoning, viz. conjunction, structural composition, and quotient, cf. [BDH+12]. On deterministic MTS, these operations can be given easily using simple structural operational rules (for such semantics of weighted systems, see for instance [KS13a]). For non-deterministic specifications this is significantly harder; in [BDF+13] it is shown that DMTS and AA permit these operations and, additionally but trivially, disjunction. Here we show how to extend these operations on non-deterministic systems to our setting with structured labels.

We remark that structural composition and quotient operators are well-known from some logics, such as, e.g., linear [Gir87] or spatial logic [CC03], see also [CLM11] for a stochastic extension. However, whereas these operators are part of the formal syntax in those logics, for us they are simply operations on logical expressions (or DMTS, or AA).

Given the equivalence of DMTS, AA and the modal ν -calculus exposed in the previous section, we will often state properties for all three types of specifications at the same time, letting \mathcal{S} stand for any of the three types. For definitions and proofs, we are free to use the type of specification which is most well suited for the context; we will use DMTS for the logical operations (Section 9.3.1) and AA for the structural operations (Sections 9.3.2 and 9.3.3).

9.3.1 Disjunction and conjunction

Disjunction of specifications is easily defined, as we allow for multiple initial states. For two DMTS $\mathcal{D}_1 = (S_1, S_1^0, -\rightarrow_1, \longrightarrow_1)$ and $\mathcal{D}_2 = (S_2, S_2^0, -\rightarrow_2, \longrightarrow_2)$, we can hence define $\mathcal{D}_1 \vee \mathcal{D}_2 = (S_1 \cup S_2, S_1^0 \cup S_2^0, -\rightarrow_1 \cup -\rightarrow_2, \longrightarrow_1 \cup \longrightarrow_2)$ (with all unions disjoint).

For conjunction, we let $\mathcal{D}_1 \wedge \mathcal{D}_2 = (S_1 \times S_2, S_1^0 \times S_2^0, \dashrightarrow, \longrightarrow)$, with

- $(s_1, s_2) \xrightarrow{a_1 \otimes a_2} (t_1, t_2)$ whenever $s_1 \xrightarrow{a_1} t_1$, $s_2 \xrightarrow{a_2} t_2$ and $a_1 \otimes a_2$ is defined.
- for all $s_1 \longrightarrow N_1$, $(s_1, s_2) \longrightarrow \{(a_1 \otimes a_2, (t_1, t_2)) \mid (a_1, t_1) \in N_1, s_2 \stackrel{a_2}{\dashrightarrow}_2 t_2, a_1 \otimes a_2 \text{ defined}\},$
- $\bullet \ \text{ for all } s_2 \longrightarrow N_2, \, (s_1,s_2) \longrightarrow \{(a_1 \otimes a_2, (t_1,t_2)) \mid (a_2,t_2) \in N_2, s_1 \stackrel{a_1}{\dashrightarrow}_1 t_1, \\ a_1 \otimes a_2 \text{ defined}\}.$

The following theorem generalizes Theorem 8.23 to structured labels. Also its proof is a generalization, but our structured labels do introduce some extra difficulties.

9.11 Theorem. For all specifications S_1 , S_2 , S_3 ,

- $S_1 \vee S_2 \leq_{\mathsf{m}} S_3$ iff $S_1 \leq_{\mathsf{m}} S_3$ and $S_2 \leq_{\mathsf{m}} S_3$,
- $S_1 \leq_m S_2 \wedge S_3$ iff $S_1 \leq_m S_2$ and $S_1 \leq_m S_3$,
- $[S_1 \vee S_2] = [S_1] \cup [S_2]$, and $[S_1 \wedge S_2] = [S_1] \cap [S_2]$.

Proof: The proof that $S_1 \vee S_2 \leq_m S_3$ iff $S_1 \leq_m S_3$ and $S_2 \leq_m S_3$ is trivial: any modal refinement $R \subseteq (S_1 \cup S_2) \times S_3$ splits into two refinements $R_1 \subseteq S_1 \times S_3$, $R_2 \subseteq S_2 \times S_3$ and vice versa.

For the proof of the second claim, which we show for DMTS, we prove the back direction first. Let $R_2 \subseteq S_1 \times S_2$, $R_3 \subseteq S_1 \times S_3$ be initialized (DMTS) modal refinements which witness $S_1 \leq_{\mathsf{m}} S_2$ and $S_1 \leq_{\mathsf{m}} S_3$, respectively. Define $R = \{(s_1, (s_2, s_3)) \mid (s_1, s_2) \in R_2, (s_1, s_3) \in R_3\} \subseteq S_1 \times (S_2 \times S_3)$, then R is initialized.

Now let $(s_1, (s_2, s_3)) \in R$, then $(s_1, s_2) \in R_2$ and $(s_1, s_3) \in R_3$. Assume that $s_1 \xrightarrow{a_1} t_1$, then by $S_1 \leq_{\mathsf{m}} S_2$, we have $s_2 \xrightarrow{a_2} t_2$ with $a_1 \preccurlyeq a_2$ and $(t_1, t_2) \in R_2$. Similarly, by $S_1 \leq_{\mathsf{m}} S_3$, we have $s_3 \xrightarrow{a_3} \xrightarrow{a_3} t_3$ with $a_1 \preccurlyeq a_3$ and $(t_1, t_3) \in R_3$. But then also $a_1 \preccurlyeq a_2 \otimes a_3$ and $(t_1, (t_2, t_3)) \in R$, and $(s_2, s_3) \xrightarrow{a_2 \otimes a_3} (t_2, t_3)$ by definition.

Assume that $(s_2, s_3) \longrightarrow N$. Without loss of generality we can assume that there is $s_2 \longrightarrow_2 N_2$ such that $N = \{(a_2 \otimes a_3, (t_2, t_3)) \mid (a_2, t_2) \in N_2, s_3 \stackrel{a_3}{--} \downarrow_3 t_3\}$. By $S_1 \leq_{\mathsf{m}} S_2$, we have $s_1 \longrightarrow_1 N_1$ such that $\forall (a_1, t_1) \in N_1 : \exists (a_2, t_2) \in N_2 : a_1 \preccurlyeq a_2, (t_1, t_2) \in R_2$.

Let $(a_1, t_1) \in N_1$, then also $s_1 \xrightarrow{a_1} t_1$, so by $S_1 \leq_{\mathsf{m}} S_3$, there is $s_3 \xrightarrow{a_3} t_3$ with $a_1 \preccurlyeq a_3$ and $(t_1, t_3) \in R_3$. By the above, we also have $(a_2, t_2) \in N_2$ such that $a_1 \preccurlyeq a_2$ and $(t_1, t_2) \in R_2$, but then $(a_2 \otimes a_3, (t_2, t_3)) \in N$, $a_1 \preccurlyeq a_2 \wedge a_3$, and $(t_1, (t_2, t_3)) \in R$.

For the other direction of the second claim, let $R \subseteq S_1 \times (S_2 \times S_3)$ be an initialized (DMTS) modal refinement which witnesses $S_1 \leq_{\mathsf{m}} S_2 \wedge S_3$. We show that $S_1 \leq_{\mathsf{m}} S_2$, the proof of $S_1 \leq_{\mathsf{m}} S_3$ being entirely analogous. Define $R_2 = \{(s_1, s_2) \mid \exists s_3 \in S_3 : (s_1, (s_2, s_3)) \in R\} \subseteq S_1 \times S_2$, then R_2 is initialized.

Let $(s_1, s_2) \in R_2$, then we must have $s_3 \in S_3$ such that $(s_1, (s_2, s_3)) \in R$. Assume that $s_1 \xrightarrow{a_1} t_1$, then also $(s_2, s_3) \xrightarrow{a} (t_2, t_3)$ for some a with $a_1 \leq a$ and $(t_1, (t_2, t_3)) \in R$. By construction we have $s_2 \xrightarrow{a_2} t_2$ and $s_3 \xrightarrow{a_3} t_3$ such that $a = a_2 \otimes a_3$, but then $a_1 \leq a_2 \otimes a_3 \leq a_2$ and $(t_1, t_2) \in R_2$.

Assume that $s_2 \longrightarrow_2 N_2$, then by construction we have $(s_2, s_3) \longrightarrow N = \{(a_2 \otimes a_3, (t_2, t_3)) \mid (a_2, t_2) \in N_2, s_3 \xrightarrow{a_3} t_3\}$. By $S_1 \leq_{\mathsf{m}} S_2 \wedge S_3$, there is $s_1 \longrightarrow_1 N_1$ such that $\forall (a_1, t_1) \in N_1 : \exists (a, (t_2, t_3)) \in N : a_1 \preccurlyeq a, (t_1, (t_2, t_3)) \in R$.

Let $(a_1, t_1) \in N_1$, then we have $(a, (t_2, t_3)) \in N$ for which $a_1 \leq a$ and $(t_1, (t_2, t_3)) \in R$. By construction of N, this implies that there are $(a_2, t_2) \in N_2$ and $s_3 \xrightarrow{a_3} t_3$ such that $a = a_2 \otimes a_3$, but then $a_1 \leq a_2 \otimes a_3 \leq a_2$ and $(t_1, t_2) \in R$.

$$\mathcal{D}_1 \xrightarrow{a} \underbrace{s_2}$$

$$\mathcal{D}_2 \xrightarrow{b} \underbrace{t_1} \xrightarrow{b} \underbrace{t_2}$$

Figure 9.1: Two simple DMTS

As to the last claims of the theorem, $[\![S_1 \land S_2]\!] = [\![S_1]\!] \cap [\![S_2]\!]$ is clear from what we just proved: for all implementations $\mathcal{I}, \mathcal{I} \leq_m \mathcal{S}_1 \land \mathcal{S}_2$ iff $\mathcal{I} \leq_m \mathcal{S}_1$ and $\mathcal{I} \leq_m \mathcal{S}_2$. For the other part, it is clear by construction that for any implementation \mathcal{I} , any witness R for $\mathcal{I} \leq_m \mathcal{S}_1$ is also a witness for $\mathcal{I} \leq_m \mathcal{S}_1 \lor \mathcal{S}_2$, and similarly for \mathcal{S}_2 , hence $[\![S_1]\!] \cup [\![S_2]\!] \subseteq [\![S_1 \lor \mathcal{S}_2]\!]$.

To show that also $[\![S_1]\!] \cup [\![S_2]\!] \supseteq [\![S_1 \vee S_2]\!]$, we note that an initialized refinement R witnessing $\mathcal{I} \leq_{\mathsf{m}} \mathcal{S}_1 \vee \mathcal{S}_2$ must relate the initial state of \mathcal{I} either to an initial state of \mathcal{S}_1 or to an initial state of \mathcal{S}_2 . In the first case, and by disjointness, R witnesses $\mathcal{I} \leq_{\mathsf{m}} \mathcal{S}_1$, in the second, $\mathcal{I} \leq_{\mathsf{m}} \mathcal{S}_2$.

With bottom and top elements given by $\bot = (\emptyset, \emptyset, \emptyset)$ and $\top = (\{s\}, \{s\}, \text{Tran}_{\top})$ with $\text{Tran}_{\top}(s) = 2^{2^{\Sigma \times \{s\}}}$, our classes of specifications form bounded distributive lattices up to \equiv_{m} .

9.3.2 Structural composition

For AA $\mathcal{A}_1 = (S_1, S_1^0, \operatorname{Tran}_1)$, $\mathcal{A}_2 = (S_2, S_2^0, \operatorname{Tran}_2)$, their structural composition is $\mathcal{A}_1 \| \mathcal{A}_2 = (S_1 \times S_2, S_1^0 \times S_2^0, \operatorname{Tran})$, with $\operatorname{Tran}((s_1, s_2)) = \{M_1 \oplus M_2 \mid M_1 \in \operatorname{Tran}_1(s_1), M_2 \in \operatorname{Tran}_2(s_2)\}$ for all $s_1 \in S_1$, $s_2 \in S_2$, where $M_1 \oplus M_2 = \{(a_1 \oplus a_2, (t_1, t_2)) \mid (a_1, t_1) \in M_1, (a_2, t_2) \in M_2, a_1 \oplus a_2 \text{ defined}\}$.

Remark a subtle difference between conjunction and structural composition, which we expose for discrete labels and CSP-style composition: for the DMTS \mathcal{D}_1 , \mathcal{D}_2 shown in Figure 9.1, both $\mathcal{D}_1 \wedge \mathcal{D}_2$ and $\mathcal{D}_1 || \mathcal{D}_2$ have only one state, but $\operatorname{Tran}(s_1 \wedge t_1) = \emptyset$ and $\operatorname{Tran}(s_1 || t_1) = \{\emptyset\}$, so that $\mathcal{D}_1 \wedge \mathcal{D}_2$ is inconsistent, whereas $\mathcal{D}_1 || \mathcal{D}_2$ is not.

This definition extends the structural composition defined for modal transition systems, with structured labels, in [FL14a]. For DMTS specifications (and hence also for ν -calculus expressions), the back translation from AA to DMTS entails an exponential explosion.

9.12 Theorem. Up to \equiv_m , the operator \parallel is associative, commutative and monotone.

Proof: Associativity and commutativity are clear by associativity and commutativity of \oplus . Monotonicity is equivalent to the assertion that (up to \equiv_{m}) \parallel distributes over the least upper bound \vee ; one easily sees that for all specifications \mathcal{S}_1 , \mathcal{S}_2 , \mathcal{S}_3 , the identity is a two-sided modal refinement $\mathcal{S}_1 \| (\mathcal{S}_2 \vee \mathcal{S}_3) \equiv_{\mathsf{m}} \mathcal{S}_1 \| \mathcal{S}_2 \vee \mathcal{S}_1 \| \mathcal{S}_3$.

9.13 Corollary (Independent implementability). For all specifications S_1 , S_2 , S_3 , S_4 , $S_1 \leq_{\mathsf{m}} S_3$ and $S_2 \leq_{\mathsf{m}} S_4$ imply $S_1 \| S_2 \leq_{\mathsf{m}} S_3 \| S_4$.

9.3.3 Quotient

Because of non-determinism, we have to use a power set construction for the quotient, as opposed to conjunction and structural composition where product is sufficient. For AA $\mathcal{A}_3=(S_3,S_3^0,\operatorname{Tran}_3),\,\mathcal{A}_1=(S_1,S_1^0,\operatorname{Tran}_1),$ the quotient is $\mathcal{A}_3/\mathcal{A}_1=(S,\{s^0\},\operatorname{Tran}),$ with $S=2^{S_3\times S_1}$ and $s^0=\{(s_3^0,s_1^0)\mid s_3^0\in S_3^0,s_1^0\in S_1^0\}$. States in S will be written $\{s_3^1/s_1^1,\ldots,s_3^n/s_1^n)\}$. Intuitively, this denotes that such state when composed with s_1^i conforms to s_3^i for each i; we call this consistency here.

We now define Tran. First, $\operatorname{Tran}(\emptyset) = 2^{\Sigma \times \{\emptyset\}}$, so \emptyset is universal. For any other state $s = \{s_3^1/s_1^1, \ldots, s_3^n/s_1^n\} \in S$, its set of *permissible labels* is defined by

$$pl(s) = \{a_2 \in \Sigma \mid \forall i = 1, \dots, n : \forall (a_1, t_1) \in \in \operatorname{Tran}_1(s_1^i) : \exists (a_3, t_3) \in \in \operatorname{Tran}_3(s_3^i) : a_1 \oplus a_2 \leq a_3 \},$$

that is, a label is permissible iff it cannot violate consistency. Here we use the notation $x \in \in \mathbb{Z}$ as a shortcut for $\exists y : x \in y \in \mathbb{Z}$.

Now for each $a \in pl(s)$ and each $i \in \{1, ..., n\}$, let $\{t_1 \in S_1 \mid (a, t_1) \in \{1, ..., t_1^{i, m_i}\}\}$ be an enumeration of all the possible states in S_1 after an a-transition. Then we define the set of all sets of possible assignments of next-a states from s_1^i :

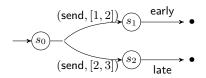
$$pt_a(s) = \{ \{ (t_3^{i,j}, t_1^{i,j}) \mid i = 1, \dots, n, j = 1, \dots, m_i \}$$
$$| \forall i : \forall j : (a, t_3^{i,j}) \in \in \text{Tran}_3(s_3^i) \}$$

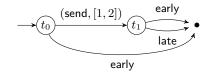
These are all possible next-state assignments which preserve consistency. Now let $pt(s) = \bigcup_{a \in pl(s)} pt_a(s)$ and define

$$\operatorname{Tran}(s) = \{ M \subseteq pt(s) \mid \forall i = 1, \dots, n : \\ \forall M_1 \in \operatorname{Tran}_1(s_1^i) : \exists M_3 \in \operatorname{Tran}_3(s_3^i) : M \triangleright M_1 \preceq_R M_3 \},$$

where $M \triangleright M_1 = \{(a_1 \oplus a, t_3^i) \mid (a, \{t_3^1/t_1^1, \dots, t_3^k/t_1^k)\}) \in M, (a_1, t_1^i) \in M_1\}$, to guarantee consistency no matter which element of $\operatorname{Tran}_1(s_1^i)$, s is composed with.

9.14 Example. Figure 9.2 shows two simple specifications and their quotient under $\dot{\oplus}$, *i.e.*, using addition of intervals for label synchronization (see Example 9.4). During the construction and the translation back to DMTS, many states were eliminated as they were inconsistent (their Tran-set was empty). For instance, there is no may transition to state $\{s_2/t_2\}$, because when it is





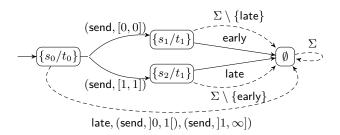


Figure 9.2: Two DMTS (top and center) and their quotient (bottom)

composed with t_2 there is no guarantee of a late-transition, hence no guarantee to refine s_2 .

Note that in order to have a finite representation of the quotient, we have to extend the label set to allow intervals which are not closed; for instance, the may-transition (send, $]1, \infty]$) from $\{s_0/t_0\}$ to \emptyset comprises the fact that $pt_a(\{s_0/t_0\}) = \emptyset$ for all $a = (\text{send}, [x, \infty])$ with x > 1. This can be formalized by introducing a (partial) label quotient operator $\emptyset : \Sigma \times \Sigma \rightharpoonup \Sigma$ which is adjoint to label synchronization \mathbb{O} , see Chapter 7.

9.15 Theorem. For all specifications S_1 , S_2 , S_3 , $S_1 \| S_2 \leq_m S_3$ iff $S_2 \leq_m S_3 / S_1$.

Proof: We show the proof for AA; for DMTS and ν -calculus expressions it will follow through the translations. Let $\mathcal{A}_1 = (S_1, S_1^0, \operatorname{Tran}_1)$, $\mathcal{A}_2 = (S_2, S_2^0, \operatorname{Tran}_2)$, $\mathcal{A}_3 = (S_3, S_3^0, \operatorname{Tran}_3)$; we show that $\mathcal{A}_1 \| \mathcal{A}_2 \leq_{\mathsf{m}} \mathcal{A}_3$ iff $\mathcal{A}_2 \leq_{\mathsf{m}} \mathcal{A}_3 / \mathcal{A}_1$.

We assume that the elements of $Tran_1(s_1)$ are pairwise disjoint for each $s_1 \in S_1$; this can be achieved by, if necessary, splitting states.

First we note that by construction, $s \supseteq t$ implies $s \leq_{\mathsf{m}} t$ for all $s, t \in S$.

Assume that $A_2 \leq_{\mathsf{m}} A_3/A_1$ and let $R = \{(s_2, s_3/s_1) \mid s_2 \leq_{\mathsf{m}} s_3/s_1\}$ be the witnessing refinement relation. Let $R' = \{(s_1||s_2, s_3) \mid (s_2, s_3/s_1) \in R\}$ (for

readability, we abuse notation here and write $(s_1||s_2, s_3)$ instead of (s_1, s_2, s_3) ; we show that R' is a witness for $A_1||A_2 \leq_m A_3$.

Let $(s_1||s_2, s_3) \in R'$ and $M_{\parallel} \in \text{Tran}_{\parallel}(s_1||s_2)$. Then $M_{\parallel} = M_1||M_2|$ with $M_1 \in \text{Tran}_1(s_1)$ and $M_2 \in \text{Tran}_2(s_2)$. As $s_2 \leq_{\mathsf{m}} s_3/s_1$, we can pair M_2 with a set $M_{\ell} \in \text{Tran}_{\ell}(s_3/s_1)$ such that $M_2 \preceq_R M_{\ell}$.

Let $M_3 = M_1 \triangleright M_1$. We show that $M_{\parallel} \preccurlyeq_{R'} M_3$:

- Let $(a, t_1 || t_2) \in M_{\parallel}$, then there are $a_1, a_2 \in \Sigma$ with $a = a_1 \oplus a_2$ and $(a_1, t_1) \in M_1$, $(a_2, t_2) \in M_2$. By $M_2 \preccurlyeq_R M_{/}$, there is $(a'_2, t) \in M_{/}$ such that $a_2 \preccurlyeq a'_2$ and $t_2 \leq_{\mathsf{m}} t$. Note that $a_3 = a_1 \oplus a'_2$ is defined and $a \preccurlyeq a_3$. Write $t = \{t_3^1/t_1^1, \ldots, t_3^n/t_1^n\}$. By construction, there is an index i for which $t_1^i = t_1$, hence $(a_3, t_3^i) \in M_3$. Also, $t \supseteq \{t_3^i/t_1^i\}$, hence $t_2 \leq_{\mathsf{m}} t_3^i/t_1^i$ and consequently $(t_1 || t_2, t_3) \in R'$.
- Let $(a_3,t_3) \in M_3$, then there are $(a_2',t) \in M_f$ and $(a_1,t_1) \in M_1$ such that $a_3 = a_1 \oplus a_2'$ and $t_3/t_1 \in t$. By $M_2 \preccurlyeq_R M_f$, there is $(a_2,t_2) \in M_2$ for which $a_2 \preccurlyeq a_2'$ and $t_2 \leq_{\mathsf{m}} t$. Note that $a = a_1 \oplus a_2$ is defined and $a \preccurlyeq a_3$. Thus $(a,t_1||t_2) \in M_{\parallel}$, and by $t \supseteq \{t_3/t_1\}$, $t_2 \leq_{\mathsf{m}} t_3/t_1$.

Assume, for the other direction of the proof, that $\mathcal{A}_1 \| \mathcal{A}_2 \leq_{\mathsf{m}} \mathcal{A}_3$ and let $R = \{(s_1 \| s_2, s_3) \mid s_1 \| s_2 \leq_{\mathsf{m}} s_2\}$ (again abusing notation) be the witnessing refinement relation. Define $R' \subseteq S_2 \times 2^{S_3 \times S_1}$ by

$$R = \{(s_2, \{s_3^1/s_1^1, \dots, s_3^n/s_1^n\}) \mid \forall i = 1, \dots, n : (s_1^i | s_2, s_3^i) \in R\};$$

we show that R' is a witness for $A_2 \leq_{\mathsf{m}} A_3/A_1$. Let $(s_2, s) \in R'$, with $s = \{s_3^1/s_1^1, \ldots, s_3^n/s_1^n\}$, and $M_2 \in \operatorname{Tran}_2(s_2)$.

For every $i = 1, \ldots, n$, write the set $\operatorname{Tran}_1(s_1^i) = \{M_1^{i,1}, \ldots, M_1^{i,m_i}\}$. By assumption, $M_1^{i,j_1} \cap M_1^{i,j_2} = \emptyset$ for $j_1 \neq j_2$, hence every $(a_1, t_1) \in \operatorname{Tran}_1(s_1^i)$ is contained in a unique $M_1^{i,\delta_i(a_1,t_1)} \in \operatorname{Tran}_1(s_1^i)$.

For every $j = 1, ..., m_i$, let $M^{i,j} = M_1^{i,j} \| M_2 \in \text{Tran}_{\parallel}(s_1^i \| s_2)$. By $s_1^i \| s_2 \leq_{\mathsf{m}} s_3^i$, we have $M_3^{i,j} \in \text{Tran}_3(s_3^i)$ such that $M^{i,j} \preccurlyeq_R M_3^{i,j}$.

Now define

$$M = \{(a_2, t) \mid \exists (a_2, t_2) \in M_2 : \forall t_3/t_1 \in t : \exists i, a_1, a_3 : (a_1, t_1) \in \in \operatorname{Tran}_1(s_1^i), (a_3, t_3) \in M_3^{i, \delta_i(a_1, t_1)}, a_1 \oplus a_2 \leq a_3, t_1 \| t_2 \leq_{\mathsf{m}} t_3 \}.$$
 (9.3)

We need to show that $M \in \text{Tran}_{\ell}(s)$.

Let $i \in \{1, \ldots, n\}$ and $M_1^{i,j} \in \operatorname{Tran}_1(s_1^i)$; we claim that $M \triangleright M_1^{i,j} \preceq_{R'} M_3^{i,j}$. Let $(a_3, t_3) \in M \triangleright M_1^{i,j}$, then $a_3 = a_1 \oplus a_2$ for some a_1, a_2 such that $t_3/t_1 \in t$, $(a_1, t_1) \in M_1^{i,j}$ and $(a_2, t) \in M$. By disjointness, $j = \delta_i(a_1, t_1)$, hence by definition of M, $(a_3, t_3) \in M_3^{i,j}$ as was to be shown.

For the reverse inclusion, let $(a_3, t_3) \in M_3^{i,j}$. By $M^{i,j} \leq_R M_3^{i,j}$ and definition of $M^{i,j}$, there are $(a_1, t_1) \in M_1^{i,j}$ and $(a_2, t_2) \in M_2$ for which $a_1 \oplus a_2 \leq a_3$

and $t_1||t_2 \leq_{\mathsf{m}} t_3$. Thus $j = \delta_i(a_1, t_1)$, so that there must be $(a_2, t) \in M$ for which $t_3/t_1 \in t$, but then also $(a_1 \oplus a_2, t_3) \in M \triangleright M_1^{i,j}$.

We show that $M_2 \preceq_{R'} M$.

- Let $(a_2, t_2) \in M_2$. For every i = 1, ..., n and every $(a_1, t_1) \in \text{Tran}_1(t_1^i)$, we can use $M^{i,j} \leq_R M_3^{i,j}$ and choose an element $(\eta_i(a_1, t_1), \tau_i(a_1, t_1)) \in M_3^{i,\delta_i(a_1,t_1)}$ for which $t_1 || t_2 \leq_m \tau_i(a_1, t_1)$ and $a_1 \oplus a_2 \leq \eta_i(a_1, t_1)$. Let $t = \{\tau_i(a_1, t_1)/t_1 \mid i = 1, ..., n, (a_1, t_1) \in \text{Tran}_1(t_1^i)\}$, then $(a_2, t) \in M$ and $(t_2, t) \in R'$.
- Let $(a_2, t) \in M$, then we have $(a_2, t_2) \in M_2$ satisfying the conditions in (9.3). Hence $t_1 || t_2 \leq_{\mathsf{m}} t_3$ for all $t_3/t_1 \in t$, so that $(t_2, t) \in R'$.

9.4 Robust Specification Theories

We proceed to lift the results of the previous sections to a quantitative setting, where the Boolean notions of modal and thorough refinement are replaced by refinement distances. We have shown in previous chapters that a good setting for quantitative analysis is given by the one of recursively specified trace distances on an abstract complete lattice \mathbb{L} . In order to extend this to specification theories, we enrich \mathbb{L} with an addition like in Chapter 7 and require it to be a (commutative) quantale, see below.

Denote by $\Sigma^{\infty} = \Sigma^* \cup \Sigma^{\omega}$ the set of finite and infinite traces over Σ .

9.4.1 Recursively specified trace distances

Recall that a *(commutative)* quantale consists of a complete lattice $(\mathbb{L}, \sqsubseteq_{\mathbb{L}})$ and a commutative, associative addition operation $\oplus_{\mathbb{L}}$ which distributes over arbitrary suprema; we denote by $\bot_{\mathbb{L}}$, $\top_{\mathbb{L}}$ the bottom and top elements of \mathbb{L} . We call a function $d: X \times X \to \mathbb{L}$, for a set X and a quantale \mathbb{L} , an \mathbb{L} -hemimetric if it satisfies $d(x,x) = \bot_{\mathbb{L}}$ for all $x \in X$ and $d(x,z) \sqsubseteq_{\mathbb{L}} d(x,y) \oplus_{\mathbb{L}} d(y,z)$ for all $x,y,z \in X$.

L-hemimetrics are generalizations of distances: for $\mathbb{L} = \mathbb{R}_{\geq 0} \cup \{\infty\}$ the extended real line, an $(\mathbb{R}_{\geq 0} \cup \{\infty\})$ -hemimetric is simply an extended hemimetric, *i.e.*, a function $d: X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ which satisfies d(x,x) = 0 for all $x \in X$ and the triangle inequality $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in X$. If d also is symmetric, *i.e.*, satisfies d(x,y) = d(y,x) for all $x,y \in X$, then d is usually called a pseudometric. If d also satisfies the principle of separability, or indiscernibility of identicals, *i.e.*, such that d(x,y) = 0 implies x = y, it is called a metric.

A recursive trace distance specification (\mathbb{L} , eval, $d_{\mathsf{tr}}^{\mathbb{L}}$, F) consists of a quantale \mathbb{L} , a quantale morphism eval : $\mathbb{L} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$, an \mathbb{L} -hemimetric $d_{\mathsf{tr}}^{\mathbb{L}} : \Sigma^{\infty} \times \Sigma^{\infty} \to \mathbb{L}$ (called *lifted trace distance*), and a *distance iterator*

function $F: \Sigma \times \Sigma \times \mathbb{L} \to \mathbb{L}$. For our purposes, F must be monotone in the third and anti-monotone in the second coordinate and satisfy an extended triangle inequality: for all $a, b, c \in \Sigma$ and $\alpha, \beta \in \mathbb{L}$, $F(a, b, \alpha) \oplus_{\mathbb{L}} F(b, c, \beta) \supseteq_{\mathbb{L}} F(a, c, \alpha \oplus_{\mathbb{L}} \beta)$.

F is to specify $d_{\mathsf{tr}}^{\mathbb{L}}$ recursively in the sense that for all $a, b \in \Sigma$ and all $\sigma, \tau \in \Sigma^{\infty}$ (and with "." denoting concatenation),

$$d_{\mathsf{tr}}^{\mathbb{L}}(a.\sigma, b.\tau) = F(a, b, d_{\mathsf{tr}}^{\mathbb{L}}(\sigma, \tau)). \tag{9.4}$$

The trace distance associated with such a distance specification is $d_{\mathsf{tr}}: \Sigma^{\infty} \times \Sigma^{\infty} \to \mathbb{R}_{>0}$ given by $d_{\mathsf{tr}} = \mathsf{eval} \circ d^{\mathbb{L}}_{\mathsf{tr}}$.

Note that $d_{\mathsf{tr}}^{\mathbb{L}}$ specializes to a distance on labels (because $\Sigma \subseteq \Sigma^{\infty}$); we require that this is compatible with label refinement in the sense that $a \preccurlyeq b$ implies $d_{\mathsf{tr}}^{\mathbb{L}}(a,b) = \bot_{\mathbb{L}}$. Then (9.4) implies that whenever $a \preccurlyeq b$, then $F(a,b,\bot_{\mathbb{L}}) = d_{\mathsf{tr}}^{\mathbb{L}}(a,b) = \bot_{\mathbb{L}}$. As an inverse property, we say that F is recursively separating if $F(a,b,\alpha) = \bot_{\mathbb{L}}$ implies that $a \preccurlyeq b$ and $\alpha = \bot_{\mathbb{L}}$.

9.16 Example. We have shown in previous chapters that all commonly used trace distances obey recursive characterizations as above. We give a few examples, all of which are recursively separating:

The *point-wise* distance from [dAFH+05] has $\mathbb{L} = \mathbb{R}_{\geq 0} \cup \{\infty\}$, eval = id and

$$d_{\mathsf{tr}}^{\mathbb{L}}(a.\sigma, b.\tau) = \max(d(a, b), d_{\mathsf{tr}}^{\mathbb{L}}(\sigma, \tau)),$$

where $d: \Sigma \times \Sigma \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is a hemimetric on labels. For the label set $\Sigma = U \times \{[l,r] \mid l \in \mathbb{R} \cup \{-\infty\}, r \in \mathbb{R} \cup \{\infty\}, l \leq r\}$ from Example 9.2, one useful example of such a hemimetric is $d((u_1,[l_1,r_1]),(u_2,[l_2,r_2])) = \sup_{x_1 \in [l_1,r_1]} \inf_{x_2 \in [l_2,r_2]} |x_1 - x_2| = \max(l_2 - l_1, r_1 - r_2, 0)$ if $u_1 = u_2$ and ∞ otherwise.

The discounting distance, also used in [dAFH⁺05], again uses $\mathbb{L} = \mathbb{R}_{\geq 0} \cup \{\infty\}$ and eval = id, but

$$d^{\mathbb{L}}_{\mathsf{tr}}(a.\sigma,b.\tau) = d(a,b) + \lambda d^{\mathbb{L}}_{\mathsf{tr}}(\sigma,\tau)$$

for a constant $\lambda \in [0, 1]$.

For the *limit-average* distance used in [ČHR12] and other papers, $\mathbb{L} = (\mathbb{R}_{\geq 0} \cup \{\infty\})^{\mathbb{N}}$, eval $(\alpha) = \liminf_{j \in \mathbb{N}} \alpha(j)$, and

$$d^{\mathbb{L}}_{\mathsf{tr}}(a.\sigma,b.\tau)(j) = \tfrac{1}{j+1}d(a,b) + \tfrac{j}{j+1}d^{\mathbb{L}}_{\mathsf{tr}}(\sigma,\tau)(j-1)\,.$$

It is clear that limit-average distance has no recursive specification which uses $\mathbb{L} = \mathbb{R}_{\geq 0} \cup \{\infty\}$ as for the other distances above. Intuitively, the quantale $(\mathbb{R}_{\geq 0} \cup \{\infty\})^{\mathbb{N}}$ has to be used to memorize how many symbols one has seen in the sequences σ , τ . This and other examples show that using general quantales in recursive trace distance specifications instead of simply $\mathbb{L} = \mathbb{R}_{\geq 0} \cup \{\infty\}$ is necessary.

The discrete trace distance is given by $d_{\mathsf{tr}}(\sigma,\tau) = 0$ if $\sigma \leq \tau$ and ∞ otherwise (here we have extended \leq to traces in the obvious way). It has a

recursive characterization with $\mathbb{L} = \mathbb{R}_{\geq 0} \cup \{\infty\}$, eval = id, and $d_{\mathsf{tr}}(a.\sigma, b.\tau) = d_{\mathsf{tr}}(\sigma, \tau)$ if $a \leq b$ and ∞ otherwise.

For the rest of this paper, we fix a recursively specified trace distance.

9.4.2 Refinement distances

We lift the notions of modal refinement, for all our formalisms, to distances. Conceptually, this is done by replacing "\formalfontarrow" quantifiers by "sup" and "\exists" by "inf" in the definitions, and then using the distance iterator to introduce a recursive functional whose least fixed point is the distance.

9.17 Definition. The *lifted refinement distance* on the states of DMTS $\mathcal{D}_1 = (S_1, S_1^0, -\rightarrow_1, \longrightarrow_1)$ and $\mathcal{D}_2 = (S_2, S_2^0, -\rightarrow_2, \longrightarrow_2)$ is the least fixed point to the equations

$$d_{\mathsf{m}}^{\mathbb{L}}(s_{1},s_{2}) = \max \begin{cases} \sup_{\substack{a_{1} \\ s_{1} - \cdots + t_{1}}} \inf_{s_{2} - \cdots + t_{2}} F(a_{1},a_{2},d_{\mathsf{m}}^{\mathbb{L}}(t_{1},t_{2})), \\ \sup_{s_{2} - \cdots + N_{2}} \inf_{s_{1} - \cdots + N_{1}} \sup_{(a_{1},t_{1}) \in N_{1}} \inf_{(a_{2},t_{2}) \in N_{2}} F(a_{1},a_{2},d_{\mathsf{m}}^{\mathbb{L}}(t_{1},t_{2})), \end{cases}$$

for $s_1 \in S_1$, $s_2 \in S_2$. For AA $\mathcal{A}_1 = (S_1, S_1^0, \operatorname{Tran}_1)$, $\mathcal{A}_2 = (S_2, S_2^0, \operatorname{Tran}_2)$, the right-hand side is replaced by

$$\sup_{M_1 \in \operatorname{Tran}_1(s_1)} \inf_{M_2 \in \operatorname{Tran}_2(s_2)} \max \begin{cases} \sup_{(a_1,t_1) \in M_1} \inf_{(a_2,t_2) \in M_2} F(a_1,a_2,d_{\mathsf{m}}^{\mathbb{L}}(t_1,t_2)) \,, \\ \sup_{(a_2,t_2) \in M_2} \inf_{(a_1,t_1) \in M_1} F(a_1,a_2,d_{\mathsf{m}}^{\mathbb{L}}(t_1,t_2)) \,, \end{cases}$$

and for ν -calculus expressions $\mathcal{N}_1=(X_1,X_1^0,\Delta_1),\,\mathcal{N}_2=(X_2,X_2^0,\Delta_2)$ in normal form, it is

$$\max \begin{cases} \sup & \inf F(a_1, a_2, d_{\mathsf{m}}^{\mathbb{L}}(y_1, y_2)), \\ a_1 \in \Sigma, y_1 \in \square_1^{a_1}(x_1) & a_2 \in \Sigma, y_2 \in \square_2^{a_2}(x_2) \\ \sup & \inf \sup \inf |\sup F(a_1, a_2, d_{\mathsf{m}}^{\mathbb{L}}(y_1, y_2)). \\ N_2 \in \lozenge_2(x_2) & N_1 \in \lozenge_1(x_1) & (a_1, y_1) \in N_1 & (a_2, y_2) \in N_2 \end{cases}$$

Using Tarski's fixed point theorem, one easily sees that the lifted refinement distances are indeed well-defined. (Here one needs monotonicity of F in the third coordinate, together with the fact that sup and inf are monotonic.)

Note that we define the distances using *least* fixed points, as opposed to the *greatest* fixed point definition of standard refinement. Informally, this is because our order is reversed: we are not interested in maximizing refinement relations, but in *minimizing* refinement distance.

The lifted refinement distance between specifications is defined by

$$d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{S}_{1},\mathcal{S}_{2}) = \sup_{s_{1}^{0} \in S_{1}^{0}} \inf_{s_{2}^{0} \in S_{2}^{0}} d_{\mathsf{m}}^{\mathbb{L}}(s_{1}^{0}, s_{2}^{0}) \,.$$

Analogously to thorough refinement, there is also a lifted thorough refinement distance, given by $d_{\mathsf{th}}^{\mathbb{L}}(\mathcal{S}_1, \mathcal{S}_2) = \sup_{\mathcal{I}_1 \in [\![\mathcal{S}_1]\!]} \inf_{\mathcal{I}_2 \in [\![\mathcal{S}_2]\!]} d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{I}_1, \mathcal{I}_2).$

Using the eval function, one gets distances $d_{\mathsf{m}} = \mathsf{eval} \circ d_{\mathsf{m}}^{\mathbb{L}}$ and $d_{\mathsf{th}} = \mathsf{eval} \circ d_{\mathsf{th}}^{\mathbb{L}}$, with values in $\mathbb{R}_{\geq 0} \cup \{\infty\}$, which will be the ones one is interested in for concrete applications.

We recall the notion of refinement family from Chapter 7 and extend it to specifications. We give the definition for AA only; for DMTS and the modal ν -calculus it is similar.

- **9.18 Definition.** A refinement family from A_1 to A_2 , for AA $A_1 = (S_1, S_1^0, \operatorname{Tran}_1)$, $A_2 = (S_2, S_2^0, \operatorname{Tran}_2)$, is an \mathbb{L} -indexed family of relations $R = \{R_\alpha \subseteq S_1 \times S_2 \mid \alpha \in \mathbb{L}\}$ with the property that for all $\alpha \in \mathbb{L}$ with $\alpha \neq \top_{\mathbb{L}}$, all $(s_1, s_2) \in R_\alpha$, and all $M_1 \in \operatorname{Tran}_1(s_1)$, there is $M_2 \in \operatorname{Tran}_2(s_2)$ such that
 - $\forall (a_1, t_1) \in M_1 : \exists (a_2, t_2) \in M_2, \beta \in \mathbb{L} : (t_1, t_2) \in R_\beta, F(a_1, a_2, \beta) \sqsubseteq \alpha,$
 - $\forall (a_2, t_2) \in M_2 : \exists (a_1, t_1) \in M_1, \beta \in \mathbb{L} : (t_1, t_2) \in R_\beta, F(a_1, a_2, \beta) \sqsubseteq \alpha.$
- **9.19 Lemma.** For all AA $A_1 = (S_1, S_1^0, \operatorname{Tran}_1)$, $A_2 = (S_2, S_2^0, \operatorname{Tran}_2)$, there exists a refinement family R from A_1 to A_2 such that for all $s_1^0 \in S_1^0$, there is $s_2^0 \in S_2^0$ for which $(s_1^0, s_2^0) \in R_{d_m^{\mathbb{L}}(A_1, A_2)}$.

We say that a refinement family as in the lemma witnesses $d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{A}_1, \mathcal{A}_2)$.

Proof: Define R by $R_{\alpha} = \{(s_1, s_2) \mid d_{\mathsf{m}}^{\mathbb{L}}(s_1, s_2) \sqsubseteq_{\mathbb{L}} \alpha\}$. First, as $(s_1^0, s_2^0) \in R_{d_{\mathsf{m}}^{\mathbb{L}}(s_1^0, s_2^0)}$ for all $s_1^0 \in S_1^0$, $s_2^0 \in S_2^0$, it is indeed the case that for all $s_1^0 \in S_1^0$, there is $s_2^0 \in S_2^0$ for which

$$(s_1^0,s_2^0) \in R_{d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{A}_1,\mathcal{A}_2)} = R_{\max_{s_1^0 \in S_1^0} \min_{s_2^0 \in S_2^0} d_{\mathsf{m}}^{\mathbb{L}}(s_1^0,s_2^0)} \,.$$

Now let $\alpha \in \mathbb{L}$ with $\alpha \neq \top_{\mathbb{L}}$ and $(s_1, s_2) \in R_{\alpha}$. Let $M_1 \in \operatorname{Tran}_1(s_1)$. We have $d_{\mathsf{m}}^{\mathbb{L}}(s_1, s_2) \sqsubseteq_{\mathbb{L}} \alpha$, hence there is $M_2 \in \operatorname{Tran}_2(s_2)$ such that

$$\alpha \sqsupseteq_{\mathbb{L}} \max \begin{cases} \sup \inf_{(a_1,t_1) \in M_1} F(a_1,a_2,d_{\mathsf{m}}^{\mathbb{L}}(t_1,t_2)), \\ \sup \inf_{(a_2,t_2) \in M_2} F(a_1,a_2,d_{\mathsf{m}}^{\mathbb{L}}(t_1,t_2)). \end{cases}$$

But this entails that for all $(a_1, t_1) \in M_1$, there is $(a_2, t_2) \in M_2$ and $\beta = d^{\mathbb{L}}_{\mathsf{m}}(t_1, t_2)$ with $F(a_1, a_2, \beta) \sqsubseteq_{\mathbb{L}} \alpha$, and that for all $(a_2, t_2) \in M_2$, there is $(a_1, t_1) \in M_1$ and $\beta = d^{\mathbb{L}}_{\mathsf{m}}(t_1, t_2)$ such that $F(a_1, a_2, \beta) \sqsubseteq_{\mathbb{L}} \alpha$.

The following quantitative extension of Theorems 9.7 and 9.10 shows that our translations preserve and reflect refinement distances.

9.20 Theorem. For all DMTS $\mathcal{D}_1, \mathcal{D}_2$, all AA \mathcal{A}_1 , \mathcal{A}_2 and all ν -calculus expressions $\mathcal{N}_1, \mathcal{N}_2$:

$$\begin{split} d^{\mathbb{L}}_{\mathsf{m}}(\mathcal{D}_1,\mathcal{D}_2) &= d^{\mathbb{L}}_{\mathsf{m}}(\mathit{da}(\mathcal{D}_1),\mathit{da}(\mathcal{D}_2)) \\ d^{\mathbb{L}}_{\mathsf{m}}(\mathcal{A}_1,\mathcal{A}_2) &= d^{\mathbb{L}}_{\mathsf{m}}(\mathit{ad}(\mathcal{A}_1),\mathit{ad}(\mathcal{A}_2)) \\ d^{\mathbb{L}}_{\mathsf{m}}(\mathcal{D}_1,\mathcal{D}_2) &= d^{\mathbb{L}}_{\mathsf{m}}(\mathit{dn}(\mathcal{D}_1),\mathit{dn}(\mathcal{D}_2)) \\ d^{\mathbb{L}}_{\mathsf{m}}(\mathcal{N}_1,\mathcal{N}_2) &= d^{\mathbb{L}}_{\mathsf{m}}(\mathit{nd}(\mathcal{N}_1),\mathit{nd}(\mathcal{N}_2)) \end{split}$$

Proof:

 $d_{\mathsf{m}}^{\mathbb{L}}(da(\mathcal{D}_1), da(\mathcal{D}_2)) \sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{D}_1, \mathcal{D}_2)$:

Let $\mathcal{D}_1 = (S_1, S_1^0, -\rightarrow_1, \longrightarrow_1)$ and $\mathcal{D}_2 = (S_2, S_2^0, -\rightarrow_2, \longrightarrow_2)$ be DMTS. There exists a DMTS refinement family $R = \{R_\alpha \subseteq S_1 \times S_2 \mid \alpha \in \mathbb{L}\}$ such that for all $s_1^0 \in S_1^0$, there is $s_2^0 \in S_2^0$ with $(s_1^0, s_2^0) \in R_{d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{D}_1, \mathcal{D}_2)}$. We show that R is an AA refinement family.

Let $\alpha \in \mathbb{L}$ and $(s_1, s_2) \in R_{\alpha}$. Let $M_1 \in \text{Tran}_1(s_1)$ and define

$$M_{2} = \{(a_{2}, t_{2}) \mid s_{2} \xrightarrow{a_{2}} t_{2}, \exists (a_{1}, t_{1}) \in M_{1} : \exists \beta \in \mathbb{L} : (t_{1}, t_{2}) \in R_{\beta}, F(a_{1}, a_{2}, \beta) \sqsubseteq_{\mathbb{L}} \alpha \}.$$

The condition

$$\forall (a_2, t_2) \in M_2 : \exists (a_1, t_1) \in M_1, \beta \in \mathbb{L} : (t_1, t_2) \in R_\beta, F(a_1, a_2, \beta) \sqsubseteq \alpha$$

is satisfied by construction. For the inverse condition, let $(a_1, t_1) \in M_1$, then $s_1 \xrightarrow{a_1} t_1$, and as R is a DMTS refinement family, this implies that there is $s_2 \xrightarrow{a_2} t_2$ and $\beta \in \mathbb{L}$ for which $(t_1, t_2) \in R_{\beta}$ and $F(a_1, a_2, \beta) \sqsubseteq_{\mathbb{L}} \alpha$, so that $(a_2, t_2) \in M_2$ by construction.

We are left with showing that $M_2 \in \text{Tran}_2(s_2)$. First we notice that by construction, indeed $s_2 \xrightarrow{a_2} t_2$ for all $(a_2, t_2) \in M_2$. Now let $s_2 \longrightarrow N_2$; we need to show that $N_2 \cap M_2 \neq \emptyset$.

We have $s_1 \longrightarrow N_1$ such that $\forall (a_1, t_1) \in N_1 : \exists (a_2, t_2) \in N_2, \beta \in \mathbb{L} : (t_1, t_2) \in R_{\beta}, F(a_1, a_2, \beta) \sqsubseteq_{\mathbb{L}} \alpha$. We know that $N_1 \cap M_1 \neq \emptyset$, so let $(a_1, t_1) \in N_1 \cap M_1$. Then there is $(a_2, t_2) \in N_2$ and $\beta \in \mathbb{L}$ such that $(t_1, t_2) \in R_{\beta}$ and $F(a_1, a_2, \beta) \sqsubseteq_{\mathbb{L}} \alpha$. But $(a_2, t_2) \in N_2$ implies $s_2 \xrightarrow{a_2} t_2$, hence $(a_2, t_2) \in M_2$. $\frac{d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{D}_1, \mathcal{D}_2) \sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}^{\mathbb{L}}(da(\mathcal{D}_1), da(\mathcal{D}_2)):}{Let \mathcal{D}_1 = (S_1, S_1^0, -\rightarrow_1, \longrightarrow_1) \text{ and } \mathcal{D}_2 = (S_2, S_2^0, -\rightarrow_2, \longrightarrow_2) \text{ be DMTS.}}$

Let $\mathcal{D}_1 = (S_1, S_1^0, - \to_1, \longrightarrow_1)$ and $\mathcal{D}_2 = (S_2, S_2^0, - \to_2, \longrightarrow_2)$ be DMTS. There exists an AA refinement family $R = \{R_\alpha \subseteq S_1 \times S_2 \mid \alpha \in \mathbb{L}\}$ such that for all $s_1^0 \in S_1^0$, there is $s_2^0 \in S_2^0$ for which $(s_1^0, s_2^0) \in R_{d_m}^{\mathbb{L}}(da(\mathcal{D}_1), da(\mathcal{D}_2))$. We show that R is a DMTS refinement family. Let $\alpha \in \mathbb{L}$ and $(s_1, s_2) \in R_\alpha$.

Let $s_1 \xrightarrow{a_1} t_1$, then we cannot have $s_1 \longrightarrow \emptyset$. Let $M_1 = \{(a_1, t_1)\} \cup \bigcup \{N_1 \mid s_1 \longrightarrow N_1\}$, then $M_1 \in \operatorname{Tran}_1(s_1)$ by construction. This implies that there is $M_2 \in \operatorname{Tran}_2(s_2)$, $(a_2, t_2) \in M_2$ and $\beta \in \mathbb{L}$ such that $(t_1, t_2) \in R_\beta$ and $F(a_1, a_2, \beta) \sqsubseteq_{\mathbb{L}} \alpha$, but then also $s_2 \xrightarrow{a_2} t_2$ as was to be shown.

Let $s_2 \longrightarrow N_2$ and assume, for the sake of contradiction, that there is no $s_1 \longrightarrow N_1$ for which $\forall (a_1,t_1) \in N_1 : \exists (a_2,t_2) \in N_2, \beta \in \mathbb{L} : (t_1,t_2) \in R_\beta, F(a_1,a_2,\beta) \sqsubseteq_{\mathbb{L}} \alpha$ holds. Then for each $s_1 \longrightarrow N_1$, there is an element $(a_{N_1},t_{N_1}) \in N_1$ such that $\exists (a_2,t_2) \in N_2, \beta \in \mathbb{L} : (t_{N_1},t_2) \in R_\beta, F(a_{N_1},a_2,\beta) \sqsubseteq_{\mathbb{L}} \alpha$ does not hold.

Let $M_1 = \{(a_{N_1}, t_{N_1}) \mid s_1 \longrightarrow N_1\}$, then $M_1 \in \text{Tran}_1(s_1)$ by construction. Hence we have $M_2 \in \text{Tran}_2(s_2)$ such that $\forall (a_2, t_2) \in M_2 : \exists (a_1, t_2) \in M_1, \beta \in \mathbb{L} : (t_1, t_2) \in R_\beta, F(a_1, a_2, \beta) \sqsubseteq \alpha$. Now $N_2 \cap M_2 \neq \emptyset$, so let $(a_2, t_2) \in N_2 \cap M_2$, then there is $(a_1, t_1) \in M_1$ and $\beta \in \mathbb{L}$ such that $(t_1, t_2) \in R_\beta$ and $F(a_1, a_2, \beta) \sqsubseteq_{\mathbb{L}} \alpha$, in contradiction to how M_1 was constructed.

 $d_{\mathsf{m}}^{\mathbb{L}}(\mathit{ad}(\mathcal{A}_1), \mathit{ad}(\mathcal{A}_2)) \sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{A}_1, \mathcal{A}_2)$:

Let $\mathcal{A}_1 = (S_1, S_1^0, \operatorname{Tran}_1)$, $\mathcal{A}_2 = (S_2, S_2^0, \operatorname{Tran}_2)$ be AA, with DMTS translations $ad(\mathcal{A}_1) = (D_1, D_1^0, \longrightarrow_1, \longrightarrow_1)$, $ad(\mathcal{A}_2) = (D_2, D_2^0, \longrightarrow_2, \longrightarrow_2)$. There is an AA refinement family $R = \{R_\alpha \subseteq S_1 \times S_2 \mid \alpha \in \mathbb{L}\}$ such that for all $s_1^0 \in S_1^0$, there is $s_2^0 \in S_2^0$ with $(s_1^0, s_2^0) \in R_{d_m}^{\mathbb{L}}(\mathcal{A}_1, \mathcal{A}_2)$.

Define a relation family $R' = \{R'_{\alpha} \subseteq D_1 \times D_2 \mid \alpha \in \mathbb{L}\}$ by

$$R'_{\alpha} = \{ (M_1, M_2) \mid \exists (s_1, s_2) \in R_{\alpha} : M_1 \in \text{Tran}_1(s_1), M_2 \in \text{Tran}(s_2), \\ \forall (a_1, t_1) \in M_1 : \exists (a_2, t_2) \in M_2, \beta \in \mathbb{L} : \\ (t_1, t_2) \in R_{\beta}, F(a_1, a_2, \beta) \sqsubseteq_{\mathbb{L}} \alpha, \\ \forall (a_2, t_2) \in M_2 : \exists (a_1, t_1) \in M_1, \beta \in \mathbb{L} : \\ (t_1, t_2) \in R_{\beta}, F(a_1, a_2, \beta) \sqsubseteq_{\mathbb{L}} \alpha \}.$$

We show that R' is a witness for $d_{\mathsf{m}}^{\mathbb{L}}(ad(\mathcal{A}_1), ad(\mathcal{A}_2)) \sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{A}_1, \mathcal{A}_2)$. Let $\alpha \in \mathbb{L}$ and $(M_1, M_2) \in R'_{\alpha}$.

Let $M_2 \longrightarrow_2 N_2$. By construction of \longrightarrow , there is $(a_2, t_2) \in M_2$ such that $N_2 = \{(a_2, M_2') \mid M_2' \in \operatorname{Tran}_2(t_2)\}$. Then $(M_1, M_2) \in R_{\alpha}'$ implies that there must be $(a_1, t_1) \in M_1$ and $\beta \in \mathbb{L}$ such that $(t_1, t_2) \in R_{\beta}$ and $F(a_1, a_2, \beta) \sqsubseteq_{\mathbb{L}} \alpha$. Let $N_1 = \{(a_1, M_1') \mid M_1' \in \operatorname{Tran}_1(t_1)\}$, then $M_1 \longrightarrow_1 N_1$.

We show that $\forall (a_1, M_1') \in N_1 : \exists (a_2, M_2') \in N_2 : (M_1', M_2') \in R_{\beta}'$: Let $(a_1, M_1') \in N_1$, then $M_1' \in \text{Tran}_1(t_1)$. From $(t_1, t_2) \in R_{\beta}$ we get $M_2' \in \text{Tran}_2(t_2)$ such that

$$\forall (b_1, u_1) \in M_1' : \exists (b_2, u_2) \in M_2', \gamma \in \mathbb{L} : (u_1, u_2) \in R_{\gamma}, F(b_1, b_2, \gamma) \sqsubseteq_{\mathbb{L}} \beta,$$

$$\forall (b_2, u_2) \in M_2' : \exists (b_1, u_1) \in M_1', \gamma \in \mathbb{L} : (u_1, u_2) \in R_{\gamma}, F(b_1, b_2, \gamma) \sqsubseteq_{\mathbb{L}} \beta,$$

hence $(M_1', M_2') \in R_\beta'$; also, $(a_2, M_2') \in N_2$ by construction of N_2 .

Let $M_1 \xrightarrow{a_1} M_1'$, then we have $M_1 \longrightarrow_1 N_1$ for which $(a_1, M_1') \in N_1$ by construction of $-\to_1$. This in turn implies that there must be $(a_1, t_1) \in M_1$ such that $N_1 = \{(a_1, M_1'') \mid M_1'' \in \operatorname{Tran}_1(t_1)\}$. By $(M_1, M_2) \in R_{\alpha}'$, we get $(a_2, t_2) \in M_2$ and $\beta \in \mathbb{L}$ such that $(t_1, t_2) \in R_{\beta}$ and $F(a_1, a_2, \beta) \sqsubseteq_{\mathbb{L}} \alpha$. Let $N_2 = \{(a_2, M_2') \mid M_2' \in \operatorname{Tran}_2(t_2)\}$, then $M_2 \longrightarrow_2 N_2$ and hence $M_2 \xrightarrow{a_2} M_2'$

for all $(a_2, M_2) \in N_2$. By the same arguments as above, there is $(a_2, M_2) \in N_2$ for which $(M'_1, M'_2) \in R'_{\beta}$.

We miss to show that R' is initialized. Let $M_1^0 \in D_1^0$, then we have $s_1^0 \in S_1^0$ with $M_1^0 \in \text{Tran}_1(s_1^0)$. As R is initialized, this entails that there is $s_2^0 \in S_2^0$ with $(s_1^0, s_2^0) \in R_{d_{\mathfrak{m}}^{\mathbb{L}}(\mathcal{A}_1, \mathcal{A}_2)}$, which gives us $M_2^0 \in \operatorname{Tran}_2(s_2^0)$ which satisfies the conditions in the definition of $R'_{d^{\perp}(A_1,A_2)}$, whence $(M_1^0, M_2^0) \in R'_{d^{\perp}(A_1,A_2)}$.

$d_{\mathbf{m}}^{\mathbb{L}}(\mathcal{A}_1, \mathcal{A}_2) \sqsubseteq_{\mathbb{L}} d_{\mathbf{m}}^{\mathbb{L}}(ad(\mathcal{A}_1), ad(\mathcal{A}_2))$:

Let $\mathcal{A}_1 = (S_1, S_1^0, \text{Tran}_1), \, \mathcal{A}_2 = (S_2, S_2^0, \text{Tran}_2)$ be AA, with DMTS translations $ad(\mathcal{A}_1) = (D_1, D_1^0, \longrightarrow_1, -\rightarrow_1), ad(\mathcal{A}_2) = (D_2, D_2^0, \longrightarrow_2, -\rightarrow_2).$ There is a DMTS refinement family $R = \{R_{\alpha} \subseteq D_1 \times D_2 \mid \alpha \in \mathbb{L}\}$ such that for all $M_1^0 \in D_1^0$, there exists $M_2^0 \in D_2^0$ with $(M_1^0, M_2^0) \in R_{d_{\mathsf{m}}^{\mathbb{L}}(ad(\mathcal{A}_1), ad(\mathcal{A}_2))}$.

Define a relation family $R' = \{R'_{\alpha} \subseteq S_1 \times S_2 \mid \alpha \in \mathbb{L}\}$ by

$$R'_{\alpha} = \{(s_1, s_2) \mid \forall M_1 \in \text{Tran}_1(s_1) : \exists M_2 \in \text{Tran}_2(s_2) : (M_1, M_2) \in R_{\alpha} \};$$

we will show that R' is a witness for $d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{A}_1, \mathcal{A}_2) \sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}^{\mathbb{L}}(ad(\mathcal{A}_1), ad(\mathcal{A}_2))$.

Let $\alpha \in \mathbb{L}$, $(s_1, s_2) \in R'_{\alpha}$ and $M_1 \in \text{Tran}_1(s_1)$, then by construction of R', we have $M_2 \in \text{Tran}_2(s_2)$ with $(M_1, M_2) \in R_{\alpha}$.

Let $(a_2, t_2) \in M_2$ and define $N_2 = \{(a_2, M_2') \mid M_2' \in \text{Tran}_2(t_2)\}$, then $M_2 \longrightarrow_2 N_2$. Now $(M_1, M_2) \in R_\alpha$ implies that there must be $M_1 \longrightarrow_1$ N_1 satisfying $\forall (a_1, M_1') \in N_1 : \exists (a_2, M_2') \in N_2, \beta \in \mathbb{L} : (M_1', M_2') \in R_\beta$, $F(a_1, a_2, \beta) \sqsubseteq_{\mathbb{L}} \alpha$. We have $(a_1, t_1) \in M_1$ such that $N_1 = \{(a_1, M_1') \mid M_1' \in$ Tran₁(t_1)}; we only miss to show that (t_1, t_2) $\in R'_{\beta}$ for some $\beta \in \mathbb{L}$ for which $F(a_1, a_2, \beta) \sqsubseteq_{\mathbb{L}} \alpha$. Let $M'_1 \in \operatorname{Tran}_1(t_1)$, then $(a_1, M'_1) \in N_1$, hence there is $(a_2, M_2') \in N_2$ and $\beta \in \mathbb{L}$ such that $(M_1', M_2') \in R_\beta$ and $F(a_1, a_2, \beta) \sqsubseteq \alpha$, but $(a_2, M_2') \in N_2$ also entails $M_2' \in \text{Tran}_2(t_2)$.

Let $(a_1, t_1) \in M_1$ and define $N_1 = \{(a_1, M_1') \mid M_1' \in \text{Tran}_1(t_1)\}$, then $M_1 \longrightarrow_1 N_1$. Now let $(a_1, M_1') \in N_1$, then $M_1 \xrightarrow{a_1} M_1'$, hence we have $M_2 \xrightarrow{a_2} M_2'$ and $\beta \in \mathbb{L}$ such that $(M_1', M_2') \in R_\beta$ and $F(a_1, a_2, \beta) \sqsubseteq_{\mathbb{L}} \alpha$. By construction of $--\rightarrow_2$, this implies that there is $M_2 \longrightarrow_2 N_2$ with $(a_2, M_2') \in N_2$, and we have $(a_2, t_2) \in M_2$ for which $N_2 = \{(a_2, M_2'') \mid M_2'' \in \text{Tran}_2(t_2)\}$. Now if $M_1'' \in \text{Tran}_1(t_1)$, then $(a_1, M_1'') \in N_1$, hence there is $(a_2, M_2'') \in N_2$ with $(M_1'', M_2'') \in R_\beta$, but $(a, M_2'') \in N_2$ also gives $M_2'' \in \text{Tran}_2(t_2)$.

We miss to show that R' is initialized. Let $s_1^0 \in S_1^0$ and $M_1^0 \in \text{Tran}_1(s_1^0)$. As R is initialized, this gets us $M_2^0 \in D_2$ with $(M_1^0, M_2^0) \in R_{d_m^{\mathbb{L}}(ad(\mathcal{A}_1), ad(\mathcal{A}_2))}$, but $M_2^0 \in \text{Tran}_2(s_2^0)$ for some $s_2^0 \in S_2^0$, and then $(s_1^0, s_2^0) \in R'_{d_m^1(ad(\mathcal{A}_1), ad(\mathcal{A}_2))}$.

 $\frac{d_{\mathsf{m}}^{\mathbb{L}}(dn(\mathcal{D}_{1}),dn(\mathcal{D}_{2})) \sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{D}_{1},\mathcal{D}_{2}):}{\text{Let } \mathcal{D}_{1} = (S_{1},S_{1}^{0},-\rightarrow_{1},\longrightarrow_{1}) \text{ and } \mathcal{D}_{2} = (S_{2},S_{2}^{0},-\rightarrow_{2},\longrightarrow_{2}) \text{ be DMTS},}$ with ν -calculus translations $dn(\mathcal{D}_1) = (S_1, S_1^0, \Delta_1)$ and $dn(\mathcal{D}_2) = (S_2, S_2^0, \Delta_2)$. There is a DMTS refinement family $R = \{R_{\alpha} \subseteq S_1 \times S_2 \mid \alpha \in \mathbb{L}\}$ such that for all $s_1^0 \in S_1^0$, there exists $s_2^0 \in S_2^0$ for which $(s_1^0, s_2^0) \in R_{d_{\infty}^{\mathbb{L}}(\mathcal{D}_1, \mathcal{D}_2)}$.

Let $\alpha \in \mathbb{L}$, $(s_1, s_2) \in R_{\alpha}$, $a_1 \in \Sigma$, and $t_1 \in \square_1^{a_1}(s_1)$. Then $s_1 \stackrel{a_1}{\dashrightarrow} t_1$, hence we have $s_2 \stackrel{a_2}{\dashrightarrow} t_2$ and $\beta \in \mathbb{L}$ with $(t_1, t_2) \in R_{\beta}$ and $F(a_1, a_2, \beta) \sqsubseteq_{\mathbb{L}} \alpha$, but then also $t_2 \in \square_2^{a_2}(s_2)$.

Let $N_2 \in \Diamond_2(s_2)$, then also $s_2 \longrightarrow_2 N_2$, so that there must be $s_1 \longrightarrow_1 N_1$ such that $\forall (a_1, t_1) \in N_1 : \exists (a_2, t_2) \in N_2, \beta \in \mathbb{L} : (t_1, t_2) \in R_\beta, F(a_1, a_2, \beta) \sqsubseteq_{\mathbb{L}} \alpha$, but then also $N_1 \in \Diamond_1(s_1)$.

$d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{D}_1, \mathcal{D}_2) \sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}^{\mathbb{L}}(dn(\mathcal{D}_1), dn(\mathcal{D}_2))$:

Let $\mathcal{D}_1 = (S_1, S_1^0, -\rightarrow_1, \longrightarrow_1)$ and $\mathcal{D}_2 = (S_2, S_2^0, -\rightarrow_2, \longrightarrow_2)$ be DMTS, with ν -calculus translations $dn(\mathcal{D}_1) = (S_1, S_1^0, \Delta_1)$ and $dn(\mathcal{D}_2) = (S_2, S_2^0, \Delta_2)$. There is a ν -calculus refinement family $R = \{R_\alpha \subseteq S_1 \times S_2 \mid \alpha \in \mathbb{L}\}$ such that for all $s_1^0 \in S_1^0$, there exists $s_2^0 \in S_2^0$ for which $(s_1^0, s_2^0) \in R_{d_m^{\mathbb{L}}(\mathcal{D}_1, \mathcal{D}_2)}$.

Let $\alpha \in \mathbb{L}$ and $(s_1, s_2) \in R_{\alpha}$, and assume that $s_1 \xrightarrow{a_1} t_1$. Then $t_1 \in \square_1^{a_1}(s_1)$, so that there is $a_2 \in \Sigma$, $t_2 \in \square_2^{a_2}(s_2)$ and $\beta \in \mathbb{L}$ for which $(t_1, t_2) \in R_{\beta}$ and $F(a_1, a_2, \beta) \sqsubseteq_{\mathbb{L}} \alpha$, but then also $s_2 \xrightarrow{a_2} t_2$.

Assume that $s_2 \longrightarrow_2 N_2$, then $N_2 \in \Diamond_2(s_2)$. Hence there is $N_1 \in \Diamond_1(s_1)$ so that $\forall (a_1, t_1) \in N_1 : \exists (a_2, t_2) \in N_2, \beta \in \mathbb{L} : (t_1, t_2) \in R_\beta, F(a_1, a_2, \beta) \sqsubseteq_{\mathbb{L}} \alpha$, but then also $s_1 \longrightarrow_1 N_1$.

$d_{\mathsf{m}}^{\mathbb{L}}(nd(\mathcal{N}_1), nd(\mathcal{N}_2)) \sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{N}_1, \mathcal{N}_2)$:

Let $\mathcal{N}_1 = (X_1, X_1^0, \Delta_1)$, $\mathcal{N}_2 = (X_2, X_2^0, \Delta_2)$ be ν -calculus expressions in normal form, with DMTS translations $nd(\mathcal{N}_1) = (X_1, X_1^0, -\rightarrow_1, \longrightarrow_1)$ and $nd(\mathcal{N}_2) = (X_2, X_2^0, -\rightarrow_2, \longrightarrow_2)$. There is a ν -calculus refinement family $R = \{R_{\alpha} \subseteq X_1 \times X_2 \mid \alpha \in \mathbb{L}\}$ such that for all $x_1^0 \in X_1^0$, there is $x_2^0 \in X_2^0$ for which $(x_1^0, x_2^0) \in R_{d_{\mathbf{m}}^{\mathbb{L}}(\mathcal{N}_1, \mathcal{N}_2)}$.

Let $\alpha \in \mathbb{L}$ and $(x_1, x_2) \in R_{\alpha}$, and assume that $x_1 \stackrel{a_1}{--} \downarrow_1 y_1$. Then $y_1 \in \square_1^{a_1}(x_1)$, hence there are $a_2 \in \Sigma$, $y_2 \in \square_2^{a_2}$ and $\beta \in \mathbb{L}$ such that $(y_1, y_2) \in R_{\beta}$ and $F(a_1, a_2, \beta) \sqsubseteq_{\mathbb{L}} \alpha$, but then also $x_2 \stackrel{a_2}{--} \downarrow_2 y_2$.

Assume that $x_2 \longrightarrow_2 N_2$, then $N_2 \in \Diamond_2(x_2)$. Hence there must be $N_1 \in \Diamond_1(x_1)$ such that $\forall (a_1, y_1) \in N_1 : \exists (a_2, y_2) \in N_2, \beta \in \mathbb{L} : (y_1, y_2) \in R_\beta, F(a_1, a_2, \beta) \sqsubseteq_{\mathbb{L}} \alpha$, but then also $x_1 \longrightarrow_1 N_1$.

$d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{N}_1, \mathcal{N}_2) \sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}^{\mathbb{L}}(\mathit{nd}(\mathcal{N}_1), \mathit{nd}(\mathcal{N}_2))$:

Let $\mathcal{N}_1 = (X_1, X_1^0, \Delta_1)$, $\mathcal{N}_2 = (X_2, X_2^0, \Delta_2)$ be ν -calculus expressions in normal form, with DMTS translations $nd(\mathcal{N}_1) = (X_1, X_1^0, -\rightarrow_1, \longrightarrow_1)$ and $nd(\mathcal{N}_2) = (X_2, X_2^0, -\rightarrow_2, \longrightarrow_2)$. There is a DMTS refinement family $R = \{R_{\alpha} \subseteq X_1 \times X_2 \mid \alpha \in \mathbb{L}\}$ such that for all $x_1^0 \in X_1^0$, there is $x_2^0 \in X_2^0$ for which $(x_1^0, x_2^0) \in R_{d_{\infty}^{\mathbb{L}}(\mathcal{N}_1, \mathcal{N}_2)}$.

Let $\alpha \in \mathbb{L}$, $(x_1, x_2) \in R_{\alpha}$, $a_1 \in \Sigma$, and $y_1 \in \square_1^{a_1}(x_1)$. Then $x_1 \xrightarrow{a_2} y_1$, hence we have $x_2 \xrightarrow{a_2} y_2$ and $\beta \in \mathbb{L}$ so that $(y_1, y_2) \in R_{\beta}$ and $F(a_1, a_2, \beta) \sqsubseteq_{\mathbb{L}} \alpha$, but then also $y_1 \in \square_2^{a_2}(x_2)$.

Let $N_2 \in \Diamond_2(x_2)$, then also $x_2 \longrightarrow_2 N_2$. Hence we must have $x_1 \longrightarrow_1 N_1$ with $\forall (a_1, y_1) \in N_1 : \exists (a_2, y_2) \in N_2, \beta \in \mathbb{L} : (y_1, y_2) \in R_\beta, F(a_1, a_2, \beta) \sqsubseteq_{\mathbb{L}} \alpha$, but then also $N_1 \in \Diamond_1(x_1)$.

9.4.3 Properties

We sum up some important properties of our distances.

9.21 Proposition. For all specifications S_1 , S_2 , $S_1 \leq_{\mathsf{m}} S_2$ implies $d^{\mathbb{L}}_{\mathsf{m}}(S_1, S_2) = \bot_{\mathbb{L}}$, and $S_1 \leq_{\mathsf{t}} S_2$ implies $d^{\mathbb{L}}_{\mathsf{th}}(S_1, S_2) = \bot_{\mathbb{L}}$. If F is recursively separating, then $d^{\mathbb{L}}_{\mathsf{m}}(S_1, S_2) = \bot_{\mathbb{L}}$ implies $S_1 \leq_{\mathsf{m}} S_2$.

Proof: We show the proposition for AA. First, if $A_1 \leq_m A_2$, with $A_1 = (S_1, S_1^0, \operatorname{Tran}_1)$, $A_2 = (S_2, S_2^0, \operatorname{Tran}_2)$, then there is an initialized refinement relation $R \subseteq S_1 \times S_2$, i.e., such that for all $(s_1, s_2) \in R$ and all $M_1 \in \operatorname{Tran}_1(s_1)$, there is $M_2 \in \operatorname{Tran}_2(s_2)$ for which

- $\forall (a_1, t_1) \in M_1 : \exists (a_2, t_2) \in M_2 : a_1 \leq a_2, (t_1, t_2) \in R$ and
- $\forall (a_2, t_2) \in M_2 : \exists (a_1, t_1) \in M_1 : a_1 \leq a_2, (t_1, t_2) \in R.$

Defining $R' = \{R'_{\alpha} \mid \alpha \in \mathbb{L}\}$ by $R'_{\alpha} = R$ for all $\alpha \in \mathbb{L}$, we see that R' is an initialized refinement family which witnesses $d^{\mathbb{L}}_{\mathsf{m}}(\mathcal{A}_1, \mathcal{A}_2) = \bot_{\mathbb{L}}$.

We have shown that $\mathcal{A}_1 \leq_{\mathsf{m}} \mathcal{A}_2$ implies $d^{\mathbb{L}}_{\mathsf{m}}(\mathcal{A}_1, \mathcal{A}_2) = \perp_{\mathbb{L}}$. Now if $\mathcal{A}_1 \leq_{\mathsf{t}} \mathcal{A}_2$ instead, then for all $\mathcal{I} \in [\![\mathcal{A}_1]\!]$, also $\mathcal{I} \in [\![\mathcal{A}_2]\!]$, hence $d^{\mathbb{L}}_{\mathsf{th}}(\mathcal{A}_1, \mathcal{A}_2) = \perp_{\mathbb{L}}$.

To show the last property, assume F to be recursively separating. Define $R \subseteq S_1 \times S_2$ by $R = \{(s_1, s_2) \mid d^{\mathbb{L}}_{\mathsf{m}}(s_1, s_2) = \bot_{\mathbb{L}}\}$; we show that R is a witness for $\mathcal{A}_1 \leq_{\mathsf{m}} \mathcal{A}_2$. By $d^{\mathbb{L}}_{\mathsf{m}}(\mathcal{A}_1, \mathcal{A}_2) = \bot_{\mathbb{L}}$, R is initialized.

Let $(s_1, s_2) \in R$ and $M_1 \in \text{Tran}_1(s_1)$, then there is $M_2 \in \text{Tran}_2(s_2)$ such that

$$\forall (a_1, t_1) \in M_1 : \exists (a_2, t_2) \in M_2, \beta_1 \in \mathbb{L} :$$

$$d_{\mathsf{m}}^{\mathbb{L}}(t_1, t_2) \sqsubseteq_{\mathbb{L}} \beta_1, F(a_1, a_2, \beta_1) = \bot_{\mathbb{L}} ,$$

$$\forall (a_2, t_2) \in M_2 : \exists (a_1, t_1) \in M_1, \beta_1 \in \mathbb{L} :$$

$$d_{\mathsf{m}}^{\mathbb{L}}(t_1, t_2) \sqsubseteq_{\mathbb{L}} \beta_1, F(a_1, a_2, \beta_1) = \bot_{\mathbb{L}} .$$

As F is recursively separating, we must have $a_1 \leq a_2$ in both these equations and $\beta_1 = \beta_2 = \bot_{\mathbb{L}}$. But then $(t_1, t_2) \in R$, hence R is indeed a witness for $A_1 \leq_{\mathsf{m}} A_2$.

9.22 Proposition. The functions $d_{\mathsf{m}}^{\mathbb{L}}$ and $d_{\mathsf{th}}^{\mathbb{L}}$ are \mathbb{L} -hemimetrics, and d_{m} , d_{th} are hemimetrics.

Proof: We show the proof for AA. The properties that $d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{A}, \mathcal{A}) = \perp_{\mathbb{L}}$ and $d_{\mathsf{th}}^{\mathbb{L}}(\mathcal{A}, \mathcal{A}) = \perp_{\mathbb{L}}$ follow from Proposition 9.21.

We show the triangle inequality for $d_{\mathsf{m}}^{\mathbb{L}}$. The triangle inequality for $d_{\mathsf{th}}^{\mathbb{L}}$ will then follow from standard arguments used to show that the Hausdorff metric satisfies the triangle inequality, see for example [AB07, Lemma 3.72]. Let $\mathcal{A}_1 = (S_1, S_1^0, \operatorname{Tran}_1)$, $\mathcal{A}_2 = (S_2, S_2^0, \operatorname{Tran}_2)$, $\mathcal{A}_3 = (S_3, S_3^0, \operatorname{Tran}_3)$ be AA and $R^1 = \{R_{\alpha}^1 \subseteq S_1 \times S_2 \mid \alpha \in \mathbb{L}\}$, $R^2 = \{R_{\alpha}^2 \subseteq S_2 \times S_3 \mid \alpha \in \mathbb{L}\}$ refinement

families such that $\forall s_1^0 \in S_1^0 : \exists s_2^0 \in S_2^0 : (s_1^0, s_2^0) \in R^1_{d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{A}_1, \mathcal{A}_2)}$ and $\forall s_2^0 \in S_2^0 : \exists s_3^0 \in S_3^0 : (s_2^0, s_3^0) \in R^2_{d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{A}_2, \mathcal{A}_3)}.$ Define $R = \{R_{\alpha} \subseteq S_1 \times S_3 \mid \alpha \in \mathbb{L}\}$ by

$$R_{\alpha} = \{(s_1, s_3) \mid \exists \alpha_1, \alpha_2 \in \mathbb{L}, s_2 \in S_2 : (s_1, s_2) \in R^1_{\alpha_1}, (s_2, s_3) \in R^2_{\alpha_2}, \alpha_1 \oplus_{\mathbb{L}} \alpha_2 = \alpha \}.$$

Then for all $s_1^0 \in S_1^0$ there is $s_3^0 \in S_3^0$ such that $(s_1^0, s_3^0) \in R_{d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{A}_1, \mathcal{A}_2) \oplus_{\mathbb{L}} d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{A}_2, \mathcal{A}_3)}$; we show that R is a refinement family from \mathcal{A}_1 to \mathcal{A}_2 .

Let $\alpha \in \mathbb{L}$ and $(s_1, s_3) \in R_{\alpha}$, then we have $\alpha_1, \alpha_2 \in \mathbb{L}$ and $s_2 \in S_2$ such that $\alpha_1 \oplus_{\mathbb{L}} \alpha_2 = \alpha$, $(s_1, s_2) \in R^1_{\alpha_1}$ and $(s_2, s_3) \in R^2_{\alpha_2}$. Let $M_1 \in \operatorname{Tran}_1(s_1)$, then we have $M_2 \in \operatorname{Tran}_2(s_2)$ such that

$$\forall (a_{1}, t_{1}) \in M_{1} : \exists (a_{2}, t_{2}) \in M_{2}, \beta_{1} \in \mathbb{L} :$$

$$(t_{1}, t_{2}) \in R_{\beta_{1}}^{1}, F(a_{1}, a_{2}, \beta_{1}) \sqsubseteq_{\mathbb{L}} \alpha_{1}, \qquad (9.5)$$

$$\forall (a_{2}, t_{2}) \in M_{2} : \exists (a_{1}, t_{1}) \in M_{1}, \beta_{1} \in \mathbb{L} :$$

$$(t_{1}, t_{2}) \in R_{\beta_{1}}^{1}, F(a_{1}, a_{2}, \beta_{1}) \sqsubseteq_{\mathbb{L}} \alpha_{1}. \qquad (9.6)$$

This in turn implies that there is $M_3 \in \text{Tran}_3(s_3)$ with

$$\forall (a_{2}, t_{2}) \in M_{2} : \exists (a_{3}, t_{3}) \in M_{3}, \beta_{2} \in \mathbb{L} :$$

$$(t_{2}, t_{3}) \in R_{\beta_{2}}^{2}, F(a_{2}, a_{3}, \beta_{2}) \sqsubseteq_{\mathbb{L}} \alpha_{2}, \qquad (9.7)$$

$$\forall (a_{3}, t_{3}) \in M_{3} : \exists (a_{2}, t_{2}) \in M_{2}, \beta_{2} \in \mathbb{L} :$$

$$(t_{2}, t_{3}) \in R_{\beta_{3}}^{2}, F(a_{2}, a_{3}, \beta_{2}) \sqsubseteq_{\mathbb{L}} \alpha_{2}. \qquad (9.8)$$

Now let $(a_1,t_1) \in M_1$, then we get $(a_2,t_2) \in M_2$, $(a_3,t_3) \in M_3$ and $\beta_1,\beta_2 \in \mathbb{L}$ as in (9.5) and (9.7). Let $\beta = \beta_1 \oplus_{\mathbb{L}} \beta_2$, then $(t_1,t_3) \in R_{\beta}$, and by the extended triangle inequality for F, $F(a_1,a_3,\beta) \sqsubseteq_{\mathbb{L}} F(a_1,a_2,\beta_1) \oplus_{\mathbb{L}} F(a_2,a_3,\beta_2) \sqsubseteq_{\mathbb{L}} \alpha_1 \oplus_{\mathbb{L}} \alpha_2 = \alpha$.

Similarly, given $(a_3, t_3) \in M_3$, we can apply (9.8) and (9.6) to get $(a_1, t_1) \in M_1$ and $\beta \in \mathbb{L}$ such that $(t_1, t_3) \in R_\beta$ and $F(a_1, a_3, \beta) \sqsubseteq_{\mathbb{L}} \alpha$.

We have shown that $d_{\mathsf{m}}^{\mathbb{L}}$ and $d_{\mathsf{tr}}^{\mathbb{L}}$ are \mathbb{L} -hemimetrics. Using monotonicity of the eval function, it follows that d_{m} and d_{tr} are hemimetrics.

9.23 Proposition. For the discrete distances, $d_{\mathsf{m}}^{\mathrm{disc}}(\mathcal{S}_1, \mathcal{S}_2) = 0$ if $\mathcal{S}_1 \leq_{\mathsf{m}} \mathcal{S}_2$ and ∞ otherwise. Similarly, $d_{\mathsf{th}}^{\mathrm{disc}}(\mathcal{S}_1, \mathcal{S}_2) = 0$ if $\mathcal{S}_1 \leq_{\mathsf{t}} \mathcal{S}_2$ and ∞ otherwise.

Proof: We show the proposition for AA. We already know that, also for the discrete distances, $\mathcal{A}_1 \leq_{\mathsf{m}} \mathcal{A}_2$ implies $d_{\mathsf{m}}(\mathcal{A}_1, \mathcal{A}_2) = 0$ and that $\mathcal{A}_1 \leq_{\mathsf{t}} \mathcal{A}_2$ implies $d_{\mathsf{th}}(\mathcal{A}_1, \mathcal{A}_2) = 0$. We show that $d_{\mathsf{m}}(\mathcal{A}_1, \mathcal{A}_2) = 0$ implies $\mathcal{A}_1 \leq_{\mathsf{m}} \mathcal{A}_2$. Let $R = \{R_{\alpha} \subseteq S_1 \times S_2 \mid \alpha \in \mathbb{L}\}$ be a refinement family such that $\forall s_1^0 \in S_1^0: \exists s_2^0 \in S_2^0: (s_1^0, s_2^0) \in R_0$. We show that R_0 is a witness for $\mathcal{A}_1 \leq_{\mathsf{m}} \mathcal{A}_2$; it is clearly initialized.

Let $(s_1, s_2) \in R_0$ and $M_1 \in \text{Tran}_1(s_1)$, then we have $M_2 \in \text{Tran}_2(s_2)$ such that

$$\forall (a_1, t_1) \in M_1 : \exists (a_2, t_2) \in M_2, \beta \in \mathbb{L} : (t_1, t_2) \in R_\beta, F(a_1, a_2, \beta) = 0, \forall (a_2, t_2) \in M_2 : \exists (a_1, t_1) \in M_1, \beta \in \mathbb{L} : (t_1, t_2) \in R_\beta, F(a_1, a_2, \beta) = 0.$$

$$(9.9)$$

Using the definition of the distance, we see that the condition $F(a_1, a_2, \beta) = 0$ is equivalent to $a_1 \leq a_2$ and $\beta = 0$, hence (9.9) degenerates to

$$\forall (a_1, t_1) \in M_1 : \exists (a_2, t_2) \in M_2 : (t_1, t_2) \in R_0, a_1 \leq a_2,$$

$$\forall (a_2, t_2) \in M_2 : \exists (a_1, t_1) \in M_1 : (t_1, t_2) \in R_0, a_1 \leq a_2,$$

which are exactly the conditions for R_0 to be a modal refinement.

Again by definition, we see that for any AA \mathcal{A}_1 , \mathcal{A}_2 , either $d_{\mathsf{m}}(\mathcal{A}_1, \mathcal{A}_2) = 0$ or $d_{\mathsf{m}}(\mathcal{A}_1, \mathcal{A}_2) = \infty$, hence $\mathcal{A}_1 \not\leq_{\mathsf{m}} \mathcal{A}_2$ implies that $d_{\mathsf{m}}(\mathcal{A}_1, \mathcal{A}_2) = \infty$.

To show the last part of the proposition, we notice that

$$\begin{split} \mathit{d}_{\mathsf{th}}(\mathcal{A}_1, \mathcal{A}_2) &= \sup_{\mathcal{I}_1 \in \llbracket \mathcal{A}_1 \rrbracket} \inf_{\mathcal{I}_2 \in \llbracket \mathcal{A}_2 \rrbracket} \mathit{d}_{\mathsf{m}}(\mathcal{I}_1, \mathcal{I}_2) \\ &= \begin{cases} 0 & \text{if } \forall \mathcal{I}_1 \in \llbracket \mathcal{A}_1 \rrbracket : \exists \mathcal{I}_2 \in \llbracket \mathcal{A}_2 \rrbracket : \mathcal{I}_1 \leq_{\mathsf{m}} \mathcal{I}_2 \,, \\ \infty & \text{otherwise} \,, \end{cases} \\ &= \begin{cases} 0 & \text{if } \llbracket \mathcal{A}_1 \rrbracket \subseteq \llbracket \mathcal{A}_2 \rrbracket \,, \\ \infty & \text{otherwise} \,. \end{cases} \end{split}$$

Hence
$$d_{\mathsf{th}}(\mathcal{A}_1, \mathcal{A}_2) = 0$$
 if $\mathcal{A}_1 \leq_{\mathsf{t}} \mathcal{A}_2$ and $d_{\mathsf{th}}(\mathcal{A}_1, \mathcal{A}_2) = \infty$ otherwise.

As a quantitative analogy to the implication from (Boolean) modal refinement to thorough refinement (see Proposition 9.6), the next theorem shows that thorough refinement distance is bounded above by modal refinement distance. Note that for the discrete trace distance (and using Proposition 9.23), this is equivalent to the Boolean statement.

9.24 Theorem. For all specifications S_1 , S_2 , $d_{\mathsf{th}}^{\mathbb{L}}(S_1, S_2) \sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}^{\mathbb{L}}(S_1, S_2)$.

Proof: We prove the statement for AA; for DMTS and ν -calculus expressions it then follows from Theorem 9.20.

Let $\mathcal{A}_1 = (S_1, S_1^0, \operatorname{Tran}_1)$, $\mathcal{A}_2 = (S_2, S_2^0, \operatorname{Tran}_2)$. We have a refinement family $R = \{R_\alpha \subseteq S_1 \times S_2 \mid \alpha \in \mathbb{L}\}$ such that for all $s_1^0 \in S_1^0$, there is $s_2^0 \in S_2^0$ with $(s_1^0, s_2^0) \in R_{d_{\mathbb{L}}^{\mathbb{L}}(\mathcal{A}_1, \mathcal{A}_2)}$. Let $\mathcal{I} = (S, S^0, T) \in [\![\mathcal{A}_1]\!]$, *i.e.*, $\mathcal{I} \leq_{\mathsf{m}} \mathcal{A}_1$.

Let $R^1 \subseteq S \times S_1$ be an initialized modal refinement, and define a relation family $R^2 = \{R^2_{\alpha} \subseteq S \times S_2 \mid \alpha \in \mathbb{L}\}$ by $R^2_{\alpha} = R^1 \circ R_{\alpha} = \{(s, s_2) \mid \exists s_1 \in S : (s, s_1) \in R^1, (s_1, s_2) \in R_{\alpha}$. We define a LTS $\mathcal{I}_2 = (S_2, S_2^0, T_2)$ as follows:

For all $\alpha \in \mathbb{L}$ with $\alpha \neq \top_{\mathbb{L}}$ and $(s, s_2) \in R^2_{\alpha}$: We must have $s_1 \in S_1$ with $(s, s_1) \in R^1$ and $(s_1, s_2) \in R_{\alpha}$. Then there is $M_1 \in \text{Tran}_1(s_1)$ such that

- for all $s \xrightarrow{a} t$, there is $(a, t_1) \in M_1$ with $(t, t_1) \in R_1$,
- for all $(a_1, t_1) \in M_1$, there is $s \xrightarrow{a} t$ with $(t, t_1) \in R_1$.

This in turn implies that there is $M_2 \in \text{Tran}_2(s_2)$ satisfying the conditions in Definition 9.18. For all $(a_2, t_2) \in M_2$: add a transition $s_2 \xrightarrow{a_2} t_2$ to T_2 .

We show that the identity relation $\{(s_2, s_2) \mid s_2 \in S_2\}$ is a witness for $\mathcal{I}_2 \leq_{\mathsf{m}} \mathcal{A}_2$. Let $s_2 \in S_2$ and $s_2 \xrightarrow{a_2} t_2$. By construction, there is an $M_2 \in \operatorname{Tran}_2(s_2)$ with $(a_2, t_2) \in M_2$, and for all $(a'_2, t'_2) \in M_2$, $s_2 \xrightarrow{a'_2} t'_2$.

We show that R^2 is a witness for $d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{I}, \mathcal{I}_2)$; clearly, R^2 is initialized. Let $\alpha \in \mathbb{L}$ with $\alpha \neq \top_{\mathbb{L}}$ and $(s, s_2) \in R_{\alpha}^2$, then there is $s_1 \in S_1$ with $(s, s_1) \in R^1$ and $(s_1, s_2) \in R_{\alpha}$. We also have $M_1 \in \mathrm{Tran}_1(s_1)$ such that

- for all $s \xrightarrow{a} t$, there is $(a, t_1) \in M_1$ with $(t, t_1) \in R^1$,
- for all $(a, t_1) \in M_1$, there is $s \xrightarrow{a} t$ with $(t, t_1) \in R^1$

and thus $M_2 \in \text{Tran}_2(s_2)$ satisfying the conditions in Definition 9.18.

Let $s \xrightarrow{a} t$, then there is $(a, t_1) \in M_1$ with $(t, t_1) \in R^1$, hence also $(a_2, t_2) \in M_2$ and $\beta \in \mathbb{L}$ with $(t_1, t_2) \in R_\beta$ and $F(a, a_2, \beta) \sqsubseteq_{\mathbb{L}} \alpha$. But then $(t, t_2) \in R_\beta^2$, and $s_2 \xrightarrow{a_2} t_2$ by construction.

Let $s_2 \xrightarrow{a_2} t_2$. By construction, there is an $M_2 \in \text{Tran}_2(s_2)$ with $(a_2, t_2) \in M_2$. This implies that there is $M_1 \in \text{Tran}_1(s_1)$, $\beta \in \mathbb{L}$ and $(a_1, t_1) \in M_1$ with $(t_1, t_2) \in R_\beta$ and $F(a_1, a_2, \beta) \sqsubseteq \alpha$. But then there is also $s \xrightarrow{a_1} t$ with $(t, t_1) \in R^1$, hence $(t, t_2) \in R^2_\beta$.

9.4.4 Disjunction and conjunction

In order to generalize the properties of Theorem 9.11 to our quantitative setting, we introduce a notion of relaxed implementation semantics:

9.25 Definition. The α -relaxed implementation semantics of \mathcal{S} , for a specification \mathcal{S} and $\alpha \in \mathbb{L}$, is

$$[\![\mathcal{S}]\!]^\alpha = \{\mathcal{I} \text{ implementation } | \ d^{\mathbb{L}}_{\mathsf{m}}(\mathcal{I},\mathcal{S}) \sqsubseteq \alpha\} \,.$$

Hence, $[\![\mathcal{S}]\!]^{\alpha}$ comprises all labeled transition systems which are implementations of \mathcal{S} up to α . Note that by Proposition 9.21 and for F recursively separating, $[\![\mathcal{S}]\!]^{\perp_{\mathbb{L}}} = [\![\mathcal{S}]\!]$.

- **9.26 Theorem.** For all specifications S_1 , S_2 , S_3 and $\alpha \in \mathbb{L}$,
 - $d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{S}_1 \vee \mathcal{S}_2, \mathcal{S}_3) = \max(d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{S}_1, \mathcal{S}_3), d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{S}_2, \mathcal{S}_3)),$
 - $d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{S}_1, \mathcal{S}_2 \wedge \mathcal{S}_3) \sqsupseteq_{\mathbb{L}} \max(d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{S}_1, \mathcal{S}_2), d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{S}_1, \mathcal{S}_3)),$
 - $[S_1 \vee S_2]^{\alpha} = [S_1]^{\alpha} \cup [S_2]^{\alpha}$, and

• $[S_1 \wedge S_2]^{\alpha} \subseteq [S_1]^{\alpha} \cap [S_2]^{\alpha}$.

Proof: We show the proof for DMTS.

The proof that $d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{D}_1 \vee \mathcal{D}_2, \mathcal{D}_3) = \max(d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{D}_1, \mathcal{D}_3), d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{D}_2, \mathcal{D}_3))$ is trivial: any refinement family witnessing $d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{D}_1 \vee \mathcal{D}_2, \mathcal{D}_3)$ splits into two families witnessing $d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{D}_1, \mathcal{D}_3)$ and $d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{D}_2, \mathcal{D}_3)$ and vice versa.

To show that $d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{D}_1, \mathcal{D}_2 \wedge \mathcal{D}_3) \supseteq_{\mathbb{L}} \max(d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{D}_1, \mathcal{D}_2), d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{D}_1, \mathcal{D}_3))$, let $R = \{R_{\alpha} \subseteq S_1 \times (S_2 \times S_3) \mid \alpha \in \mathbb{L}\}$ be a witness for $d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{D}_1, \mathcal{D}_2 \wedge \mathcal{D}_3)$ and define $R^2 = \{R_{\alpha}^2 \subseteq S_1 \times S_2 \mid \alpha \in \mathbb{L}\}$ by $R_{\alpha}^2 = \{(s_1, s_2) \mid \exists s_3 \in S_3 : (s_1, (s_2, s_3)) \in R_{\alpha}\}$ for all $\alpha \in \mathbb{L}$.

Let $s_1^0 \in S_1^0$, then we have $(s_2^0, s_3^0) \in S_2^0 \times S_3^0$ so that $(s_1^0, (s_2^0, s_3^0)) \in R_{d_{\mathbb{D}}^{\mathbb{D}}(\mathcal{D}_1, \mathcal{D}_2 \wedge \mathcal{D}_3)}$, hence also $(s_1^0, s_2^0) \in R_{d_{\mathbb{D}}^{\mathbb{D}}(\mathcal{D}_1, \mathcal{D}_2 \wedge \mathcal{D}_3)}^{d_{\mathbb{D}}(\mathcal{D}_1, \mathcal{D}_2 \wedge \mathcal{D}_3)}$.

Let $\alpha \in \mathbb{L}$ and $(s_1, s_2) \in R_{\alpha}^2$, then we have $s_3 \in S_3$ for which $(s_1, (s_2, s_3)) \in R_{\alpha}$. Assume first that $s_1 \stackrel{a_1}{\longrightarrow} t_1$, then there is $(s_2, s_3) \stackrel{a}{\longrightarrow} (t_2, t_3)$ and $\beta \in \mathbb{L}$ such that $F(a_1, a, \beta) \sqsubseteq_{\mathbb{L}} \alpha$ and $(t_1, (t_2, t_3)) \in R_{\beta}$, hence $(t_1, t_2) \in R_{\beta}^2$. By construction of $\mathcal{D}_2 \wedge \mathcal{D}_3$, there are $s_2 \stackrel{a_2}{\longrightarrow} t_2$ and $s_3 \stackrel{a_3}{\longrightarrow} t_3$ such that $a = a_2 \otimes a_3$, but then by anti-monotonicity, $F(a_1, a_2, \beta) \sqsubseteq_{\mathbb{L}} F(a_1, a, \beta) \sqsubseteq \alpha$.

Now assume $s_2 \longrightarrow N_2$, then, by construction, $(s_2, s_3) \longrightarrow N = \{(a_2 \otimes a_3, (t_2, t_3)) \mid (a_2, t_2) \in N_2, s_3 \stackrel{a_3}{\dashrightarrow} t_3\}$. Hence we have $s_1 \longrightarrow_1 N_1$ such that $\forall (a_1, t_1) \in N_1 : \exists (a, (t_2, t_3)) \in N, \beta \in \mathbb{L} : F(a_1, a, \beta) \sqsubseteq_{\mathbb{L}} \alpha, (t_1, (t_2, t_3)) \in R_{\beta}$.

Let $(a_1, t_1) \in N_1$, then we have $(a, (t_2, t_3)) \in N$ and $\beta \in \mathbb{L}$ for which $F(a_1, a, \beta) \sqsubseteq_{\mathbb{L}} \alpha$ and $(t_1, (t_2, t_3)) \in R_{\beta}$, hence $(t_1, t_2) \in R_{\beta}^2$. By construction of N, this implies that there are $(a_2, t_2) \in N_2$ and $s_3 \xrightarrow{a_3} t_3$ such that $a = a_2 \otimes a_3$, but then by anti-monotonicity, $F(a_1, a_2, \beta) \sqsubseteq_{\mathbb{L}} F(a_1, a, \beta) \sqsubseteq \alpha$.

We have shown that $d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{D}_1, \mathcal{D}_2 \wedge \mathcal{D}_3) \sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{D}_1, \mathcal{D}_2)$. The proof of $d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{D}_1, \mathcal{D}_2 \wedge \mathcal{D}_3) \sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{D}_1, \mathcal{D}_3)$ is entirely analogous.

The inclusion $[\![\mathcal{D}_1 \wedge \mathcal{D}_2]\!]^{\alpha} \subseteq [\![\mathcal{D}_1]\!]^{\alpha} \cap [\![\mathcal{D}_2]\!]^{\alpha}$ is clear now: If $\mathcal{I} \in [\![\mathcal{D}_1 \wedge \mathcal{D}_2]\!]^{\alpha}$, i.e., $d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{I}, \mathcal{D}_1 \wedge \mathcal{D}_2) \sqsubseteq_{\mathbb{L}} \alpha$, then also $d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{I}, \mathcal{D}_1) \sqsubseteq_{\mathbb{L}} \alpha$ and $d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{I}, \mathcal{D}_2) \sqsubseteq_{\mathbb{L}} \alpha$, thus $\mathcal{I} \in [\![\mathcal{D}_1]\!]^{\alpha} \cap [\![\mathcal{D}_2]\!]^{\alpha}$.

To show that $[\![\mathcal{D}_1 \vee \mathcal{D}_2]\!]^{\alpha} = [\![\mathcal{D}_1]\!]^{\alpha} \cup [\![\mathcal{D}_2]\!]^{\alpha}$, one notices, like in the proof of Theorem 9.11, that for any LTS \mathcal{I} , any refinement family witnessing $d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{I}, \mathcal{D}_1)$ or $d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{I}, \mathcal{D}_2)$ is also a witness for $d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{I}, \mathcal{D}_1 \vee \mathcal{D}_2)$ and vice versa.

The below example shows why the inclusions above cannot be replaced by equalities. To sum up, disjunction is quantitatively sound and complete, whereas conjunction is only quantitatively sound.

9.27 Example. For the point-wise or discounting distances, the DMTS in Figure 9.3 are such that $d_{\mathsf{m}}(\mathcal{I}, \mathcal{D}_1) = 1$ and $d_{\mathsf{m}}(\mathcal{I}, \mathcal{D}_2) = 1$, but $d_{\mathsf{m}}(\mathcal{I}, \mathcal{D}_1 \wedge \mathcal{D}_2) = \infty$. Hence $d_{\mathsf{m}}(\mathcal{I}, \mathcal{D}_1 \wedge \mathcal{D}_2) \neq \max(d_{\mathsf{m}}(\mathcal{I}, \mathcal{D}_1), d_{\mathsf{m}}(\mathcal{I}, \mathcal{D}_2))$, and $\mathcal{I} \in [\![\mathcal{D}_1]\!]^1 \cap [\![\mathcal{D}_2]\!]^1$, but $\mathcal{I} \notin [\![\mathcal{D}_1 \wedge \mathcal{D}_2]\!]^1$.

Figure 9.3: LTS \mathcal{I} together with DMTS \mathcal{D}_1 , \mathcal{D}_2 and their conjunction. For the point-wise or discounting distances, $d_{\mathsf{m}}(\mathcal{I}, \mathcal{D}_1) = d_{\mathsf{m}}(\mathcal{I}, \mathcal{D}_2) = 1$, but $d_{\mathsf{m}}(\mathcal{I}, \mathcal{D}_1 \wedge \mathcal{D}_2) = \infty$

9.4.5 Structural composition and quotient

We proceed to devise a quantitative generalization of the properties of structural composition and quotient exposed in Section 9.3. To this end, we need to use a *uniform composition bound* on labels:

Let $P: \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ be a function which is monotone in both coordinates, has $P(\alpha, \perp_{\mathbb{L}}) = P(\perp_{\mathbb{L}}, \alpha) = \alpha$ and $P(\alpha, \top_{\mathbb{L}}) = P(\top_{\mathbb{L}}, \alpha) = \top_{\mathbb{L}}$ for all $\alpha \in \mathbb{L}$. We require that for all $a_1, b_1, a_2, b_2 \in \Sigma$ and $\alpha, \beta \in \mathbb{L}$ with $F(a_1, a_2, \alpha) \neq \top_{\mathbb{L}}$ and $F(b_1, b_2, \beta) \neq \top_{\mathbb{L}}$, $a_1 \oplus b_1$ is defined iff $a_2 \oplus b_2$ is, and if both are defined, then

$$F(a_1 \oplus b_1, a_2 \oplus b_2, P(\alpha, \beta)) \sqsubseteq_{\mathbb{L}} P(F(a_1, a_2, \alpha), F(b_1, b_2, \beta)). \tag{9.10}$$

Note that (9.10) implies that

$$d_{\mathsf{tr}}^{\mathbb{L}}(a_1 \oplus a_2, b_1 \oplus b_2) \sqsubseteq_{\mathbb{L}} P(d_{\mathsf{tr}}^{\mathbb{L}}(a_1, b_1), d_{\mathsf{tr}}^{\mathbb{L}}(a_2, b_2)). \tag{9.11}$$

Hence P provides a *uniform bound* on distances between synchronized labels, and (9.10) extends this property so that it holds recursively. Also, this is a generalization of the condition that we imposed on \oplus in Section 9.1; it is shown in [FL14a, p. 18] that it holds for all common label synchronizations.

Remark that P can be understood as a (generalized) modulus of continuity [Con] for the partial function $f: \Sigma \times \Sigma \to \Sigma$ given by label synchronization $f(a,b) = a \oplus b$: with that notation, (9.11) asserts that the distance from $f(a_1,a_2)$ to $f(b_1,b_2)$ is bounded by P applied to the distance from (a_1,a_2) to (b_1,b_2) .

The following theorems show that composition is uniformly continuous (*i.e.*, a quantitative generalization of independent implementability; Corollary 9.13) and that quotient preserves and reflects refinement distance (a quantitative generalization of Theorem 9.15).

9.28 Theorem (Independent implementability). For all specifications S_1 , S_2 , S_3 , S_4 , $d_{\mathsf{m}}^{\mathbb{L}}(S_1 \| S_2, S_3 \| S_4) \sqsubseteq_{\mathbb{L}} P(d_{\mathsf{m}}^{\mathbb{L}}(S_1, S_3), d_{\mathsf{m}}^{\mathbb{L}}(S_2, S_4))$.

Proof: We show the proof for AA. For i = 1, 2, 3, 4, let $\mathcal{A}_i = (S_i, S_i^0, \operatorname{Tran}_i)$. Let $R^1 = \{R_{\alpha}^1 \subseteq S_1 \times S_3 \mid \alpha \in \mathbb{L}\}$, $R^2 = \{R_{\alpha}^2 \subseteq S_2 \times S_4 \mid \alpha \in \mathbb{L}\}$ be refinement families such that $\forall s_1^0 \in S_1^0 : \exists s_3^0 \in S_3^0 : (s_1^0, s_3^0) \in R_{d_{\mathfrak{m}}^{\mathbb{L}}(\mathcal{A}_1, \mathcal{A}_3)}^1$ and $\forall s_2^0 \in S_2^0 : \exists s_4^0 \in S_4^0 : (s_2^0, s_4^0) \in R_{d_{\mathfrak{m}}^{\mathbb{L}}(\mathcal{A}_2, \mathcal{A}_4)}^2$. Define $R = \{R_{\alpha} \subseteq (S_1 \times S_2) \times (S_3 \times S_4) \mid \alpha \in \mathbb{L}\}$ by

$$R_{\alpha} = \{ ((s_1, s_2), (s_3, s_4)) \mid \exists \alpha_1, \alpha_2 \in \mathbb{L} : (s_1, s_3) \in R^1_{\alpha_1}, (s_2, s_4) \in R^2_{\alpha_2}, P(\alpha_1, \alpha_2) \sqsubseteq_{\mathbb{L}} \alpha \} ,$$

then it is clear that $\forall (s_1^0, s_2^0) \in S_1^0 \times S_2^0 : \exists (s_3^0, s_4^0) \in S_3^0 \times S_4^0 : ((s_1^0, s_2^0), (s_3^0, s_4^0)) \in R_{P(d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{A}_1, \mathcal{A}_3), d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{A}_2, \mathcal{A}_4))}.$ We show that R is a refinement family from $\mathcal{A}_1 \| \mathcal{A}_2$ to $\mathcal{A}_3 \| \mathcal{A}_4$.

Let $\alpha \in \mathbb{L}$ and $((s_1, s_2), (s_3, s_4)) \in R_{\alpha}$, then we have $\alpha_1, \alpha_2 \in \mathbb{L}$ with $(s_1, s_3) \in R^1_{\alpha_1}$, $(s_2, s_4) \in R^2_{\alpha_2}$ and $P(\alpha_1, \alpha_2) \sqsubseteq_{\mathbb{L}} \alpha$. Let $M_{12} \in \text{Tran}((s_1, s_2))$, then there must be $M_1 \in \text{Tran}_1(s_1)$, $M_2 \in \text{Tran}_2(s_2)$ for which $M_{12} = M_1 \oplus M_2$. Thus we also have $M_3 \in \text{Tran}_3(s_3)$ and $M_4 \in \text{Tran}_4(s_4)$ such that

$$\forall (a_1, t_1) \in M_1 : \exists (a_3, t_3) \in M_3, \beta_1 \in \mathbb{L} :$$

$$(t_1, t_3) \in R^1_{\beta_1}, F(a_1, a_3, \beta_1) \sqsubseteq_{\mathbb{L}} \alpha_1,$$
 (9.12)

 $\forall (a_3, t_3) \in M_3 : \exists (a_1, t_1) \in M_1, \beta_1 \in \mathbb{L} :$

$$(t_1, t_3) \in R^1_{\beta_1}, F(a_1, a_3, \beta_1) \sqsubseteq_{\mathbb{L}} \alpha_1,$$
 (9.13)

 $\forall (a_2, t_2) \in M_2 : \exists (a_4, t_4) \in M_4, \beta_2 \in \mathbb{L} :$

$$(t_2, t_4) \in R^2_{\beta_2}, F(a_2, a_4, \beta_2) \sqsubseteq_{\mathbb{L}} \alpha_2,$$
 (9.14)

 $\forall (a_4, t_4) \in M_4 : \exists (a_2, t_2) \in M_2, \beta_2 \in \mathbb{L} :$

$$(t_2, t_4) \in R^2_{\beta_2}, F(a_2, a_4, \beta_2) \sqsubseteq_{\mathbb{L}} \alpha_2.$$
 (9.15)

Let $M_{34} = M_3 \oplus M_4$, then $M_{34} \in \text{Tran}((s_3, s_4))$. Let $(a_{12}, (t_1, t_2)) \in M_{12}$, then there are $(a_1, t_1) \in M_1$ and $(a_2, t_2) \in M_2$ for which $a_{12} = a_1 \oplus a_2$. Using (9.12) and (9.14), we get $(a_3, t_3) \in M_3$, $(a_4, t_4) \in M_4$ and $\beta_1, \beta_2 \in \mathbb{L}$ such that $(t_1, t_3) \in R^1_{\beta_1}$, $(t_2, t_4) \in R^2_{\beta_2}$, $F(a_1, a_3, \beta_1) \sqsubseteq_{\mathbb{L}} \alpha_1$, and $F(a_2, a_4, \beta_2) \sqsubseteq_{\mathbb{L}} \alpha_2$.

Let $a_{34} = a_3 \oplus a_4$ and $\beta = P(\beta_1, \beta_2)$, then we have $(a_{34}, (t_3, t_4)) \in M_{34}$. Also, $(t_1, t_3) \in R^1_{\beta_1}$ and $(t_2, t_4) \in R^2_{\beta_2}$ imply that $((t_1, t_2), (t_3, t_4)) \in R_{\beta}$, and

$$F(a_{12}, a_{34}, \beta) = F(a_1 \oplus a_2, a_3 \oplus a_4, P(\beta_1, \beta_2))$$

$$\sqsubseteq P(F(a_1, a_3, \beta_1), F(a_2, a_4, \beta_2))$$

$$\sqsubseteq_{\mathbb{L}} P(\alpha_1, \alpha_2) \sqsubseteq_{\mathbb{L}} \alpha.$$

We have shown that for all $(a_{12}, (t_1, t_2)) \in M_{12}$, there exists $(a_{34}, (t_3, t_4)) \in M_{34}$ and $\beta \in \mathbb{L}$ such that $((t_1, t_2), (t_3, t_4)) \in R_{\beta}$ and $F(a_{12}, a_{34}, \beta) \sqsubseteq_{\mathbb{L}} \alpha$. To show the reverse property, starting from an element $(a_{34}, (t_3, t_4)) \in M_{34}$, we can proceed entirely analogous, using (9.13) and (9.15).

9.29 Theorem. For all specifications S_1 , S_2 , S_3 , $d_{\mathsf{m}}^{\mathbb{L}}(S_1 || S_2, S_3) = d_{\mathsf{m}}^{\mathbb{L}}(S_2, S_3 / S_1)$.

Proof: We show the proof for AA. Let $\mathcal{A}_i = (S_i, S_i^0, \operatorname{Tran}_i)$ for $i = 1, \ldots, 3$; we prove that $d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{A}_1 || \mathcal{A}_2, \mathcal{A}_3) = d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{A}_2, \mathcal{A}_3 / \mathcal{A}_1)$.

We assume that the elements of $Tran_1(s_1)$ are pairwise disjoint for each $s_1 \in S_1$; this can be achieved by, if necessary, splitting states.

Define $R = \{R_{\alpha} \subseteq S_1 \times S_2 \times S_3 \mid \alpha \in \mathbb{L}\}$ by $R_{\alpha} = \{(s_1 | s_2, s_3) \mid d_{\mathsf{m}}^{\mathbb{L}}(s_2, s_3/s_1) \sqsubseteq_{\mathbb{L}} \alpha\}$. (We again abuse notation and write $(s_1 | s_2, s_3)$ instead of (s_1, s_2, s_3) .) We show that R is a witness for $d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{A}_1 | \mathcal{A}_2, \mathcal{A}_3)$.

Let $s_1^0 \| s_2^0 \in S_1^0 \times S_2^0$, then there is $s_3^0 / s_1^0 \in s^0$ for which it holds that $d_{\mathsf{m}}^{\mathbb{L}}(s_2^0, s_3^0 / s_1^0) \sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{A}_2, \mathcal{A}_3 / \mathcal{A}_1)$, hence $(s_1^0 \| s_1^0, s_3^0) \in R_{d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{A}_2, \mathcal{A}_3 / \mathcal{A}_1)}$.

Let $\alpha \in \mathbb{L} \setminus \{\top_{\mathbb{L}}\}$, $(s_1||s_2,s_3) \in R_{\alpha}$ and $M_{\parallel} \in \operatorname{Tran}_{\parallel}(s_1||s_2)$. Then $M_{\parallel} = M_1||M_2$ with $M_1 \in \operatorname{Tran}_1(s_1)$ and $M_2 \in \operatorname{Tran}_2(s_2)$. As $d_{\mathsf{m}}^{\mathbb{L}}(s_2,s_3/s_1) \sqsubseteq_{\mathbb{L}} \alpha$, we can pair M_2 with an $M_{/} \in \operatorname{Tran}_{/}(s_3/s_1)$, *i.e.*, such that the conditions in Definition 9.18 are satisfied.

Let $M_3 = M_/ \triangleright M_1$. We show that the conditions in Definition 9.18 are satisfied for the pair M_{\parallel}, M_3 :

- Let $(a, t_1 || t_2) \in M_{\parallel}$, then there are $a_1, a_2 \in \Sigma$ with $a = a_1 \oplus a_2$ and $(a_1, t_1) \in M_1$, $(a_2, t_2) \in M_2$. Hence there is $(a'_2, t) \in M_{/}$ and $\beta \in \mathbb{L}$ such that $F(a_2, a'_2, \beta) \sqsubseteq_{\mathbb{L}} \alpha$ and $d^{\mathbb{L}}_{\mathbb{L}}(t_2, t) \sqsubseteq_{\mathbb{L}} \beta$.
 - Note that $a_3 = a_1 \oplus a_2'$ is defined and $F(a, a_3, \beta) \sqsubseteq \alpha$. Write $t = \{t_3^1/t_1^1, \ldots, t_3^n/t_1^n\}$. By construction, there is an index i for which $t_1^i = t_1$, hence $(a_3, t_3^i) \in M_3$. Also, $t \supseteq \{t_3^i/t_1^i\}$, hence $d_{\mathsf{m}}^{\mathbb{L}}(t_2, t_3^i/t_1^i) \sqsubseteq \beta$ and consequently $(t_1||t_2, t_3) \in R_{\beta}$.
- Let $(a_3, t_3) \in M_3$, then there are $(a'_2, t) \in M_f$ and $(a_1, t_1) \in M_1$ such that $a_3 = a_1 \oplus a'_2$ and $t_3/t_1 \in t$. Hence there are $(a_2, t_2) \in M_2$ and $\beta \in \mathbb{L}$ for which $F(a_2, a'_2, \beta) \sqsubseteq_{\mathbb{L}} \alpha$ and $d^{\mathbb{L}}_{\mathsf{m}}(t_2, t) \sqsubseteq_{\mathbb{L}} \beta$. Note that $a = a_1 \oplus a_2$ is defined and $F(a, a_3, \beta) \sqsubseteq_{\mathbb{L}} \alpha$. Thus $(a, t_1 || t_2) \in M$, and by $t \supseteq \{t_3/t_1\}$, $d^{\mathbb{L}}_{\mathsf{m}}(t_2, t_3/t_1) \sqsubseteq \beta$.

Assume, for the other direction of the proof, that $\mathcal{A}_1 \| \mathcal{A}_2 \leq_{\mathsf{m}} \mathcal{A}_3$. Define $R = \{R_\alpha \subseteq S_2 \times 2^{S_3 \times S_1} \mid \alpha \in \mathbb{L}\}$ by

$$R_{\alpha} = \left\{ (s_2, \{s_3^1/s_1^1, \dots, s_3^n/s_1^n\}) \mid \forall i = 1, \dots, n : d_{\mathbf{m}}^{\mathbb{L}}(s_1^i \| s_2, s_3^i) \sqsubseteq_{\mathbb{L}} \alpha \right\};$$

we show that R is a witness for $d_{\mathbf{m}}^{\mathbb{L}}(\mathcal{A}_2, \mathcal{A}_3/\mathcal{A}_1)$.

Let $s_2^0 \in S_2^0$. We know that for every $s_1^0 \in S_1^0$, there exists $\sigma(s_1^0) \in S_3^0$ such that $d_{\mathsf{m}}^{\mathbb{L}}(s_1^0 \| s_2^0, s_3^0) \sqsubseteq_{\mathbb{L}} d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{A}_1 \| \mathcal{A}_2, \mathcal{A}_3)$. By $s^0 \supseteq \{\sigma(s_1^0)/s_1^0 \mid s_1^0 \in S_1^0\}$, we see that $(s_2^0, s^0) \in R_{d_{\mathsf{m}}^{\mathbb{L}}(\mathcal{A}_1 \| \mathcal{A}_2, \mathcal{A}_3)}$.

Let $\alpha \in \mathbb{L} \setminus \{\top_{\mathbb{L}}\}$ and $(s_2, s) \in R_{\alpha}$, with $s = \{s_3^1/s_1^1, \dots, s_3^n/s_1^n\}$, and $M_2 \in \operatorname{Tran}_2(s_2)$.

For every i = 1, ..., n, let us write $\operatorname{Tran}_1(s_1^i) = \{M_1^{i,1}, ..., M_1^{i,m_i}\}$. By assumption, $M_1^{i,j_1} \cap M_1^{i,j_2} = \emptyset$ for $j_1 \neq j_2$, hence every $(a_1, t_1) \in \operatorname{Tran}_1(s_1^i)$ is contained in a unique $M_1^{i,\delta_i(a_1,t_1)} \in \operatorname{Tran}_1(s_1^i)$.

For every $j=1,\ldots,m_i$, let $M^{i,j}=M_1^{i,j}\|M_2\in\operatorname{Tran}_{\parallel}(s_1^i\|s_2)$. From $d_{\mathsf{m}}^{\mathbb{L}}(s_1^i\|s_2,s_3^i)\sqsubseteq_{\mathbb{L}}\alpha$ we have $M_3^{i,j}\in\operatorname{Tran}_3(s_3^i)$ such that the conditions in Definition 9.18 hold for the pair $M^{i,j},M_3^{i,j}$.

Now define

$$M = \{(a_{2}, t) \mid \exists (a_{2}, t_{2}) \in M_{2} : \forall t_{3}/t_{1} \in t : \exists i, a_{1}, a_{3}, \beta :$$

$$(a_{1}, t_{1}) \in \in \operatorname{Tran}_{1}(s_{1}^{i}), (a_{3}, t_{3}) \in M_{3}^{i, \delta_{i}(a_{1}, t_{1})},$$

$$F(a_{1} \oplus a_{2}, a_{3}, \beta) \sqsubseteq_{\mathbb{L}} \alpha, d_{\mathbf{m}}^{\mathbb{L}}(t_{1} || t_{2}, t_{3}) \sqsubseteq_{\mathbb{L}} \beta \}.$$
 (9.16)

We need to show that $M \in \text{Tran}_{/}(s)$.

Let $i \in \{1, \ldots, n\}$ and $M_1^{i,j} \in \operatorname{Tran}_1(s_1^i)$; we claim that $M \triangleright M_1^{i,j} \preceq_R M_3^{i,j}$. Let $(a_3, t_3) \in M \triangleright M_1^{i,j}$, then $a_3 = a_1 \oplus a_2$ for some a_1, a_2 such that $t_3/t_1 \in t$, $(a_1, t_1) \in M_1^{i,j}$ and $(a_2, t) \in M$. By disjointness, $j = \delta_i(a_1, t_1)$, hence by definition of M, $(a_3, t_3) \in M_3^{i,j}$ as was to be shown.

For the reverse inclusion, let $(a_3,t_3) \in M_3^{i,j}$. By definition of $M^{i,j}$, there are $(a_1,t_1) \in M_1^{i,j}$, $(a_2,t_2) \in M_2$ and $\beta \in \mathbb{L}$ for which $F(a_1 \oplus a_2,a_3,\beta) \sqsubseteq_{\mathbb{L}} \alpha$ and $d_{\mathsf{m}}^{\mathbb{L}}(t_1||t_2,t_3) \sqsubseteq_{\mathbb{L}} \beta$. Thus $j = \delta_i(a_1,t_1)$, so that there must be $(a_2,t) \in M$ for which $t_3/t_1 \in t$, but then also $(a_1 \oplus a_2,t_3) \in M \triangleright M_1^{i,j}$.

We show that the pair M_2 , M satisfies the conditions of Definition 9.18.

- Let $(a_2, t_2) \in M_2$. For every i = 1, ..., n and every $(a_1, t_1) \in \text{Tran}_1(t_1^i)$, we can use Definition 9.18 applied to the pair $M_1^{i, \delta_i(a_1, t_1)} \| M_2, M_3^{i, \delta_i(a_1, t_1)}$ to choose an element $(\eta_i(a_1, t_1), \tau_i(a_1, t_1)) \in M_3^{i, \delta_i(a_1, t_1)}$ and $\beta_i(a_1, t_1) \in \mathbb{L}$ for which $d_{\mathsf{m}}^{\mathbb{L}}(t_1 \| t_2, \tau_i(a_1, t_1)) \sqsubseteq_{\mathbb{L}} \beta_i(a_1, t_1)$ and $F(a_1 \oplus a_2, \eta_i(a_1, t_1), \beta_i(a_1, t_1)) \sqsubseteq_{\mathbb{L}} \alpha$. Let $t = \{\tau_i(a_1, t_1)/t_1 \mid i = 1, ..., n, (a_1, t_1) \in \text{Tran}_1(t_1^i)\}$, then $(a_2, t) \in M$ and $(t_2, t) \in R_{\beta}$.
- Let $(a_2,t) \in M$, then we have $(a_2,t_2) \in M_2$ satisfying the conditions in (9.16). Hence for all $t_3/t_1 \in t$, there are i, a_1 , a_3 , and $\beta(t_3/t_1)$ such that $(a_3,t_3) \in M_3^{i,\delta_i(a_1,t_1)}$, $F(a_1 \oplus a_2,a_3,\beta(t_3/t_1)) \sqsubseteq_{\mathbb{L}} \alpha$, and $d_{\mathbf{m}}^{\mathbb{L}}(t_1||t_2,t_3) \sqsubseteq_{\mathbb{L}} \beta(t_3/t_1)$. Let $\beta = \sup\{\beta(t_3/t_1) \mid t_3/t_1 \in t\}$, then $d_{\mathbb{L}}^{\mathbb{L}}(t_1||t_2,t_3) \sqsubseteq_{\mathbb{L}} \beta$ for all $t_3/t_1 \in t$, hence $(t_2,t) \in R_{\beta}$.

9.5 Conclusion

We have presented a framework for compositional and iterative design and verification of systems which supports quantities and system and action refinement. Moreover, it is robust, in that it uses distances to measure quantitative refinement and the operations preserve distances.

The framework is very general. It can be applied to a large variety of quantities (energy, time, resource consumption etc.) and implement the robustness notions associated with them. It is also agnostic with respect to the

type of specifications used, as it applies equally to behavioral and logical specifications. This means that logical and behavioral quantitative specifications can be freely combined in quantitative system development.

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