GPS or not GPS

that is the question

LRE
LABORATOIRE DE RECHERCHE DE L'EPITA

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2 July 2024

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Motivations:

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Motivations & Goals

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- Provide a better understanding of a specific class of posets
- Interesting for its use in concurrency theory
- Higher Dimentional Automata’s languages are pomsets

Goals:

- Enumerate gps-posets
- Find a combinatorial proof of a conjecture
- Finish an ongoing work started few years ago

What is a Partially Ordered Set ?

Definition

A partially ordered set (**poset**) is a set with a partial order relation.
An order relation is a relation that is:

- Reflexive: $\forall x \in E, xRx$
- Transitive: $x, y, z \in E, xRy \land yRz \implies xRz$
- Antisymmetric: $x, y \in E, xRy \land yRx \implies x = y$

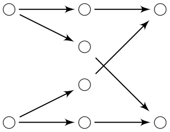

Figure: Example of a poset

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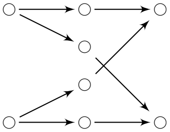

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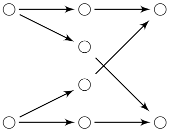

Figure: Example of a poset

- ○ : represents an event
- ⟶: represents the order relation between the event on the left and the item on the right

What is an iposet ?

Definition

A **poset with interfaces** (iposet) is a poset together with 2 injections:

$$[n] \xrightarrow{s} P \xleftarrow{t} [m], \qquad n, m \geqslant 0$$

such that the image of $s[n]$ is minimal and the image of $t[m]$ is maximal.

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- n is an "**unstarted**" event - ◗

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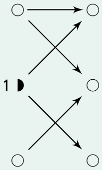
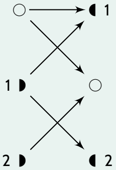
- t is a **terminating interface** -
- n is an "**unstarted**" event -
- m is an "**unfinished**" event -

Examples of iposets

Valid iposets





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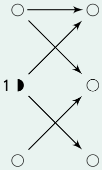

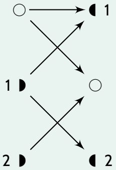

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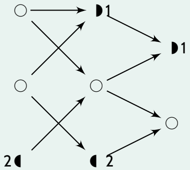

How can we analyze those posets ?

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- **Gluing-Parallel** posets (gp-posets)

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Here are some classes of posets:

- **Series-Parallel** posets (sp-posets)
- **Gluing-Parallel** posets (gp-posets)
- **Gluing-Parallel-Symmetric** posets (gps-posets)

Definition

The **parallel composition** $P \otimes Q$ is defined as the coproduct $P \sqcup Q$ as carrier set together with the order defined as:

$$(p, i) < (q, j) \iff i = j \wedge p <_i q, \qquad i, j \in \{1, 2\}$$

Another way to consider this definition is the following:
With:

$$[n_1] \to (P) \leftarrow [m_1], [n_2] \to (Q) \leftarrow [m_2]$$

$P \otimes Q$ is defined by:

$$[n_1 + n_2] \to [P \otimes Q] \leftarrow [m_1 + m_2]$$

Examples of parallel compositions

Warning

The parallel composition is not commutative!

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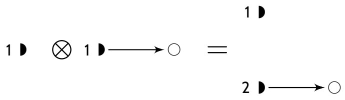

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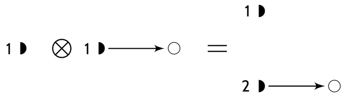
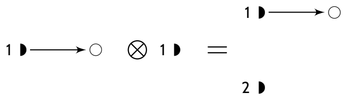


Case 2:



Gluing composition

Definition

The **gluing composition** $P \triangleright Q$ of two iposets

$$[n_1] \xrightarrow{s_1} (P, <_1) \xleftarrow{t_1} [m_1]$$

and

$$[n_2] \xrightarrow{s_2} (Q, <_2) \xleftarrow{t_2} [m_2]$$

is defined as:

$$P \triangleright Q = \begin{cases} (P \sqcup Q) / t_1(i) = s_2(i) \\ (<_1 \cup <_2 \cup (P \,/\, t_1[m]) \times (Q \,/\, s_2[m]))^+ \end{cases}$$

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The gluing composition is defined only if $m_1 = n_2$! The number of starting interfaces of P must be equal to the number of terminating interfaces of Q.

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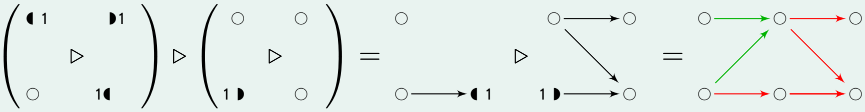
Valid gluing composition



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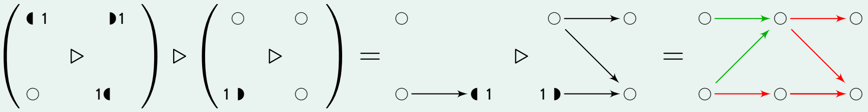

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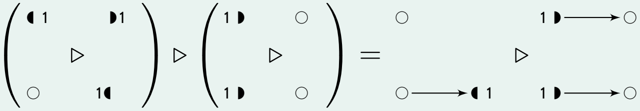
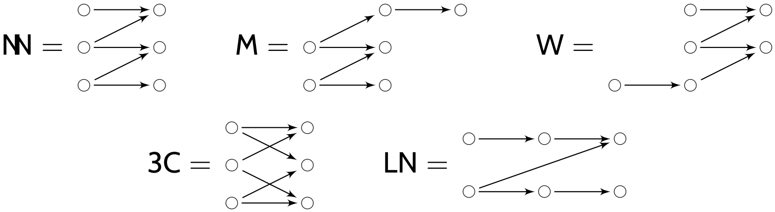


Definition

An iposet is *gluing-parallel-symmetric* (*gps*) if it is empty or can be obtained from the elements:

- ○
- ◖ 1
- 1 ◗
- 1 ⧓ 1
- $\begin{smallmatrix} 1 & ⧓ & 2 \\ 2 & ⧓ & 1 \end{smallmatrix}$ where $\begin{smallmatrix} 1 & ⧓ & 2 \\ 2 & ⧓ & 1 \end{smallmatrix} = (s, [2], t) : 2 \to 2$ is the non-trivial symmetry on 2.

by finitely many applications of $\triangleright$ and $\otimes$.


NN =
M =
W =
3C =
LN =

Conjecture

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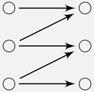
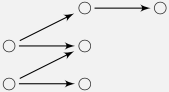
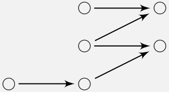
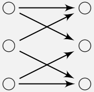
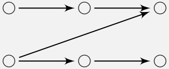
A poset is *gps* $\iff$ it does not contain one the five forbidden structures

NN =


M =


W =


3C =


LN =


First way

Proposition

A poset is *gps* $\implies$ it does not contain one of the forbidden five as an induced substructure.

Main proof argument

NN, M, W, 3C, LN do not admit non-trivial gluing decompositions.

Work

Possible paths

- Study the interval representations of the forbidden five (Quentin)

Possible paths

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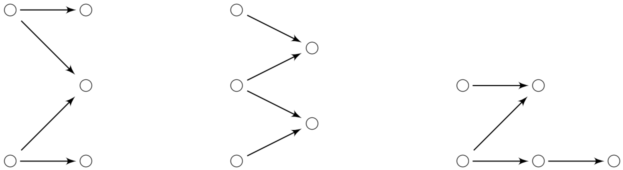
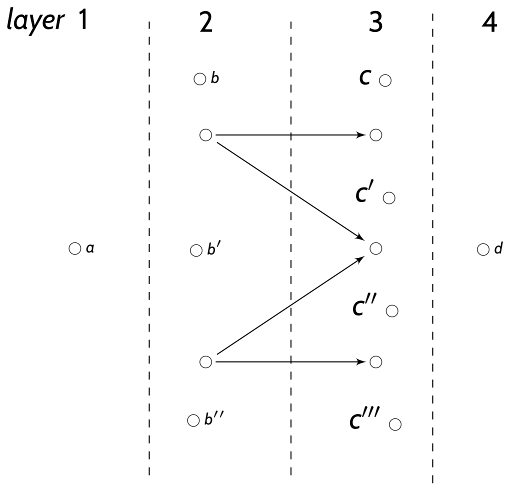
- Generate some posets based on some predicates and observe what we get when we try to add a new event

Posets construction

- We identified 3 basic structures on which we could be able to rely to generate posets.



Example of posets construction



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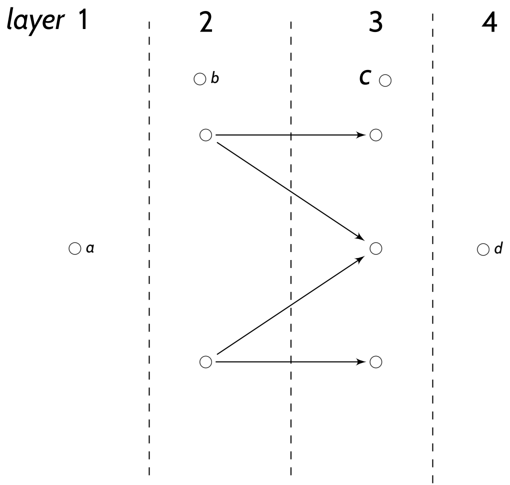

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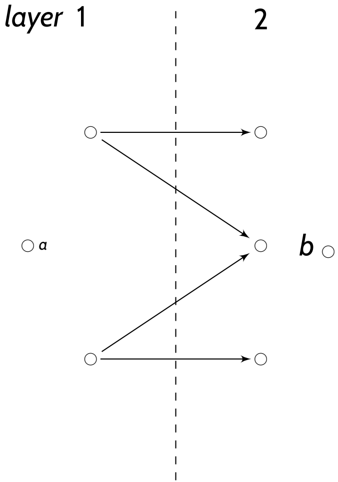

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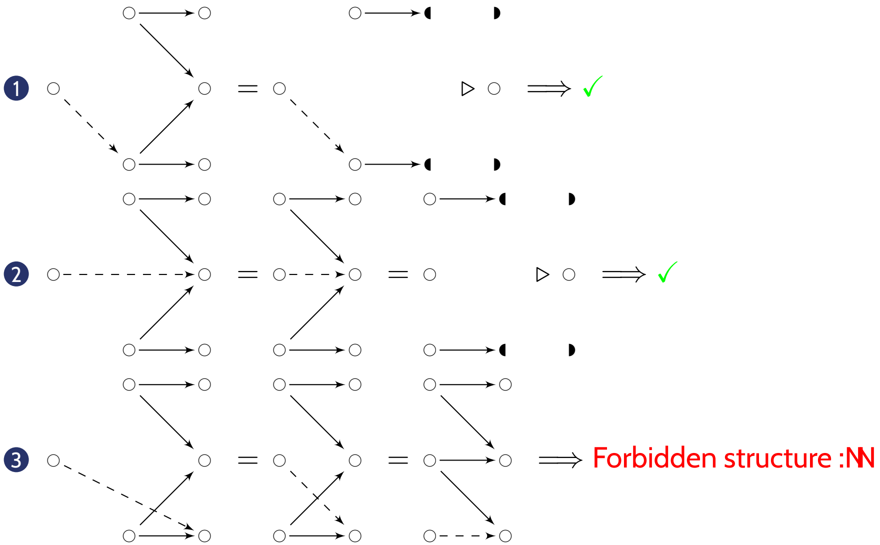
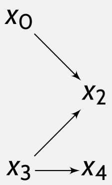


Definition

Let P be a poset. Let x, y be two points such that $x, y \in P$.
The *zigzag distance* $d_{zz}(x, y)$ is defined as the length n of a shortest *zigzag* x.

$$x = x_0 < x_2 > x_3 < \ldots x_n = y$$



Beginning of the proof

- Let P be a poset which is not GPS.

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- if $d_{zz}(a, b) > 2$, then $d_{zz}(a, b) \geqslant 4$

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- if $d_{zz}(a,b) \leqslant 2$, then $d_{zz}(a,b) = 2, hence there is z \in P_{min}$ for which $a > z < b$.

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- Let $m \in P$ such that $m < a$ and $m \not< b$.

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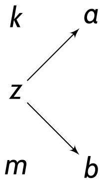
- Let $k \in P$ such that $k \not< a, k \neq a, k \neq b$.



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- For now, 5 forbidden substructures of 6 points are known

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Conclusion

- GPS-iposets: generated from $\circ$, ◗, ◖, $1 \bowtie 1$, $\begin{smallmatrix} 1 \bowtie 2 \\ 2 \bowtie 1 \end{smallmatrix}$ using $\triangleright$ and $\otimes$
- For now, 5 forbidden substructures of 6 points are known
- We explored several paths to deal with the proof of the conjecture
- Unfortunately, for the time being the proof is still uncomplete

Acknowledgements

- Quentin HAY–KERGROHENN
- Uli FAHRENBERG
- Hugo BAZILLE
- Krzysztof ZIEMIANSKI

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