GPS-Posets and the Forbidden Five

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Technical Report n^o202406-techrep-fournillon, July 2024
revision d8d7195

We introduce in this report a specific class of partially ordered sets. The main purpose is to enumerate all the partially ordered sets from this class. To do so, we try to find a proof for a conjecture that has already been put forward.

Nous introduisons dans ce rapport une classe spécifique d'ensembles partiellement ordonnés. L'objectif principal est d'énumérer tous les ensembles partiellement ordonnés de cette classe. Pour ce faire, nous essayons de trouver une preuve pour une conjecture qui a déjà été émise.

Keywords
posets, combinatorics, concurrency theory, gps-posets, forbidden structures

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Chapter 1

Introduction

In concurrency theory, which deals with modeling and analyzing systems with parallelism, partially ordered sets (posets) are employed to represent the executions of programs composed of both sequential and concurrent events. While series-parallel posets have been extensively studied due to their straightforward algebraic properties—being generated solely by serial and parallel compositions—they fall short in representing real-world computations. This limitation arises because series-parallel posets only permit decomposition by cutting through edges, whereas practical applications often require cutting through events. To address this, a new gluing composition extending the serial composition was introduced for posets with interfaces (iposets). When combined with parallel composition, this defines the class of gluing-parallel (gp) posets. However, it was observed that certain posets do not belong to this class. The objective thus shifted towards enumerating gp-posets to gain a deeper understanding of their algebraic properties.

The task of enumerating gp-posets proved to be challenging. Generation efforts managed to produce all gp-posets up to a certain size, identifying numerous forbidden structures along the way. This led to the idea of introducing a new operation to create an even larger class of posets that would be easier to study. The non-trivial symmetry operation, together with the existing gluing and parallel compositions, defines the class of gluing-parallel-symmetric (gps) posets.

The statement that a gps-poset does not contain any of these five forbidden substructures has already been proven. The main challenge is to demonstrate the converse: that any poset not containing one of these forbidden substructures is indeed a gps-poset.

In this report, we begin by defining all the necessary concepts related to posets, iposets, and gps-posets. We then explore a potentially significant theorem and provide a detailed analysis of the proof of this theorem. Finally, we discuss possible paths to prove the conjecture and demonstrate the second statement.

I am grateful to Krzysztof Ziemiański (University of Warsaw, Poland) for discussions, intuitions and advices on the subject of this work.

1.1 Contribution

The main contribution in this report is several paths for the proof of the conjecture. For now, the proof remains incomplete as there are still many questions to be answered about the intermediate results found. The different paths are detailed starting in Chapter 4. The main idea of the first path is to take a closer look at the interval representations of the forbidden substructures in order to see whether we could find some nice properties and whether it would be easier to focus on the analysis of those representations or not. The idea of the second path is to look for a more structural approach by analyzing the obtained posets after adding a point to an already gps-poset. The idea is to observe what creates non-gps structures.

Chapter 2

Preliminaries

In this chapter, we define notions and mathematical objects that have a significant role in the understanding of this report. We highlight specific vocabulary, as well as the choices of poset representation for this research. We will first define general notions about posets and how to classify them before emphasizing an important result about the class of gps-posets. The term *poset* is an acronym used to refer to *partially ordered set*. Most of these premiliminaries can be found in Fahrenberg et al. (2019), Fahrenberg et al. (2022), Äikäs et al. (2022).

2.1 Posets

A poset is a set that has a specific ordering of its elements. We first recall the properties of order relations:

Definition 2.1.0.1 (Reflexive, Symmetric, Transitive Relation). *Let E be a set and $\mathcal{R}$ a binary relation on E.*

- *$\mathcal{R}$ is reflexive: $\forall x \in E, x\mathcal{R}x$*
- *$\mathcal{R}$ is transitive: $x, y, z \in E, x\mathcal{R}y \wedge y\mathcal{R}z \implies x\mathcal{R}z$*
- *$\mathcal{R}$ is antisymmetric: $x, y \in E, x\mathcal{R}y \wedge y\mathcal{R}x \implies x = y$*

Definition 2.1.0.2 (Order Relation). *Let E be a set. $\prec$ (resp. $\preceq$) is a **partial order** relation on E if it is reflexive, anti-symmetric, transitive.*

Remark 2.1.1. *It exists two notations which are equivalent to write an order relation: $(x, y) \in \prec$ is equivalent to $x \prec y$. The last one seems more understandable when the relation is not denoted by a comparison symbol.*

Based on the given definition of an order relation, we can now define posets by using a specific order relation:

Definition 2.1.0.3 (Poset). *A partially ordered set (**poset**) is a tuple $P = (E, \prec)$ with E a set and $\prec$ a partial order relation on E.*

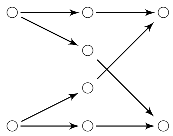


Figure 2.1: Example of a poset

Definition 2.1.0.4 (Minimal and Maximal points)**.** *Let $P = (E, \prec)$ be a poset. A point x of P is a minimal (resp. maximal) point if there is no point y of P, such that $y \prec x$ (resp. $x \prec y$). We note P^{min} (resp. P^{max}) the set of all minimal (resp. maximal) points.*

2.2 SP-Posets

While the general concept of posets is versatile for representing order relations, its broad scope can sometimes pose challenges in terms of analysis and application. The wide variety of posets makes it difficult to analyze these objects. To address these complexities and harness more practical applications, specific subclasses of posets have been studied with more manageable and well-defined properties. One such subclass is the series-parallel posets, or SP-posets. SP-posets introduce an easierstructure through the operations of series and parallel compositions.

Definition 2.2.0.1 (Parallel composition)**.** *Let $P = (E_1, <_1)$ and $Q = (E_2, <_2)$ be two posets. The **parallel composition** $P \otimes Q$ is defined as the coproduct $E_1 \sqcup E_2$ as carrier set together with the order defined as:*

$$(p, i) < (q, j) \iff i = j \land p <_i q, \qquad i, j \in \{1, 2\}$$

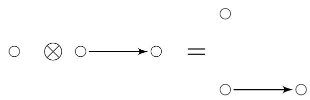


Figure 2.2: Example of a parallel composition.

Definition 2.2.0.2 (Serial composition)**.** *Let $P = (E_1, <_1)$ and $Q = (E_2, <_2)$ be two posets. The **serial composition** $P * Q$ is defined as the coproduct $E_1 \sqcup E_2$ as carrier set together with the order defined as:*

$$(p, i) < (q, j) \iff (i = j \land p <_i q) \lor i < j, \qquad i, j \in \{1, 2\}$$

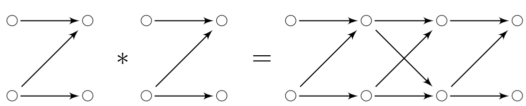


Figure 2.3: Example of a serial composition.

Remark 2.2.1. *The serial composition is not commutative.*

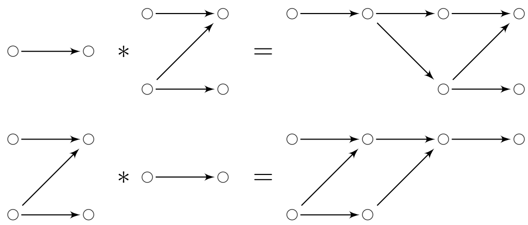


Figure 2.4: Example of the non-commutativity of the serial composition.

Based on the definitions of parallel and the serial compositions, we can now define the class of series-parallel posets.

Definition 2.2.0.3 (SP-Poset). *A poset P is **series-parallel** (an sp-poset) if it is empty or can be obtained from the singleton poset $\circ$ by applying a finite number of times the serial $(*)$ and parallel $(\otimes)$ compositions.*

Remark 2.2.2. *A poset is series-parallel if and only if it does not contain a* N.

Therefore the poset in Fig. 2.1 is not series-parallel whereas the poset in Fig. 2.2 is.

2.3 Iposets

With sp-posets, several issues arise. First, despite the easiness of its algebraic properties, the class of sp-posets appear to be too restrictive for many real-world applications. The second issue is that its algebra does not allow to split events when decomposing posets whereas it would be desirable as it is rather intuitive to start an event in the first component that would finish in the second component. To solve these issues, *posets with interfaces* (iposets) were introduced.

Definition 2.3.0.1 (Iposet). *A **poset with interfaces** (iposet) P is a poset together with two injective functions (s, t):*

$$[n] \xrightarrow{s} P \xleftarrow{t} [m], \qquad n, m \geqslant 0$$

such that the image $s[n]$ is minimal and the image $t[m]$ is maximal.

The two injections $s[n]$ and $t[m]$ represent the *source* and the *target interfaces* of P, respectively. Another equivalent notation is also used: $(s, P, t) : n \to m$.

Remark 2.3.1. *The notation S_P (resp. T_P) is equivalent to $s_P([n])$ (resp. $t_P([m])$).*

Remark 2.3.2. *Usually iposets are also called posets as they are an extension of posets without interfaces. When we want to specifically consider posets without interfaces, it is explicitly indicated.*

With this new type of poset being defined, the definition of the serial composition can be refined to create a new operation called *gluing composition*.

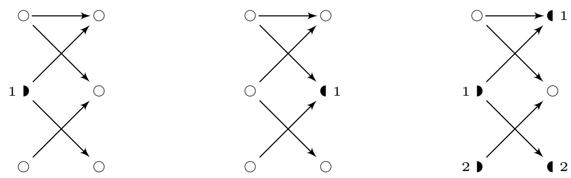


Figure 2.5: Example of iposets.

Definition 2.3.0.2 (Gluing composition). *The **gluing composition** $P \triangleright Q$ of two iposets $[n_1] \xrightarrow{s_1} (P, <_1) \xleftarrow{t_1} [m_1]$ and $[n_2] \xrightarrow{s_2} (Q, <_2) \xleftarrow{t_2} [m_2]$ is defined as:*

$$P \triangleright Q = \begin{cases} (P \sqcup Q)/t_1(i) = s_2(i) \\ (<_1 \cup <_2 \cup (P \ / \ t_1[m]) \times (Q \ / \ s_2[m]))^+ \end{cases}$$

The operation is defined if and only if $m_1 = n_2$.

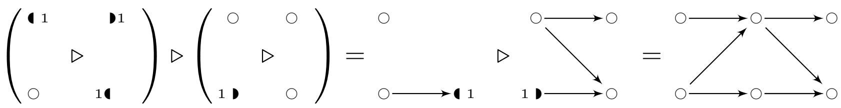


Figure 2.6: Example of a gluing composition.

Definition 2.3.0.3 (Characteristic Function). *Let $Q : n \to m$ and $R : m \to k$ be iposets. The **characteristicfunction** of the decomposition $Q \triangleright R$ is $\varphi_{Q \triangleright R} : Q \triangleright R \to \{0, *, 1\}$ defined by*

$$\varphi_{Q \triangleright R}(x) = \begin{cases} 1 & \text{for } x \in Q \setminus T_Q \\ * & \text{for } x \in T_Q = S_R \\ 0 & \text{for } x \in R \setminus S_R \end{cases}$$

Lemma 2.3.1 (Decomposition Lemma). *The characteristic function $\varphi = \varphi_{Q \triangleright R}$ satisfies the following:*

- *If $(\varphi(x), \varphi(y)) = (1, 0)$, then $x < y$*
- *If $(\varphi(x), \varphi(y)) \in \{(1, *), (*, 0), (1, 0)\}$, then $y \not< x$*
- *If $(\varphi(x), \varphi(y)) = (*, *)$, then $x \not< y$and $y \not< x$*
- *If $x < y$and $\varphi(y) \neq 0$, then $\varphi(x) = 1$. If $x < y$and $\varphi(x) \neq 1$, then $\varphi(y) = 0$*

If the decomposition $P = Q \triangleright R$ is non-trivial, then there exist $x, y \in P$ such that x is minimal, y is maximal, $\varphi(x) = 1$, and $\varphi(y) = 0$.

2.4 GP-Iposets

Definition 2.4.0.1 (GP-iposet)**.** *An iposet is **gluing-parallel** (gp) if it is empty or can be obtained from the following elements:*

by applying a finite number of times the gluing ($\triangleright$) *and parallel* ($\otimes$) *compositions.*

Remark 2.4.1. *The gp-iposets is a bigger class which includes the sp-posets. If we want to consider posets without interfaces then the set of gp-posets is equal to the set of sp-posets.*

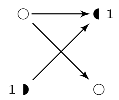


Figure 2.7: Example of a gp-iposet.

Despite the class of gp-iposets being larger than sp-posets and therefore less restrictive, many posets still do not belong to this class. These structures are called *forbidden structures*. These posets are impossible to construct or decompose based on the rules defined previously. One major problem is then to be able to enumerate the posets from this class. In the recent years, many research works about generating iposets have been done (see Äikäs et al. (2022)). Iposets were generated until 16 points and many forbidden structures were found. However it is still unknown whether other exists or not.

2.5 Forbidden Substructures

From this generation work, it has been found eleven forbidden substructures: five of them have 6 points (Fig. 2.8) and the six remaining have at least 8 points (Fig. 2.9).

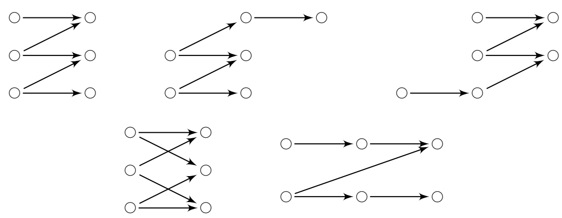


Figure 2.8: The five forbidden substructures of 6 points for gp-posets.

By conducting an in-depth analysis of these posets, we can notice that the posets from Fig. 2.9 are not gluing-parallel because one of their substructures has its interfaces permuted as shown in the example in Fig. 2.10.

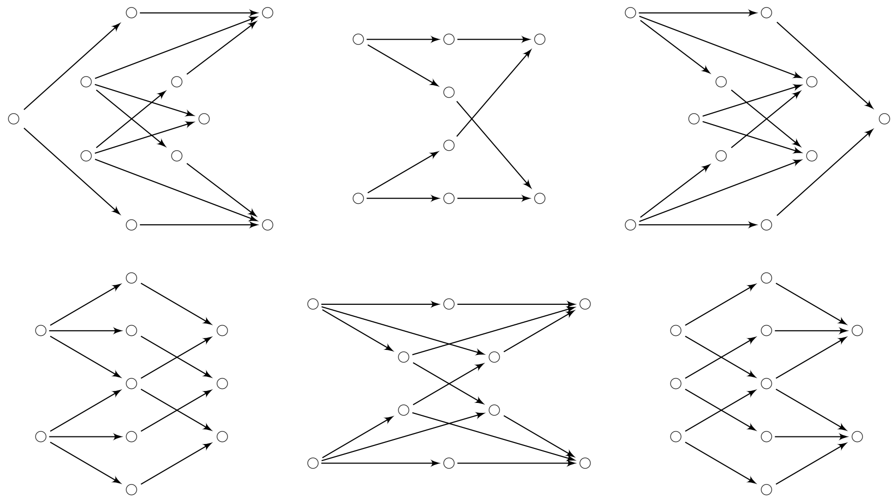


Figure 2.9: Additional forbidden substructures for gp-posets.

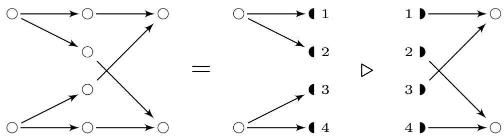


Figure 2.10: Gluing decomposition of the 8-point forbidden substructures.

The first of the iposets on the right-hand side is gluing-parallel as it can be decomposed into singletons using only the parallel and gluing decompositions (it is the parallel product of two three-point iposets). However, the second iposet is not gluing-parallel. It has its interfaces swapped and therefore it is not a parallel composition and it does not exist any working decomposition.

This is a similar case for the other forbidden substructures of at least 8 points.

2.6 GPS-Posets

From this result, we can be eager to define a new class of posets which would allow interface permutation and therefore make our study of these posets easier.

Definition 2.6.0.1 (Non-trivial Symmetry Operation)**.** *Let* $(s, P, t) : n \to m$ *be an iposet.*
 $= (s, [2], t) : 2 \to 2$ *defines the non-trivial symmetry on* 2*, with* $s(i) = i$ *and* $t(i) = 3 - i$*.*

This operation basically permutes the labelled interfaces. The operation of symmetry together with the parallel and the gluing compositions define the gps-iposets.

Definition 2.6.0.2 (GPS-Iposet)**.** *An iposet is gluing-parallel-symmetric (gps) if it is empty or can be obtained from the elements:*

by applying a finite number of times the gluing ($\triangleright$) *and parallel* ($\otimes$) *compositions.*

Hence gps-iposets contain all the gp-iposets, but also, for any n, all the symmetries $n \to n$. It therefore becomes the largest class of posets. With this new definition, the number of known forbidden substructures for this new class is reduced and it only remains the five forbidden substructures of 6 points (Fig. 2.8) that are still forbidden for gps-posets.

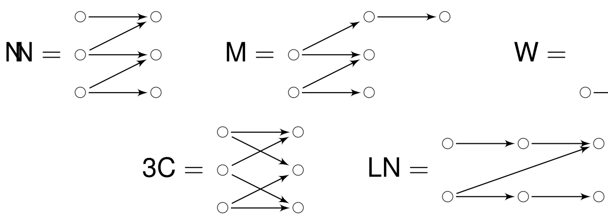


Figure 2.11: The five known forbidden substructures for gps-iposets (with their names).

The proof that these posets are not gluing-parallel nor gluing-parallel-symmetric can be found in Fahrenberg et al. (2022).

2.7 Ziemiański's Theorem

An important result regarding the gps-iposets is the Ziemiański's theorem. It is, for now, the only theorem we have on this class of posets. This theorem reveals a powerful link between gps-iposets and sp-posets.

Theorem 2.7.1 (Ziemiański)**.** *A poset is* gluing-parallel-symmetric $\iff$ *it has an interval representation in a sp-poset*

For now, no paper presenting a demonstration of this theorem is available on internet and therefore, this is why the demonstration will be detailed in this report. To demonstrate this theorem, we can first split it into two lemmas.

Lemma 2.7.2 (Direct Sense)**.** *Every gps-iposet P has an inteval representation in a sp-poset V.*

Proof. We proceed by structural induction.

- **Base case**: P is a generator, $P \in \{\varepsilon,$ ○ , ◖ , ◗ , ⧓ $\}$

  So P has an interval representation in a sp-poset because all those five elements are ○ -interval. Indeed, P admits an interval representation in itself, so in particular in a sp-poset.

- **Induction case**:

  Here, we have two subcases to consider:

  If $P = Q \otimes R$ and $(f_Q, g_Q), (f_R, g_R)$ are interval representations in sp-posets V and W, respectively, then by applying the definition of the parallel composition to both interval representations of Q and R, we have that $(f_Q \sqcup f_R, g_Q \sqcup g_R)$ (the coproduct of $(f_Q, g_Q), (f_R, g_R)$) is a representation of $Q \otimes R$.

  This representation $(f_Q \sqcup f_R, g_Q \sqcup g_R)$ is an interval representation of P in $V \otimes W$.

  This statement seems pretty straight forward as the parallelization of two posets does not create any order relation between them, therefore they remain completely independent of each other as we can see in Fig. 2.2. So taking the coproduct of $(f_Q, g_Q), (f_R, g_R)$ does not change the overall properties. Then $(f_Q \sqcup f_R, g_Q \sqcup g_R)$ is an interval representation of $V \otimes W$. For this pair to be an interval representation, we have to show:

  1. $f(p) \leqslant g(p)$ for all $p \in P$
  2. $p < q \iff g(p) < f(q)$ for all $p, q \in P$

  With:

  $$f(p) = (f_Q \sqcup f_R)(p) = \begin{cases} f_Q(p) & \text{for } p \in Q \\ f_R(p) & \text{for } p \notin Q \end{cases}$$

  $$g(p) = (g_Q \sqcup g_R)(p) = \begin{cases} g_R(p) & \text{for } p \in R \\ g_Q(p) & \text{for } p \notin R \end{cases}$$

  To show (1):

  Let $p \in P$.

  - If $p \in Q \setminus R$, then we have $f(p) = f_Q(p) \leqslant g_Q(p) = g(p)$
  - If $p \in R \setminus Q$, then we have $f(p) = f_R(p) \leqslant g_R(p) = g(p)$
  - The case where $p \in Q \cap R$ cannot occur as $Q \cap R = \emptyset$ by definition of the coproduct.

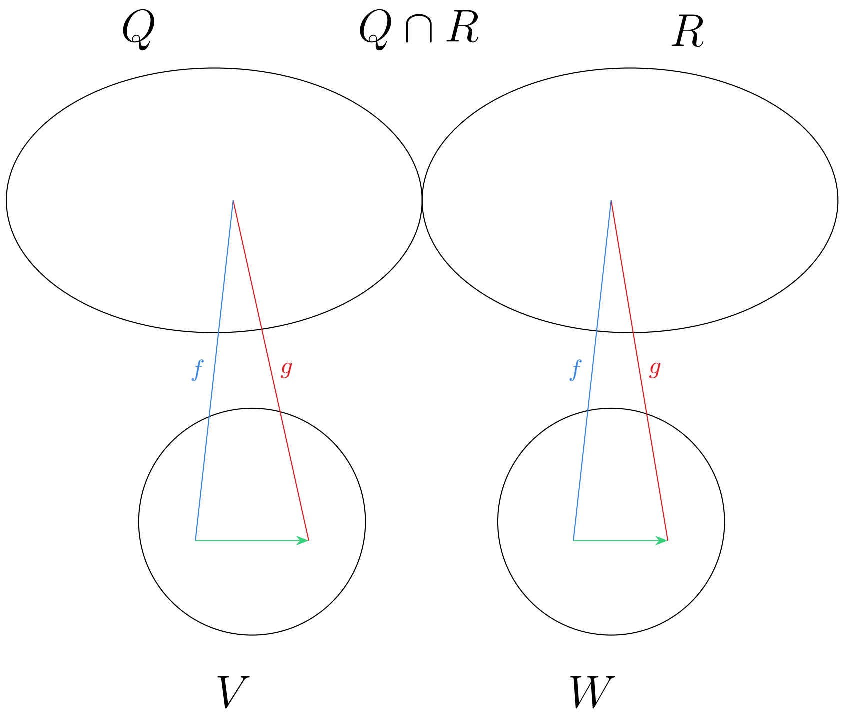


Figure 2.12: Diagram which illustrates the demonstration of (1) if $P = Q \otimes R$

To show (2):

Let $p, p' \in P$ and assume that $p < p'$.

- If both $p, p' \in Q$, then $p \notin R$ because $p <_Q p'$ and $p' \notin R$ because $Q \cap R = \emptyset$. Hence $g(p) = g_Q(p) < f_Q(p') = f(p')$
- If both $p, p' \in R$, then $p' \notin Q$ because $p <_R p'$ and $p \notin Q$ because $Q \cap R = \emptyset$. Hence $g(p) = g_R(p) < f_R(p') = f(p')$
- If $p \in Q \setminus R$ and $p' \in R \setminus Q$ then $g(p) = g_Q(p) < f_R(p') = f(p')$ because $g_Q(p) \in V$ and $f_R(p') \in W$.
- If $p \in R \setminus Q$ and $p' \in Q \setminus R$ then there is a contradiction because we assumed that $p < p'$. So this case cannot occur.
- The case where $p \in Q \cap R$ and $p' \in Q \cap R$ cannot occur as $Q \cap R = \emptyset$..

Because (2) is an equivalence, we also have to show the second implication:

We assume that $g(p) < f(p')$.

- If $p \notin R$ and $p' \notin Q$, then $p \in Q$ and $p' \in R$ because $Q \cap R = \emptyset$. So $p < p'$
- If $p \notin R$ and $p' \in Q$, then $g_Q(p) = g(p) < f(p') = f_Q(p')$, so $p < p'$
- If $p \in R$ and $p' \notin Q$, then $g_R(p) = g(p) < f(p') = f_R(p')$, so $p < p'$
- If $p \in R$ and $p' \in Q$ cannot occur. Otherwise, it would mean that $f_Q(p') < g_R(p) < f_Q(p')$ which is absurd.

$(f_Q \sqcup f_R, g_Q \sqcup g_R)$ is indeed an interval representation of P in $V \otimes W$.

If $P = Q \triangleright R$ and $(f_Q, g_Q), (f_R, g_R)$ are interval representations in sp-posets V and W, respectively, then we want to show that:

$$f(p) = \begin{cases} f_Q(p) & \text{for } p \in Q \\ f_R(p) & \text{for } p \notin Q \end{cases}$$

$$g(p) = \begin{cases} g_R(p) & \text{for } p \in R \\ g_Q(p) & \text{for } p \notin R \end{cases}$$

is an interval representation of P in $V \triangleright W$.

To show (1):

Let $p \in P$.

- If $p \in Q \setminus R$, then we have $f(p) = f_Q(p) \leqslant g_Q(p) = g(p)$
- If $p \in R \setminus Q$, then we have $f(p) = f_R(p) \leqslant g_R(p) = g(p)$
- If $p \in Q \cap R$, then $f(p) = f_Q(p) < g_R(p) = g(p)$ because $f_Q(p) \in V$ and $g_R(p) \in W$.

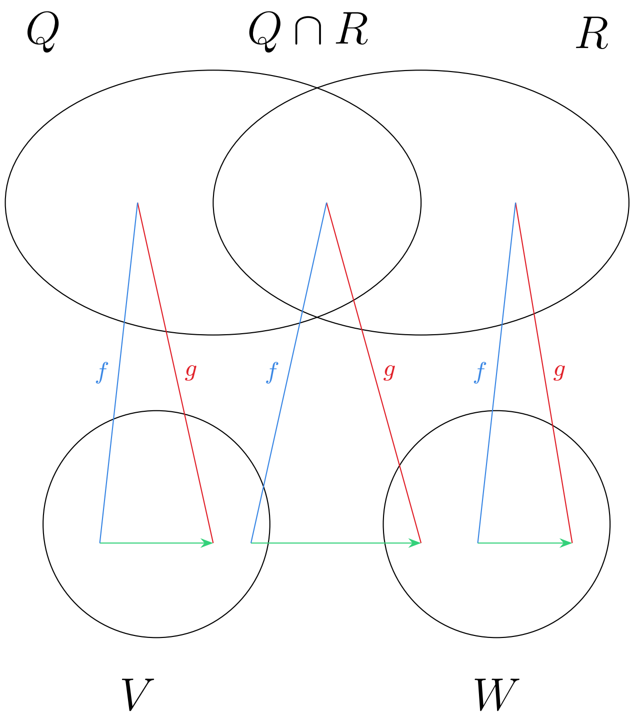


Figure 2.13: Diagram which illustrates the demonstration of (1) if $P = Q \triangleright R$

To show (2):

Let $p, p' \in P$ and assume that $p < p'$.

- If both $p, p' \in Q$, then $p \notin R$ because $p <_Q p'$. $Q \cap R$ does only contain maximal points of Q and minimal points of R so p is not a maximal point of Q whereas p' is. Hence $g(p) = g_Q(p) < f_Q(p') = f(p')$
- If both $p, p' \in R$, then $p' \notin Q$ because $p <_R p'$. $Q \cap R$ does only contain minimal points of R and maximal points of Q so p' is not a minimal point of R whereas p is. Hence $g(p) = g_R(p) < f_R(p') = f(p')$
- If $p \in Q \setminus R$ and $p' \in R \setminus Q$ then $g(p) = g_Q(p) < f_R(p') = f(p')$ because $g_Q(p) \in V$ and $f_R(p') \in W$.
- If $p \in R \setminus Q$ and $p' \in Q \setminus R$ then there is a contradiction because we assumed that $p < p'$. So this case cannot occur.
- If $p \in Q \cap R$ and $p' \in Q \cap R$ then there is a contradiction because we assumed $p < p'$ while elements from $Q \cap R$ cannot be ordered. So this case cannot occur.

Because (2) is an equivalence, we also have to show the second implication:

We assume that $g(p) < f(p')$.

- If $p \notin R$ and $p' \notin Q$ then by definition of $Q \triangleright R$, $p < p'$
- If $p \notin R$ and $p' \in Q$, then $g_Q(p) = g(p) < f(p') = f_Q(p')$, so $p < p'$
- If $p \in R$ and $p' \notin Q$, then $g_R(p) = g(p) < f(p') = f_R(p')$, so $p < p'$
- If $p \in R$ and $p' \in Q$ cannot occur. Otherwise, it would mean that $f_Q(p') < g_R(p) < f_Q(p')$ which is absurd.

□

As this part can be a bit difficult to understand, an advice is to refer to Fig. 2.12 and to Fig. 2.13 to visualize the different steps of the demonstration.

Lemma 2.7.3 (Indirect Sense)**.** *If an iposet P has an interval representation (f, g) in some sp-poset V, then P is* gluing-parallel-symmetric*.*

Proof. We proceed by structural induction.

- **Base case**: V is a generator, $V \in \{\varepsilon, \circ\}$

  So P is a parallelization of a finite number of some of the following elements $\{\varepsilon, \circ, ◖, ◗\}$ and so because points all have an interval representation in themselves, therefore several points in parallel can all be mapped to the same point.

- **Induction case** We assume that $V = W \otimes Y$.

  Let $Q = f^{-1}(W) = g^{-1}(W)$ and $R = f^{-1}(Y) = g^{-1}(Y)$ These equalities turn out to be true because $V = W \otimes Y$ is disconnected.

  Proof. Asssume that $V = W \otimes Y$ is not disconnected. Then, it exists $p \in W \cap Y$. But by definition of the parallel operation (which is defined as the coproduct, the disjoint union of two posets), if $p \in W$, then $p \notin Y$. So we have a contradiction. Therefore, $V = W \otimes Y$ is disconnected. □

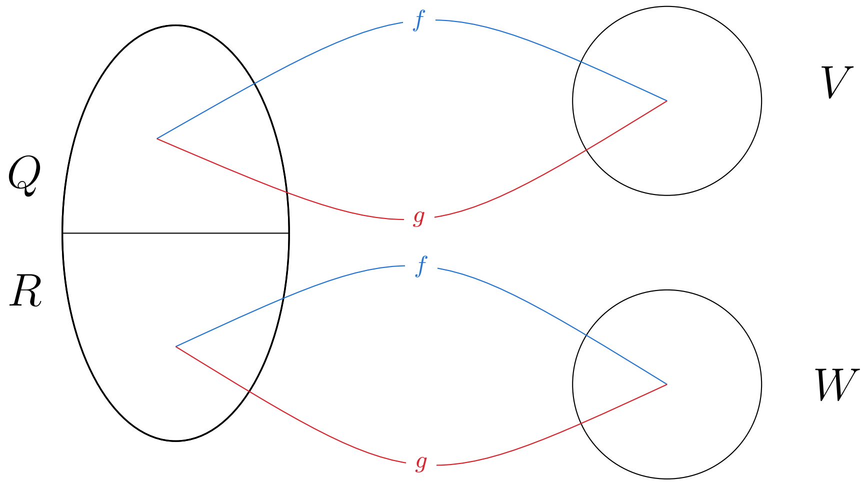


Figure 2.14: Diagram which illustrates that $W \otimes Y$ is disconnected.

Indeed, when W and Y are put in parallel: $W \cap Y = \emptyset$ as we can notice on the above figure. Then, we note $f|_Q$ the restriction of f on Q. It means that we only consider elements $p \in Q \setminus R$. We have that $(f|_Q, g|_Q)$ is an interval representation of Q in W with $f|_Q$ and $g|_Q$ being respectively, the inverse functions of $f^{-1}(W)$ and $g^{-1}(W)$ defined above. Symmetrically, we have that $(f|_R, g|_R)$ is an interval representation of R in Y with $f|_R$ and $g|_R$ being respectively, the inverse functions of $f^{-1}(Y)$ and $g^{-1}(Y)$ defined above.

So by induction hypothesis, Q and R are gps-posets and so is P.

Then, we assume that $V = W \triangleright Y$.

Let $Q = f^{-1}(W)$ and $R = g^{-1}(Y)$. We can note that, here the equalities $Q = g^{-1}(W)$ and $R = f^{-1}(Y)$ do not hold because $Q \cap R \neq \emptyset$ in this case. Indeed, a point can be decomposed into two interfaces and therefore belongs to both Q and R as we can see in Fig. 2.6 or in Fig. 2.15.

Define $f_Q, g_Q : Q \to W$ as follows:

Let $q \in Q$:

- $f_Q(q) = f(q)$
- $g_Q(q) = \begin{cases} g(q) & \text{if } g(q) \in W \\ w & \text{with } w \in W^{\max} \text{ such that } f(q) \leqslant w \end{cases}$

We can define $f_R, g_R : R \to Y$ symmetrically:

Let $q \in R$:

- $f_R(q) = \begin{cases} f(q) & \text{if } f(q) \in Y \\ y & \text{with } y \in Y^{\min} \text{ such that } g(q) \geqslant y \end{cases}$
- $g_R(q) = g(q)$

Such values of w exist for every poset $P \neq \emptyset$ as they are composed of minimal points and maximal points. For a poset $P \neq \emptyset$, $|P^{\max}| \geqslant 1$ and $|P^{\min}| \geqslant 1$.

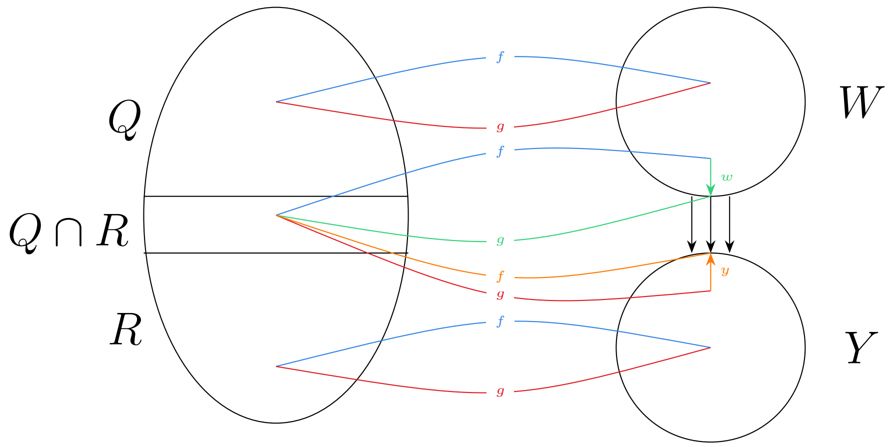


Figure 2.15: Diagram which illustrates the different cases for defining (f_Q, g_Q) and (f_R, g_R)

We want to show that (f_Q, g_Q) is an interval representation for Q in W:

To show (1):

Let $q \in Q$.

- If $g(q) \in W$, then $f_Q(q) = f(q) \leqslant g(q) = g_Q(q)$.
- Otherwise we have $f_Q(q) \leqslant g_Q(q)$ by definition of g_Q.

To show (2):

Let $q, q' \in Q$. We assume that $q < q'$.

- If $g(q) \in W$, then $g_Q(q) = g(q) < f(q') = f_Q(q')$.
- Otherwise, we have $g_Q(q) \in W$ and $g(q) \in Y$ which implies $g_Q(q) < g(q) < f(q') = f_Q(q')$.

Once again, because (2) is an equivalence, we still have to show the back implication.

We assume $g_Q(q) < f_Q(q')$, then $f_Q(q') \in W$ implies $g_Q(q) \notin W^{\max}$, hence $g(q) = g_Q(q) < f_Q(q') = f(q')$ and thus $q < q'$.

The proof that (f_R, g_R) is an interval reprseentation for R in Y is symmetric.

We proceed by induction to assert that Q and R are gps and so is $P = Q \triangleright R$.

□

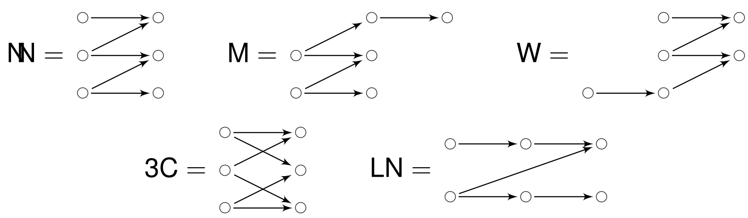
As with the demonstration of the Theorem 2.7.2, the best way to understand this part is to look at the Fig. 2.14 and Fig. 2.15 to better visualize the different steps.

All this work of classifying and analysing posets leads to the following conjecture which, if it is true, would allow to enumerate gps-iposets.

Chapter 3

The Conjecture

A poset is *gps* $\iff$ it does not contain one the five forbidden structures



The statement: *A poset is gps* $\implies$ *it does not contain one of the five forbidden substructures* has already been proven. The main core of the work is to find a proof for the other statement: *A poset does not contain one of the five forbidden structures* $\implies$ *it is gps*.

Chapter 4

Contribution

To find a proof for this conjecture, several paths were considered. We first considered analysing the interval representations of the forbidden five (Fig. 2.11). Afterward, we worked on a more structural approach whose aim was to consider posets of 5 points and to add points to observe which phenomenon creates forbidden structures.

4.1 Interval Representation Analysis

The idea of analysing the interval representations of the forbidden structures comes from the Ziemiański theorem. As it highlights a strong link between gps-posets and sp-intervals, we think analysing these interval representations would tell more about these forbidden structures. To study these reprentations, we start from the interval representation of a M:

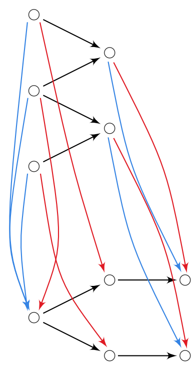


Figure 4.1: An interval representation of a poset M.

If we add a point to the M, we can get some of the forbidden structures ($\mathsf{M}_{\leftarrow}, {}^{\leftarrow}\mathsf{W}, \mathsf{NN}$), therefore we would like to know if we can get an interval representation of a forbidden structure only by adding a point to the interval representation of a M.

By analysing these two interval representations, we can notice that one inteval representation of a ${}^{\leftarrow}\mathsf{W}$ does preserve the structure of the interval representation of a W which is a good result as we would like to observe a preservation of the structure. However, when we take a look at the interval representation of a NN, we can see that the structure of the interval representation

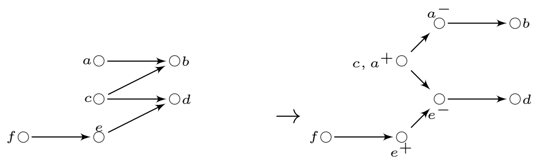


Figure 4.2: Interval Representation of the ←W.

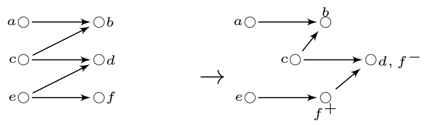


Figure 4.3: Interval Representation of the NN.

of a W is not preserved. Therefore, our first assumption turns out to be incorrect. There is a possibility that this path could still lead to a proof however, for now, we do not how to take these results any further.

4.2 Structural Analysis

Another path we considered was to try to construct forbidden structures. To do that, we first tried to add a point to a 5-point poset (a W mainly) to see if we can highlight interesting phenomenon regarding the construction of these forbidden structures.

4.2.1 Construction of 6-point Posets

We started this path by trying to construct posets of six points to see in which cases we get forbidden structures and otherwise to see whether we could get forbidden structures by adding other points or not.

We start from the poset W and define different zones where we can add points.

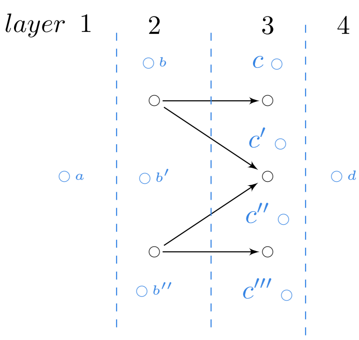


Figure 4.4: Representation of the different zones for adding points.

After that, we make some simplifications to refine our representation and therefore reduce the number of possible zones to add points.

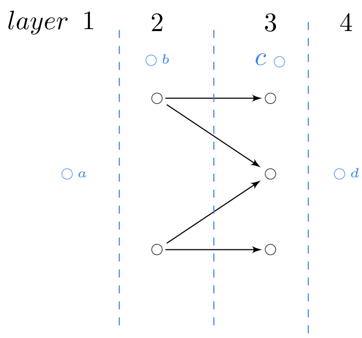


Figure 4.5: Simplified representation of the different zones where we can add points.

We finish our reduction with only two layer on which we can add one point.

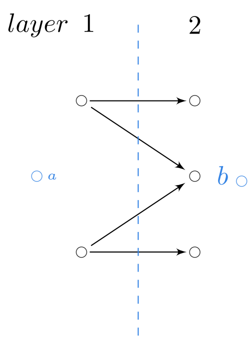


Figure 4.6: The most simplified representation of the different zones where we can add points.

The obtained posets can be found in Appendix A. We realize this process of construction only from the poset W because it already raises many questions. What we can observe from the results is that we do not necessarily obtain forbidden structures by adding a sixth point to a W. Sometimes, we get posets that are gluing-parallel-symmetric and therefore decomposable and on the other hand, we sometimes get forbidden structures. These results raise therefore a new interrogation, do we nesseraily get forbidden structures if we add another points to these obtained posets.

As this process of construction consumes a lot of time, we have decided to start the beginning of the proof.

4.2.2 Construction of 5-point Reference Substructures

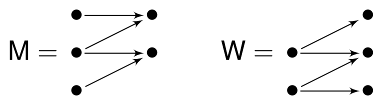
Together with *Uli FAHRENBERG*, we started to design the proof of the conjecture that is: *if a poset is not GPS, then it contains one onf the five forbidden induced substructures (Fig. 2.11)*. The idea is to create a link be finding the reference structures used in the previous section. First, an important and helpful notion for the rest of the demonstration is the definition of the *zigzag distance*.

Definition 4.2.0.1 (Zigzag Distance)**.** *Let P be a poset. The* zigzag distance $d_{zz}(x, y)$ *between two points $x, y \in P$ is the length n of a shortest zigzag $x = x_0 < x_2 > x_3 < ...x_n = y$.*

Lemma 4.2.1. *d_{zz} defines a metric on P.*

Proof. $d_{zz}(x, y) = 0$ iff $x = y$; symmetry is obvious; and if $d_{zz}(x, y) = n$ and $d_{zz}(y, z) = m$, then the two zigzags may be concatenated (and the concatenation possibly be shortened), showing that $d_{zz}(x, z) \leq n + m$. □

We will also use some other special posets below:



Lemma 4.2.2. *P contains an induced M or an induced W iff there are $x, y \in P$ with $d_{zz}(x, y) \geq 4$. P contains an induced NN iff there are $x, y \in P$ with $d_{zz}(x, y) \geq 5$.*

Proof. We only show the second claim; the first uses similar arguments. If P contains an induced NN, then the zigzag distance between its extremal points is equal to 5. Conversely, let $x, y \in P$ with $d_{zz}(x, y) = 5$, then the zigzag $x = x_1 < x_2 > x_3 < x_4 > x_5 < x_6 = y$ is an NN. (The other case is symmetric.) Any relation $x_i \leq x_j$ would shorten the zigzag, so the NN is induced. □

Lemma 4.2.3. *Let P be weakly connected. If P contains an induced M and an induced W, then P contains an induced NN.*

Remark 4.2.1. *For now, we did not find any proof for this lemma and therefore we cannot be sure that it is correct.*

Now let P be a poset which is not GPS. We may assume that P is $\otimes$-indecomposable (*i.e.*, weakly connected) and $*$-indecomposable; otherwise we can decompose P and have a smaller poset to start with.

We denote by P_{max} the subset of maximal elements and by $P_{\text{MAX}} = \{y \in P_{\text{max}} \mid \forall z \in P : |\{x : x < y\}| \geq |\{x : x < z\}|\}$ the subset of right-extreme elements.

Let $a \in P_{\text{MAX}}$. If $P_{\text{max}} = \{a\}$, then $P = (P \setminus \{a\}) * \{a\}$, in contradiction to P being indecomposable.

Assume that $P_{\text{max}} \subsetneq P_{\text{MAX}}$ and let $b \in P_{\text{max}} \setminus P_{\text{MAX}}$.

If $d_{zz}(a, b) > 2$, then $d_{zz}(a, b) \geq 4$ as both a and b are maximal. By Lemma 4.2.2 and again because of maximality, P contains an induced W.

Assume $d_{zz}(a, b) \leq 2$, then $d_{zz}(a, b) = 2$ by maximality of both, hence there is $z \in P_{\text{min}}$ for which $a > z < b$.

Let $m \in P$ such that $m < a$ and $m \not< b$; this exists because $a \in P_{\text{MAX}}$ and $b \notin P_{\text{MAX}}$. We know that $m \not< z$ because of minimality of z, but we may or may not have $z < m$.

If $\{k \in P \setminus \{a, b\} \mid k \not< a\} = \emptyset$, then $P = (P \setminus \{a\}) * \{a, b\}$ (with b in the interface), in contradiction to indecomposability. Let $k \in P \setminus \{a, b\}$ such that $k \not< a$. We know that $k \not< z$ because of minimality of z, but we may or may not have $z < k$.

1. If $k < b$:
   (a) If $z < m$: This do not lead to a M or W but to an N with an extra arc $m \to a$, which is an interesting structure in order to build the LN.
   (b) If $z \not< m$:

i. If $z < k$: This do not lead to a M or W but to an N with an extra arc $k \to b$, which is an interesting structure in order to build the LN.
ii. If $z \not< k$: then $m < a > z < b > k$ forms an induced M.

2. If $k \not< b$: We cannot have $k < m$ as this would imply $k < a$.

   (a) If $m < k$:
      i. If $z < m$: ???
      ii. If $z \not< m$:
         A. If $z < k$: ???
         B. If $z \not< k$: then $k > m < a > z < b$ forms an induced W.

   (b) If $m \not< k$:
      i. If $z < k$: ???
      ii. If $z \not< k$: then $d_{zz}(z, k) \geq 2$, hence $d_{zz}(m, k) \geq 4$ and P contains an induced M.

This proof remains for now, largely incomplete and it still raise several issues. In some cases, we get structures that are not forbidden structures but that have extra edges. This result is disturbing and remains blocking for now as it seems difficult to get forbidden structures from posets that have more edges than the non-gps posets.

Chapter 5

Conclusion

While the goal of finding a proof for the conjecture has not been reached this semester, we nevertheless managed to find some interesting intermediate results. These results could in the future lead to a complete version of the proof. For now, they are still unresolved issues and grey areas regarding the demonstration of the conjecture. Looking at several paths may not have led to the proof yet but it was for sure interesting and, in my opinion, useful for future works on this subject.

I would like to thank *Quentin HAY–KERGROHENN* with who I worked during this semester and *Uli FAHRENBERG* and *Hugo BAZILLE* for supervising us during this semester.

Chapter 6

Bibliography

Fahrenberg, U., Johansen, C., Struth, G., and Thapa, R. B. (2019). Generating posets beyond n. (page 6)

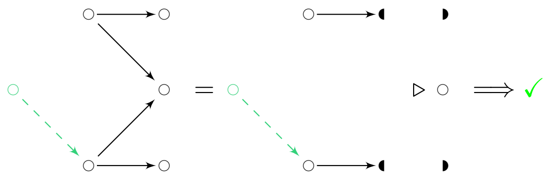
Fahrenberg, U., Johansen, C., Struth, G., and Ziemiański, K. (2022). Posets with interfaces as a model for concurrency. (pages 6 and 12)

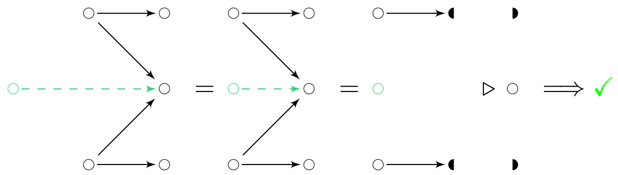
Äikäs, O., Fahrenberg, U., Johansen, C., and Ziemiański, K. (2022). Generating posets with interfaces. (pages 6 and 10)

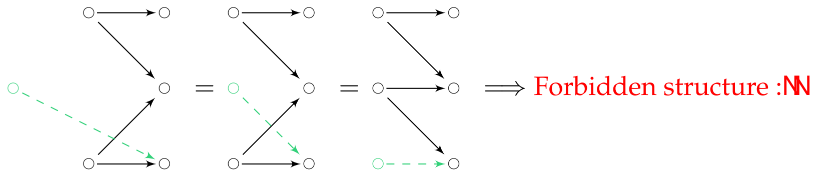
Appendix A

Posets Construction

A.1 Cases With One Added Edge

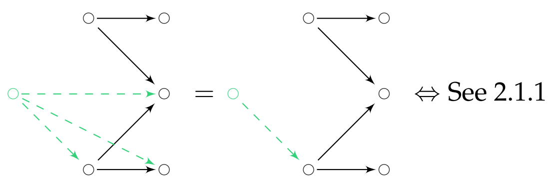
1. 

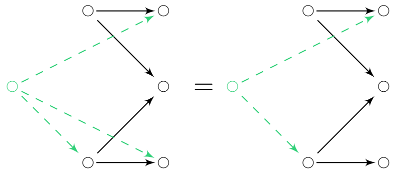
2. 

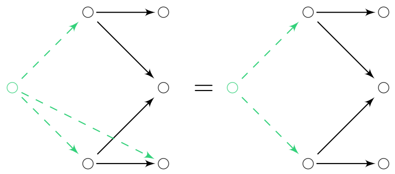
3. 

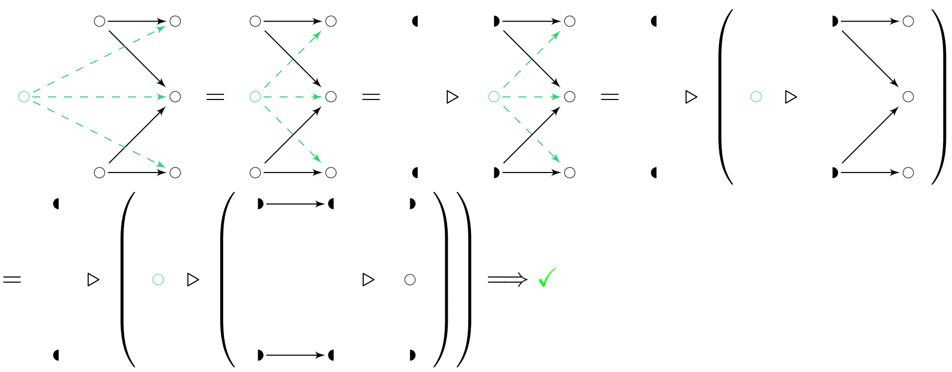
A.2 Case With Two Edges Added

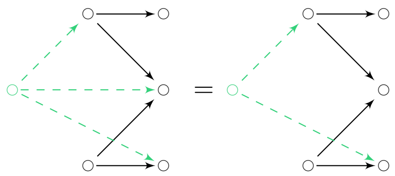
A.3 Case With Three Edges Added

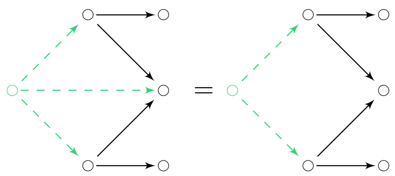
1. 

2.  ⟹ See 2.2.7

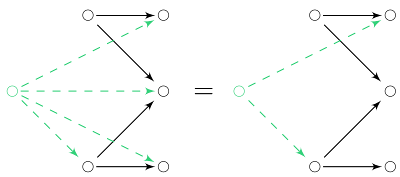
3.  ⟹ See 2.2.4

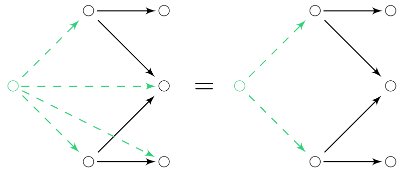
4. 

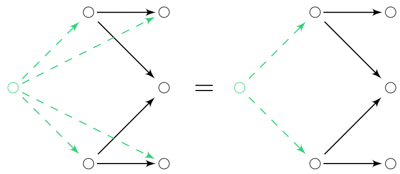
5.  ⟹ See 2.2.7

6.  ⟹ See 2.2.4

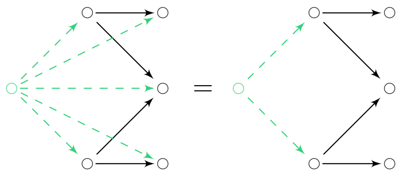
A.4 Case With Four Added Edges

1.  ⟹ See 2.2.3

2.  $\Longrightarrow$ See 2.2.4

3.  $\Longrightarrow$ See 2.2.4

A.5 Case With Five Added Edges

 $\Longrightarrow$ See 2.2.4

A.6 Symetric Case

Other cases are handled whether by the symmetric operation on gps-iposets or by the horizontal symmetry axis of the initial poset.

A.6.1 Case of Layer 2

Every possible cases are already equivalent to some cases of the layer 1. Therefore, there is no new case.