GPS Posets Report

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Abstract

A poset is a set together with a relation that is reflexive, transitive and antisymmetric. The GP-Poset (*Gluing-Parallel* Poset) is a family of posets with the *gluing* composition and the *parallel* composition. It has been discovered that GP-Posets cannot contain some structures. However, the exponentional growth of the number of posets prevents us of performing an exhausive verification with more than 16 points, so we do not know if there are more than 11 forbidden structures, 5 with six points, 1 with eight points and 5 with ten points. Adding the *symmetric* composition, we enter in the GPS-poset(*Gluing-Parallel-Symmetric* poset) world, reducing the forbidden structures to only the ones which have six points. So, we obtain this conjecture:

A poset is GPS if and only if it does not contain any of the five induced sub-structures.

The proof that the five forbidden structures are not GPS-Posets has been done, and our goal is to prove the other direction.

1 Introduction

Poset partially ordered set Analysis of program execution, SP-Posets interesting cause have good algebraic properties Interval representation, present for Petri Net, Higher dimensional automata, useful for sequence analysis However, SP-Poset and Interval representation are incompatible Reconciliation of both for the study of GPS-posets, more complex structures but with some variations to add coloration and deepness at the program execution analysis. We continue the work of [], to prove equivalence between be a GPS poset and doesn't contain some induced sub-structures. If the theorem can be proved, GPS would become the largest countable subclass.

2 Preliminaries

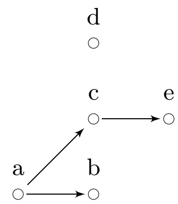
Definition: Poset

A **Poset** is a partially ordered set. So, each element must satisfy:

- Reflexivity : $\forall x \in E,\ xRx$
- Transitivity : $x, y, z \in E,\ xRy \wedge yRz \implies xRz$
- Anti-symmetric : $x, y \in E,\ xRy \wedge yRx \implies x = y$

A poset is noted as $(P, \leq_P)$ with P the set of points and $\leq_P$ the relation order. The interest of the partial order will appear with the introduction of interfaces. For the poset representation, a point in a poset is represented by a circle:" $\circ$ ". The order relation between two points is a simple arrow:" $\longrightarrow$" . When it is possible, the order relation will be depicted from left to right. Two points that are not in relation are said to be incomparable. In the following example:

Example:



We have $b > a < c < e \Longrightarrow a < e$. But $b \nleq c$ and $b \ngeq c$ so b and c are incomparable, noted $b||c$. Obviously, $\forall p \in \{a, b, c, e\}$, $d||p$. Two operations, based on the disjoint union, equip the posets for govern interactions between two of them. Let two posets $(P_1, \leq_1)$ and $(P_2, \leq_2)$,

Definition: Operators

The **Parallel composition** $P_1 \otimes P_2$ is the coproduct with $P_1 \sqcup P_2$ as carrier set and the order is defined like:

$$p \in P_i,\ q \in P_j,\ p < q \Longleftrightarrow i = j \land p <_i q \qquad i, j \in \{1, 2\}$$

The **Serial composition** $P_1 * P_2$ is the ordinal sum with the disjoint union as carrier set again but defined like:

$$p \in P_i,\ q \in P_j,\ p < q \Longleftrightarrow (i = j \land p <_i q) \lor i < j \qquad i, j \in \{1, 2\}$$

Only the *serial composition* is **not commutative** because it alters the precedence order.

Example:

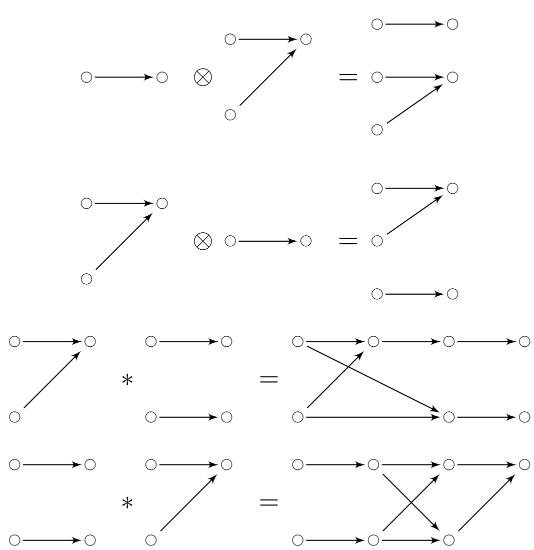
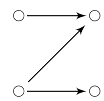


Figure 1: Parallel compositions and serial compositions

Definition: SP-Poset

A poset is **Serial-Parallel** (SP) if it is empty or it can be obtained from the singleton (○) and a finite number of application of the *Parallel composition* and the *Serial composition*. It is known that SP-posets is free of the following induced subposet:



This poset class is well known[2]. This simplicity of construction and this good algebraic properties make it a good object and find numerous applications in computer science and concurrent Klenee algebra.
Another interesting notion imported of Petri Nets and Higher Dimensional Automata is interval order. Interval orders form another class of posets relevant to concurrent and distributed computing. It is used to represent a "temporal relationship" between elements of a poset on a real line. Each interval representation of an element is characterized with an interval $[x^+, x^-]$ for all $x \in P$. x^+ represents the beginning of the point x and x^- the end of the point x. If we consider the points as process, we can assess the execution time of the process in relation to the others, while preserving the order.

Definition: Interval Order

A poset $(P, \leq)$ is an interval order if and only if there exists a mapping of each element $x \in P$ to a real interval $I(x) = [x^+, x^-]$ such that for any $x, y \in P$:

$$x \leq y \Longleftrightarrow x^- \leq y^+$$

Example:

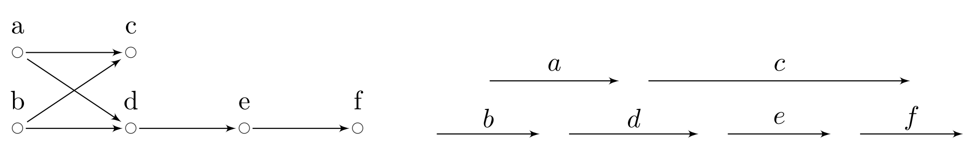


Figure 2: A poset and its interval representation

3 Poset with interfaces

Before we define the interfaces, we need to introduce two simple notions: the maximality and the minimality.

Lemma 1. *Let $p \in P$, p is said **minimal** if for all $q \in P$, $p \leq q \vee p || q$.*
*Similarly, p is **maximal** if for all $q \in P$, $p \geq q \vee p || q$*

$P^{\min}$ (resp. $P^{\max}$) is the set which contains all minimal (resp. maximal) point in P. The set, $[n] = \{1, 2, ..., n\}$ with $n \in \mathbb{N}$, allows us to number interfaces and distinguish them from each other.

Definition: IPoset

A **poset with interfaces (IPoset)** is a poset together with two injected set:

$$[n] \xrightarrow{s} P \xleftarrow{t} [m], \quad n, m \geq 0$$

such the image of $s([n])$ is minimal and the image of $t([m])$ is maximal

The injection $s : [n] \to P$ represents the *starting interfaces* symbolized by a right half-circle. This picture means the "unstarted" nature of the point, in this case comparable to an event which have a end but no start. The injection $t : P \to [m]$ represents the *terminating interfaces* symbolized by a left half-circle. Unlike the injection s, this picture means the "unfinished" nature of the point; a point with a start but no end.

Example:

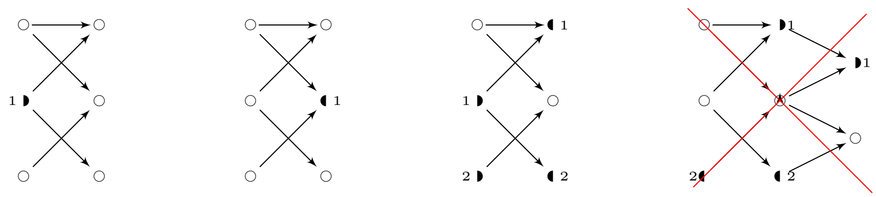


Figure 3: Some iposets possible at 6 points and one wrong usage of interfaces

With the introduction of interfaces, we can redefine the parallel composition and the serial composition. The serial composition is renamed in *gluing composition* for a better fit with the idea of the operation.

Definition: Parallel Composition

The **Parallel composition** of two iposets $[n_1] \to (P_1, \leq_1) \leftarrow [m_1]$ and $[n_2] \to (P_2, \leq_2) \leftarrow [m_2]$ is defined by:

$$[n_1 + n_2] \to (P_1 \otimes P_2,\ \leq_1 \cup \leq_2) \leftarrow [m_1 + m_2]$$

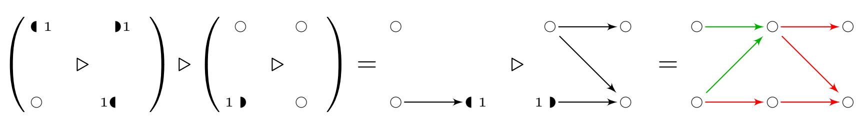
This definition of the *parallel composition* keeps the no commutativity. All P_2 interfaces have their numbering reassigned such as: $\forall i \in [n_2],\ n_1 + i$ for starting interfaces and such as: $\forall i \in [m_2],\ m_1 + i$ for terminating interfaces.

Definition: Gluing Composition

Only if $m_1 = n_2$, the **Gluing composition** of two iposets $[n_1] \xrightarrow{s_1} (P_1, <_1) \xleftarrow{t_1} [m_1]$ and $[n_2] \xrightarrow{s_2} (P_2, <_2) \xleftarrow{t_2} [m_2]$ is:

$$P_1 \triangleright P_2 = \begin{cases} (P_1 \sqcup P_2)/t_1(i) = s_2(i) \\ (<_1 \cup <_2 \cup\ (P_1\ /\ t_1[m]) \times (P_2\ /\ s_2[m]))^+ \end{cases}$$

Example:



The glue of two interface, one terminating in P_1 and one starting in P_2 permit to complete the event and the event inherit to the ordered relation of both interfaces. In the above example, all arrow of $(P_1\ /\ t_1[m]) \times (P_2\ /\ s_2[m])$ are not added because they are trivial or implied. So, to keep a readable poset, we don't draw its.

Example:

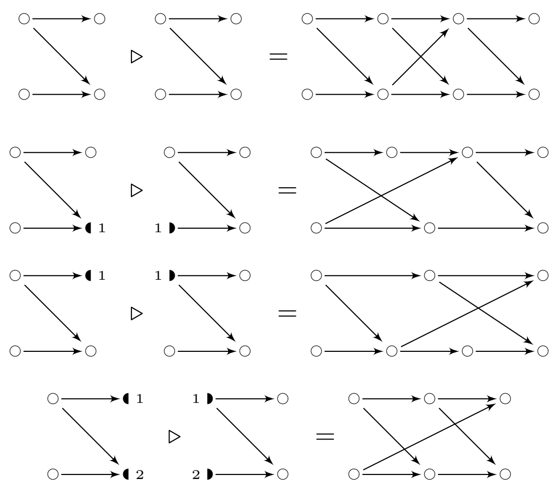


Figure 4: Some gluing compositions

With this two newly defined operations, we can create a new subclass of posets.

Definition: GP-Poset

An (i)poset is **Gluing-Parallel** (GP-(i)poset) if it is empty or if it is a finite combination of the following singletons thanks to the *Parallel composition* and the *Gluing composition*.

Figure 5: The four singletons

The GP-Posets is a larger, more complex set containing SP-Posets (as its operations are extensions of SP-Posets operations). Its total comprehension would allow the representation of more complex systems in computer science and concurentential theory.

4 Forbidden structures

After several attemps of generating the maximum of GP-Posets[3], the bound of posets with 16 points seems to be too hard to reach. Some forbidden induced substructures has been discovered: 5 with 6 points, 1 with 8 points and 5 with 10 points.

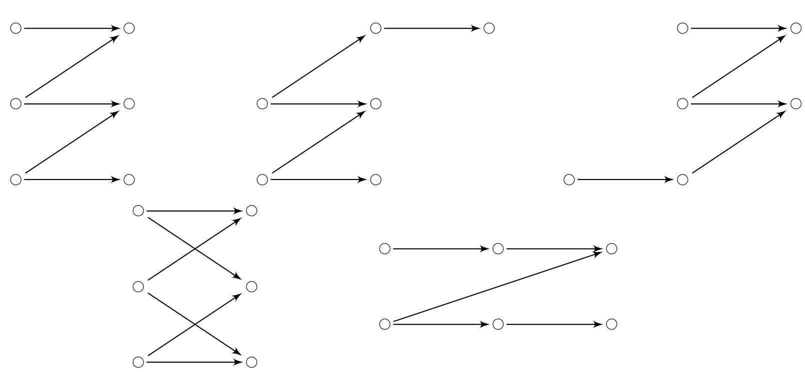


Figure 6: Forbidden substructures with 6 points.

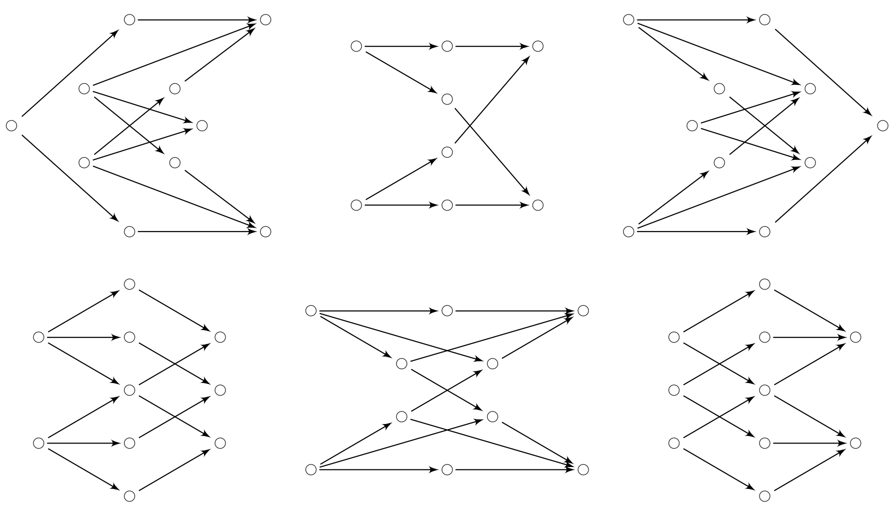
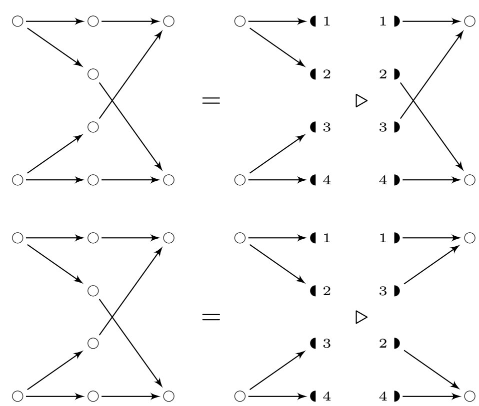


Figure 7: Additional forbidden substructures for gp-posets.

However, we don't know if there is more sub-structures due to the exponential growth of the number of posets prevents us of performing an exhaustive verification up to 16 points.

5 Gluing-Parallel-Symmetric IPosets

The 8-points forbidden substructure in the centre of the top row of Figure 6 has a gluing decomposition along a maximal antichain (subset of a partially ordered set such that any two distinct elements in the subset are incomparable).



We can make an observation on the decomposition: it is easy to see that the left poset is a GP-Poset. But the right poset is not; a switch between the interface in the middle and we obtain a poset which will be the parallel composition of two GP-Poset, so it will be GP too. Moreover, we can apply the same process to convert all forbidden structures into gp-poset. It is for this reason that we introduce the *symmetric composition* and another class of posets.

Definiton: Symmetric Element

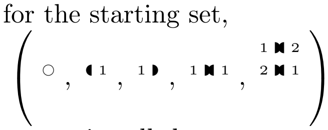
Let P, a poset, such as $P = \circ \otimes \circ$.
The **Symmetric Element** is the non-trivial symmetry on 2:

$$\mathfrak{a}_2 = [2] \xrightarrow{s} P \xleftarrow{t} [2] \qquad s(i) = i \text{ and } t(i) = 3 - i$$

Definition: GPS-Poset

An iposet is **gluing-parallel-symmetric** (*gps*) if it is empty or can be obtained from the singletons and $\mathfrak{a}_2$ by finitely many applications of $\otimes$ and $\triangleright$.

To get a view for the starting set,

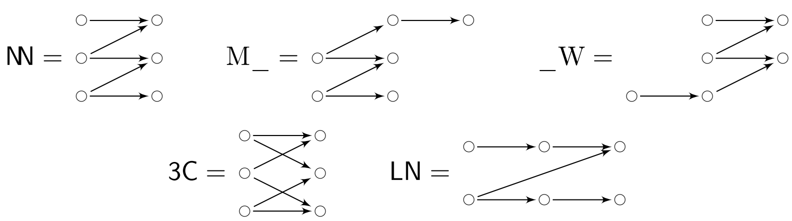
singletons $\cup\ \mathfrak{a}_2 =$ 

The gps-iposets set contains all the gp-posets and also all the symmetries $n \to n$ for any n. We can define the gps-posets class in a similar way we defined the gps-iposets. All forbidden induced sub-structures with eight points or more are gps-posets. It remains the five forbidden induced substructures at six points. It is these substructures that are at the heart of the conjecture we are interested in.

6 The Five Forbidden Structures

Conjecture

A poset is GPS if and only if it does not contain any of the five induced sub-structures:



The implication: "If a poset is GPS then it does not contain any of the five induced sub-structures" is already proved [1]. The goal is to prove the other direction of the equivalence.

6.1 IO(SP) way

The first way that we have explored, is the Corollary 34 of the article "Generating Posets with Interfaces"[3]. We need to introduce a notion to fully understand th corollary

Definition: Interval Representation

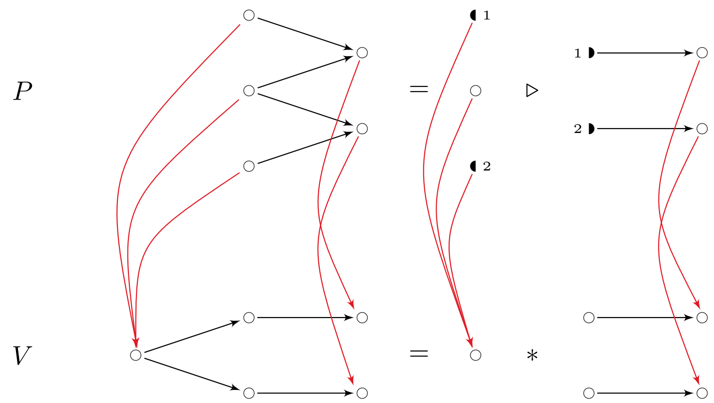
An **Interval representation** of a P poset in a V poset is a pair of functions $f,\ g : P \to V$ such that:

- $f(p) \leq g(p), \forall p \in P$
- $p <_P q \leftrightarrow g(p) <_V f(q), \forall p, q \in P$

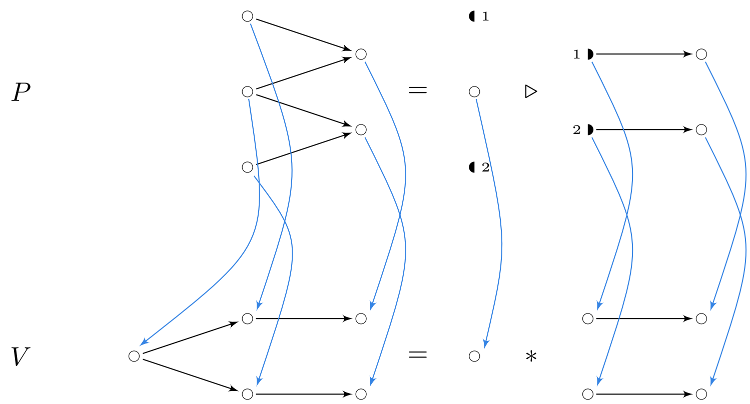
Vocabulary: If a P poset has an interval representation in a V poset then P is V-interval

Example:
Interval representation (f, g) of an P in a V. The equation (1) represents the function f and the equation (2) represents

the function g.



(1)



(2)

Corollary

A poset P is gps if and only if it admits an interval representation in an sp-poset.

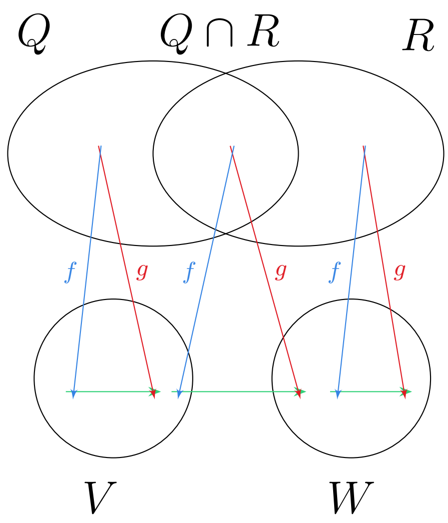
Proof:

6.1.1 Direct sense

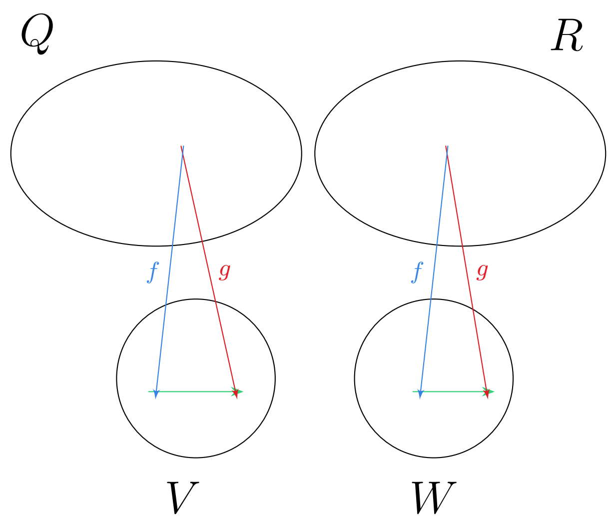
Let P be a GPS-poset.
We will proceed by structural induction on P:
For $|P| = 1$ (Base case), P is one of the four singletons.
So P can be mapped by itself in SP-interval. For $|P| > 1$ and $P = Q \otimes R$ and $(f_Q, g_Q), (f_R, g_R)$ are interval representation in W and Y (resp.). $Q \otimes R$ is disconnected so Q is W-interval and R is Y-interval. So the interval representation of P is $(f_Q \sqcup f_R, g_Q \sqcup g_R)$ and P is $Q \otimes R$-interval.
For $|P| > 1$, $P = Q \triangleright R$, with (f_Q, g_Q) and (f_R, g_R) interval representations of Q and R,

$$f(p) = \begin{cases} f_Q(p) & p \in Q \\ f_R(p) & p \notin Q \end{cases} \qquad g(p) = \begin{cases} g_R(p) & p \in R \\ g_Q(p) & p \notin R \end{cases}$$

is representation of P in $W * Y$.



Visualisation of the case $W * Y$.



Visualisation of the case $W \otimes Y$.

We want to show the first condition to be an interval representation: $\forall p \in P, f(p) \leq g(p)$.

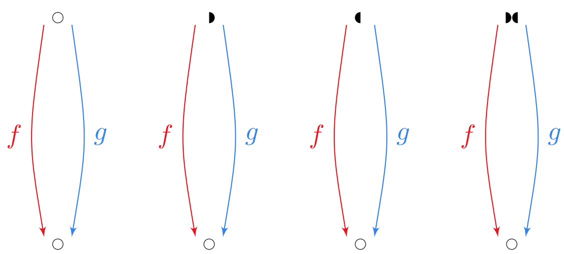
Let $p \in P$,
If $p \in Q \backslash R$, then $f(p) = f_Q(p) \leq_V g_Q(p) = g(p)$. If $p \in R \backslash Q$, then similarly to above. If $p \in R \cap Q$, $f(p) = f_Q(p) <_V g_R(p) = g(p)$, then by definition of the gluying composition (because $f_Q(p) \in W$ and $g_R(p) \in W$).

Now the second condition: $\forall p, q \in P$, $p <_P q \iff g(p) <_V f(q)$

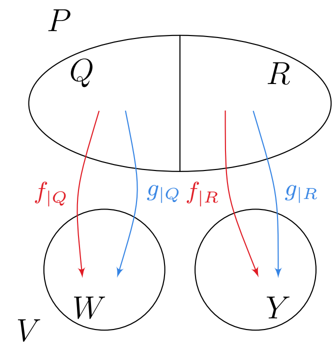
Let $p, q \in P$,
We suppose $p <_P q$. If $p, q \in Q$, like $p <_Q q$, $p \notin R$, so $g_Q(p) = g(p) <_V f(q) = f_R(q)$. If $p \notin R$ and $q \in Q$, $g_Q(p) <_V f_Q(q)$ by induction. So $g_Q(p) = g(p)$ and $f_Q(q) = f(q)$ then $g(p) <_V f(q)$. If $p \in R$ and $q \notin Q$, $g_R(p) <_V f_R(q)$ by induction. So $g_R(p) = g(p)$ and $f_Q(q) = f(q)$ then $g(p) <_V f(q)$.
Now, we suppose $g(p) <_V f(q)$ If $p \notin R$ and $q \notin Q$, so $g(p) \in W$ and $f(q) \in Y$. By definition of the gluying composition, $g(p) <_V f(q) \implies p <_P q$. If $p \notin R$ and $q \in Q$, $g_Q(p) = g(p) <_V f(q) = f_Q(q)$, so by definition $p <_P q$. If $p \in R$ and $q \notin Q$, $g_R(p) = g(p) <_V f(q) = f_R(q)$, same as above. If $p \in R$ and $q \in Q$, $g(p) = g_R(p)$ and $f(q) = f_Q(q)$, so we have $f_R(p) <_V g_R(p) <_V f_Q(q)$. Then by transitivity $f_R(p) <_V f_Q(q)$. It's absurd !! So this case is not possible.

6.1.2 Indirect sense

Let P admits an interval representation (f, g) in a SP-poset V.
We will also proceed by structural induction on V:
For $|V| = 1$ (Base case), V is the singleton so $P = f^{-1}(V)$. So P is one of the four singleton of GPS and obviously P is GPS.
For $|V| > 1$ and $V = W \otimes Y$, as $W \otimes Y$ is disconnected, we can define $Q = f^{-1}(W) = g^{-1}(W)$ and $R = f^{-1}(Y) = g^{-1}(Y)$.
We have $(f_{|Q}, g_{|Q})$ an interval representation of Q and $(f_{|R}, g_{|R})$ an interval representation of R. Q and R are GPS by induction, so $P = Q \otimes R$ is GPS.



Visualisation of the base case.



Visualisation of the case $W \otimes Y$.

For $|V| > 1$ and $V = W * Y$, Let $Q = f^{-1}(W)$ and $R = g^{-1}(Y)$. We arbitrarily define an interval representation of Q as $(f_Q, g_Q) : Q \to W$ as follow.
Let $p \in Q$:

$$\left(f_Q(p) = f(p) \ , \qquad g_Q(p) = \begin{cases} g(p) & \text{if } g(p) \in W \\ w & \text{with } w \in W^{max} \text{ such as } f(p) \leq w \end{cases} \right)$$

After define both interval representation, we show the first condition: $\forall p \in Q, f_Q(p) \leq_W g_Q(p)$,

Let $p \in Q$, If $g(p) \in W$, then $f_Q(p) = f(p) \leq g(p) = g_Q(p)$. Otherwise, $f_Q(p) \leq g_Q(p)$ by definition of g_Q above.

Now we show the second condtion: $\forall p, q \in Q$, $p <_Q q \leftrightarrow g_Q(p) <_W f_Q(q)$

Let $p, q \in Q$.
We suppose $p <_Q q$. If $g(p) \in W$, then $g(p) = g_Q(p) \leq f_Q(q) = f(q)$. Otherwise, $g_Q(p) \in W$ by definition. So we can deduce that $g_Q(p) < g(p) < f(q) = f_Q(q)$.
We suppose $g_Q(p) <_W f_Q(q)$. So $g_Q(p) \notin W^{max}$, $g_Q(p) <_W f_Q(q)$, but we also have $g_Q(p) = g(p)$ and $f_Q(q) = f(q)$ then $g(p) <_W f(q)$ and $p <_Q q$.

The same process is applied to R with an interval representation defined as $p \in R$ and $(f_R, g_R) : R \to Y$ such as:

$$\left(f_R(p) = \begin{cases} f(p) & \text{if } g(p) \in Y \\ y & \text{with } y \in Y^{min} \text{ such as } f(p) \geq y \end{cases} \ , \qquad g_R(p) = g(p) \right)$$

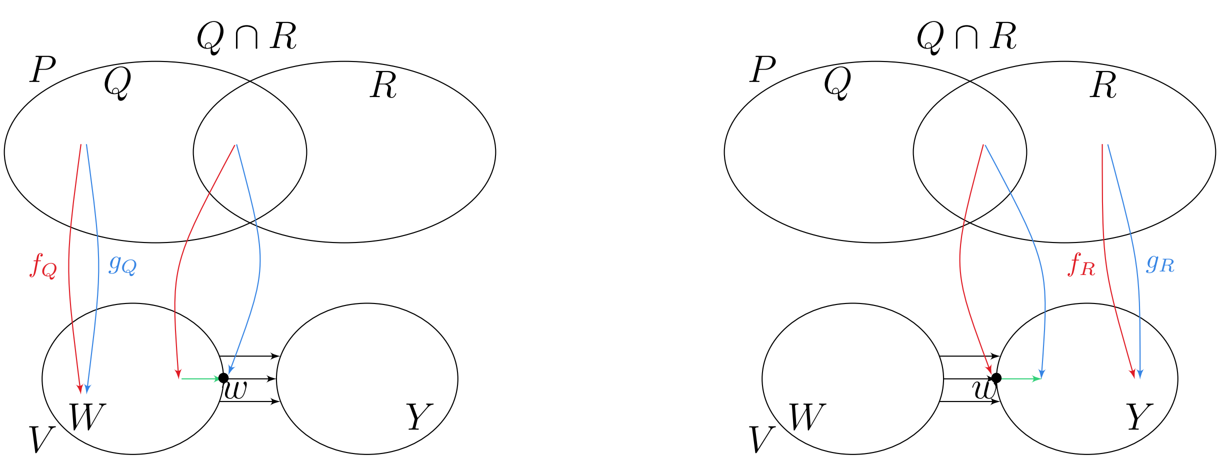
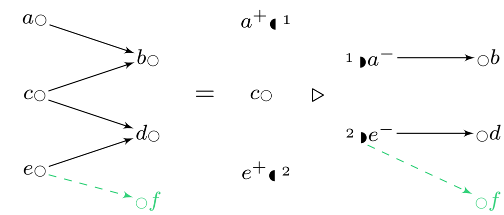


Figure 10: Visualisation of the case $W * Y$.

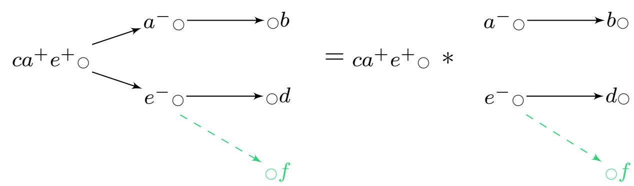
6.1.3 Interests and Obstacles

The biggest advantage of the corollary is that we can move from a complex structure to a simpler, better understood one, the SP-Poset. Now we try to build the interval representation of forbidden structures to interval based on the representations of GPS-Poset to see how the structure of the interval representation of forbidden structures can differ and if we can deduce conditions on SP-Poset interval. Take the NN and remove the right bottom point. We call it M. M is GPS so he has an interval representation in an sp-poset V(See example of Interval Representation). Now add the previously removed point to get NN to W and on V.

M



V



However, in M, $c < f$ and $\forall x \in P/\{c\}|\ x||c$, it's impossible due to the second property of the interval representation of poset. We need to split the point ca^+e^+ in M, to get a valid interval representation of NN.

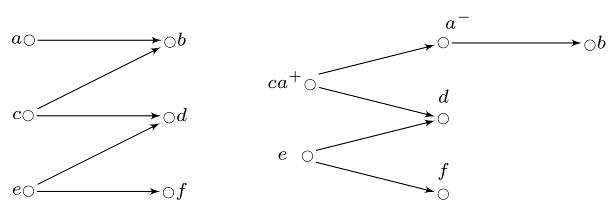
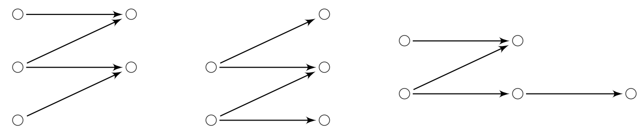


Figure 11: NN and this interval representation

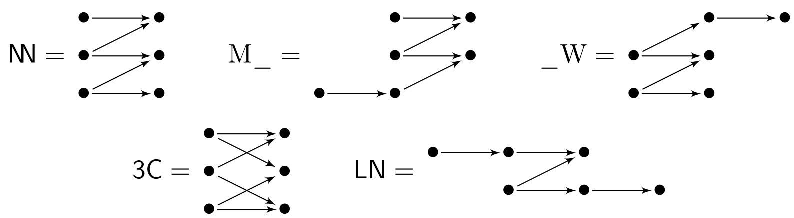
Nevertheless, we need to make a huge modification of the structure of the interval representation. This lack of consistency with the add of point means that we can't easily use and block us. So we decided to go down another way.

7 Beginning of the proof

This is another approach, based on the structure of the poset, and in particular we are trying to build the five forbidden structures from a non-gps poset because we noticed that 3 structures were at the base of the prohibited structures.



We show the start of a proof that if a poset is not GPS, then it contains one of the five forbidden induced substructures

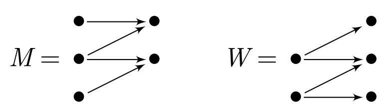


Before that, a helpful notion: the *zigzag distance* $d_{zz}(x,y)$ between two points $x, y \in P$ in a poset P is the length n of a shortest zigzag $x = x_0 < x_2 > x_3 < \cdots x_n = y$. (There are really four cases to consider here depending on whether x_0 is below x_1 and on whether x_{n-1} is below x_n.)

Lemma 2. *d_{zz} defines a metric on P.*

Proof. $d_{zz}(x,y) = 0$ iff $x = y$; symmetry is obvious; and if $d_{zz}(x,y) = n$ and $d_{zz}(y,z) = m$, then the two zigzags may be concatenated (and the concatenation possibly be shortened), showing that $d_{zz}(x,z) \leq n + m$. □

We will also use some other special posets below:



Lemma 3. *P contains an induced M or an induced W if and only if there are $x, y \in P$ with $d_{zz}(x,y) \geq 4$. P contains an induced NN if and only if there are $x, y \in P$ with $d_{zz}(x,y) \geq 5$.*

Proof. We only show the second claim; the first uses similar arguments. If P contains an induced NN, then the zigzag distance between its extremal points is equal to 5. Conversely, let $x, y \in P$ with $d_{zz}(x,y) = 5$, then the zigzag $x = x_1 < x_2 > x_3 < x_4 > x_5 < x_6 = y$ is an NN. (The other case is symmetric.) Any relation $x_i \leq x_j$ would shorten the zigzag, so the NN is induced. □

Now let P be a poset which is not GPS. We may assume that P is $\otimes$-indecomposable (*i.e.*, weakly connected) and $*$-indecomposable; otherwise we can decompose P and have a smaller poset to start with.

As usual we denote by $P_{\max}$ the subset of maximal elements and by $P_{\text{MAX}} = \{y \in P_{\max} \mid \forall z \in P : |\{x : x < y\}| \geq |\{x : x < z\}|\}$ the subset of right-extreme elements.

Let $a \in P_{\text{MAX}}$. If $P_{\max} = \{a\}$, then $P = (P \setminus \{a\}) * \{a\}$, in contradiction to P being indecomposable.

Assume that $P_{\max} \subsetneq P_{\text{MAX}}$ and let $b \in P_{\text{MAX}} \setminus P_{\max}$.

If $d_{zz}(a,b) > 2$, then $d_{zz}(a,b) \geq 4$ as both a and b are maximal. By Lemma 3 and again because of maximality, P contains an induced W.

Assume $d_{zz}(a,b) \leq 2$, then $d_{zz}(a,b) = 2$ by maximality of both, hence there is $z \in P_{\min}$ for which $a > z < b$.

Let $m \in P$ such that $m < a$ and $m \not< b$; this exists because $a \in P_{\text{MAX}}$ and $b \notin P_{\text{MAX}}$. We know that $m \not< z$ because of minimality of z, but we may or may not have $z < m$.

If $\{k \in P \setminus \{a,b\} \mid k \not< a\} = \emptyset$, then $P = (P \setminus \{a\}) * \{a,b\}$ (with b in the interface), in contradiction to indecomposability. Let $k \in P \setminus \{a,b\}$ such that $k \not< a$. We know that $k \not< z$ because of minimality of z, but we may or may not have $z < k$.

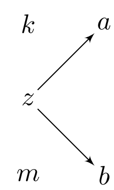


Figure 12: State of the poset P

1. If $k < b$:
   (a) If $z < m$: ???
   (b) If $z \not< m$:
      i. If $z < k$:
      ii. If $z \not< k$: then $m < a > z < b > k$ forms an induced M.
2. If $k \not< b$: We cannot have $k < m$ as this would imply $k < a$.
   (a) If $m < k$:
      i. If $z < m$: ???
      ii. If $z \not< m$:
         A. If $z < k$: ???
         B. If $z \not< k$: then $k > m < a > z < b$ forms an induced W.
   (b) If $m \not< k$:
      i. If $z < k$: ???
      ii. If $z \not< k$: then $d_{\mathrm{zz}}(z, k) \geq 2$, hence $d_{\mathrm{zz}}(m, k) \geq 4$ and P contains an induced M.

Here is the actual state of the proof. There are a few conditions missing, but we didn't have enough time to complete it.

References

References

[1] Uli Fahrenberg, Christian Johansen, Georg Struth, and Krzysztof Ziemiański. Posets with interfaces as a model for concurrency, 2022.

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[3] Olavi Äikäs, Uli Fahrenberg, Christian Johansen, and Krzysztof Ziemiański. Generating posets with interfaces, 2022.