One more step towards well-composedness of cell complexes over nD pictures

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Abstract. An nD pure regular cell complex K is weakly well-composed (wWC) if, for each vertex v of K, the set of n-cells incident to v is face-connected. In previous work we proved that if an nD picture I is digitally well composed (DWC) then the cubical complex $Q(I)$ associated to I is wWC. If I is not DWC, we proposed a combinatorial algorithm to "locally repair" $Q(I)$ obtaining an nD pure simplicial complex $P_S(I)$ homotopy equivalent to $Q(I)$ which is always wWC. In this paper we give a combinatorial procedure to compute a simplicial complex $P_S(\bar{I})$ which decomposes the complement space of $|P_S(I)|$ and prove that $P_S(\bar{I})$ is also wWC. This paper means one more step on the way to our ultimate goal: to prove that the nD repaired complex is continuously well-composed (CWC), that is, the boundary of its continuous analog is an $(n-1)$-manifold.

1 Introduction

Ensuring that the celullar decomposition K of an object is continuously well-composed (CWC) allows to later work in a more efficient way. For example in the 3-dimensional (3D) setting, it allows to work with surface parameterizations [9]. Besides, K being CWC has also topological computation benefits in the sense that the homology of the object can be deduced from the homology of its boundary [7,8]. Moreover, if the boundary of the object is not a manifold, then it is well-known that the multigrid convergence of some geometrical estimators is slower (see, for example, [14]).

The authors have been working for several years in providing algorithms to "locally repair" the cubical complex $Q(I)$ canonically associated to a given picture I, in order to obtain a cell complex homotopic to $Q(I)$ and CWC.

A first attempt to solve the problem in the 3-dimensional setting (3D) was presented in [10]. In [11,12,13], the authors provided a satisfactory solution in the 3D case to locally repair $Q(I)$ following a purely combinatorial procedure, obtaining a polyhedral $P(I)$ that is CWC.

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thickening
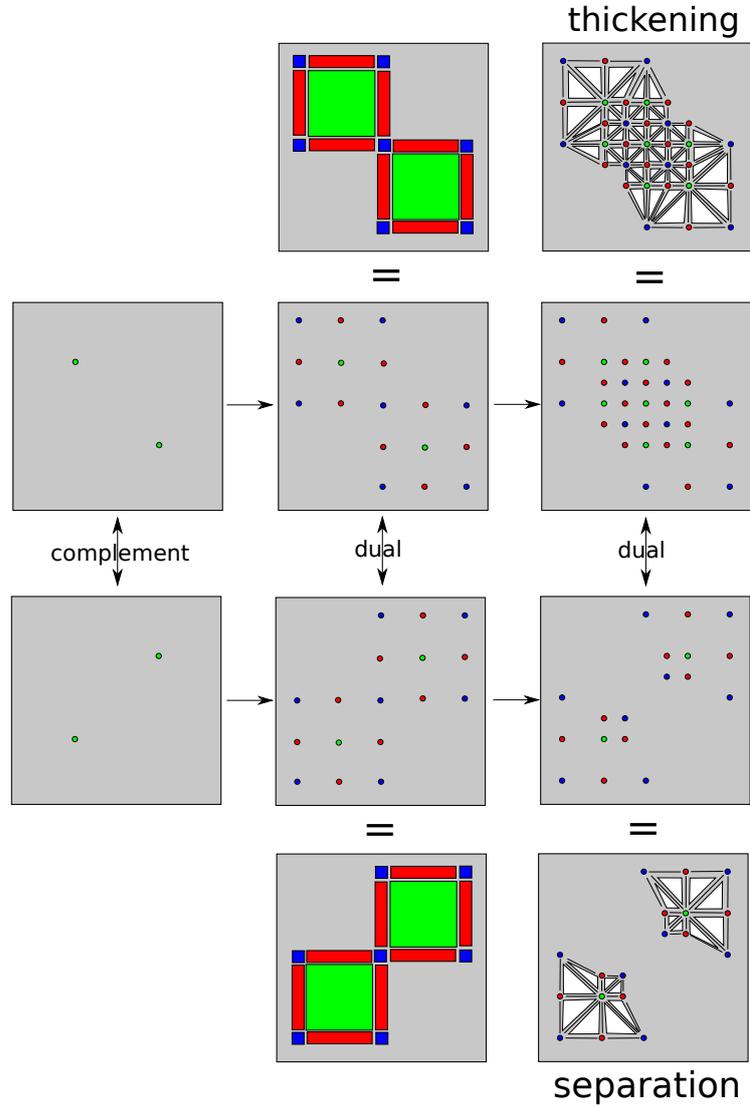
separation

Fig. 1. $P_S(I)$ versus $P_S(\bar{I})$. In the upper half, starting from an image I made of two (green) points (on the left), we compute its associated cubical complex $Q(I)$ (in the middle column). When a critical configuration is detected in I, a reparation procedure is applied on $Q(I)$, leading to a well-composed simplicial complex $P_S(I)$ (on the right). In the lower half, we compute the complement $\bar{I}$ of the image I which leads to the dual cubical complex $Q(\bar{I})$ (in the middle). Now, the reparation procedure (which is described in this paper) "separates" the complex $Q(\bar{I})$ to yield a new simplicial complex $P_S(\bar{I})$ (on the right), which is also weakly well-composed. Notice that the picture of $\bar{I}$ (and hence, $Q(\bar{I})$ and $P_S(\bar{I})$) has been restricted to a two-by-two pixels context and the infinite set of pixels around them are missed.

In [3], we extend to any dimension (nD) the previous method: Given an nD picture I, we detect non-well-composed configurations (which cause $Q(I)$ not to be CWC) and apply on those configurations a repairing process (see the upside part of Figure 1 for a sketch of the procedure) consisting in "thickening" the picture representation with some new cells in order to remove the "pinches". The resulting computed complex $P_S(I)$ is a simplicial complex homotopic to $Q(I)$. In that paper, as a first step towards continuous well-composedness, we proved that $P_S(I)$ is weakly well-composed (wWC), what means that there always exists a face-connected path in $P_S(I)$ of n-simplices incident to a common vertex v, joining any two n-simplices incident to v. Some other recent works following the idea of the reparation method are [1] (using the inter-simplex Chebyshev distance), [5] (for face centered cubic grids) and [6] (for body centered cubic grids). A tutorial revisiting the different concepts on well-composedness depending on the nature of the object of study is [2].

Proving that $P_S(I)$ is CWC in any dimension is not an easy task and is still an open problem. CWCness is *self-dual*, that is, the boundary of a compact subset X of $\mathbb{R}^n$ is a manifold iff the boundary of its complement $X^c := \mathbb{R}^n \setminus X$ is a manifold too. Hence, proving that $P_S(I)$ and its *dual* $P_S(\bar{I})$, where $\bar{I}$ is the complement of I (see the lower half of Figure 1), are both wWC, shows that the type of well-composedness of $P_S(I)$ we are proving here is also self-dual and then stronger than simply weak well-composedness. Consequently, in this paper we go one step towards the final goal, by proving that $P_S(\bar{I})$ is also wWC. The paper is organized as follows: Section 2 is devoted to recall the main concepts needed to understand the rest of the paper and we sketch the method developed in [3] to compute $P_S(I)$. In Section 3 we introduce the complement of an nD picture denoted by $\bar{I}$ and its canonically associated cubical complex $Q(\bar{I})$. In Section 4, we explain the procedure to locally repair $Q(\bar{I})$ to obtain the wWC simplicial complex $P_S(\bar{I})$, satisfying that its continuous analog is the closure of the complement space of $P_S(I)$. This procedure is a modification of the one given in [3] to compute $P_S(I)$. Section 5 is devoted to conclusions and future works.

2   Background

Let $n \geq 2$ be an integer and $\mathbb{Z}^n$ the set of points with integer coordinates in nD space $\mathbb{R}^n$. An *nD binary image* is a pair $I = (\mathbb{Z}^n, F_I)$ where F_I is a finite subset of $\mathbb{Z}^n$ called *foreground* of I. If $I = (\mathbb{Z}^n, F_I)$, with $F_I \subset 4\mathbb{Z}^n$, then we say that I is an *nD picture*. This is only a technical issue: we can encode the cubical complex associated to I in $2\mathbb{Z}^n$, but we need the foreground F_I to be included in $4\mathbb{Z}^n$ so that we have "extra" space to encode the repairing cells that will be added to "thicken" the non-manifold parts of such cubical complex.

For two integers $k \leq k'$, let $[\![k, k']\!]$ denote the set $\{k, k+1, \ldots, k'-1, k'\}$ and $\mathbb{B} = \{e^1, \ldots, e^n\}$ the canonical basis of $\mathbb{Z}^n$.

Given a point $z \in 4\mathbb{Z}^n$ and a family of vectors $\mathcal{F} = \{f^1, \ldots, f^k\} \subseteq \mathbb{B}$, the *block of dimension k* associated to the couple $(z, \mathcal{F})$ is the set defined as:

$$B(z, \mathcal{F}) = \left\{ z + \sum_{i \in [\![1,k]\!]} \lambda_i \, f^i \ : \ \lambda_i \in \{0, 4\}, \forall i \in [\![1, k]\!] \right\}.$$

This way, a 0-block is a point, a 1-block is an edge, a 2-block is a square, and so on. A subset $B \subset 4\mathbb{Z}^n$ is called a *block* if there exists a couple $(z, \mathcal{F}) \in 4\mathbb{Z}^n \times \mathcal{P}(\mathbb{B})^3$ such that $B = B(z, \mathcal{F})$. We will denote the set of blocks of $4\mathbb{Z}^n$ by $\mathcal{B}(4\mathbb{Z}^n)$.

Two points p, q belonging to a block $B \in \mathcal{B}(4\mathbb{Z}^n)$ are said to be *antagonists* in B if their distance equals the maximum distance using the L^1-norm[4] between two points in B:

$$\|p - q\|_1 = \max \left\{ \|r - s\|_1 \ ; \ r, s \in B \right\}.$$

The antagonist of a point p in a block $B \in \mathcal{B}(4\mathbb{Z}^n)$ containing p exists and is unique. It is denoted by $\mathrm{antag}_B(p)$. Note that when two points $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are antagonists in a block of dimension $k \in [\![0, n]\!]$, then $|x_i - y_i| = 4$ for $i \in \{i_1, \ldots, i_k\} \subseteq [\![1, n]\!]$ and $x_i = y_i$ otherwise.

Now, let $I = (\mathbb{Z}^n, F_I)$ be an nD picture and $B \in \mathcal{B}(4\mathbb{Z}^n)$ a block of dimension $k \in [\![2, n]\!]$. We say that I contains a *critical configuration* (CC) in the block B if $F_I \cap B = \{p, p'\}$ or $F_I \cap B = B \setminus \{p, p'\}$, with p, p' being two antagonists in B.

DWCness. The nD picture $I = (\mathbb{Z}^n, F_I)$ is said to be *digitally well-composed* (DWC) if for any block B of dimension $k \in [\![2, n]\!]$, the set $F_I \cap B$ is not a critical configuration. Additionally, $\bar{I} = (\mathbb{Z}^n, 4\mathbb{Z}^n \setminus F_I)$ is DWC iff I is DWC (we call this property *self-duality* [4] of DWCness).

We say that two elements x, y of $4\mathbb{Z}^n$ are *$2n$-neighbors* in $4\mathbb{Z}^n$ when there exists some $i \in [\![1, n]\!]$ verifying that $x_i = y_i \pm 4$ and for any $j \in [\![1, n]\!] \setminus \{i\}$, $x_j = y_j$. Also, we call *$2n$-path* joining p, q in $S \subseteq 4\mathbb{Z}^n$ a sequence $(p^0 = p, p^1, \ldots, p^k = q)$ of elements of S which verifies for any $i \in [\![0, k-1]\!]$ that p^i and p^{i+1} are $2n$-neighbours in $4\mathbb{Z}^n$.

Proposition 1 ([3]). *If $I = (\mathbb{Z}^n, F_I)$ is DWC then, for any block $B \in \mathcal{B}(4\mathbb{Z}^n)$ and for any two points p, q in $F_I \cap B$, there exists a $2n$-path in $F_I \cap B$ joining p and q.*

We use the term *nD cell complex K* to refer to a pure regular cell complex of rank n embedded in $\mathbb{R}^n$. The underlying space (i.e., the union of the n-cells as subspaces of $\mathbb{R}^n$) will be denoted by $|K|$. Regular cell complexes have particularly nice properties, for example, their homology is effectively computable (see [15, p. 243]).

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[3] The expression $\mathcal{P}(\mathbb{B})$ represents the set of all the subsets of $\mathbb{B}$.
[4] The L^1-norm of a vector $\alpha = (x_1, \ldots, x_n)$ is $\|\alpha\|_1 = \sum_{i \in [\![1,n]\!]} |x_i|$.

The *boundary surface* ∂K of an nD cell complex K is the (n-1)D cell complex composed by the $(n-1)$-cells of K that are faces of exactly one n-cell, together with all their faces.

The *cone (join)* on a simplicial complex K with vertex v, denoted by $v * K$ is the simplicial complex whose simplices have the form $\langle v_0, \ldots, v_\ell, v \rangle$ (where $\langle v_0, \ldots, v_\ell \rangle$ is a simplex of K spanned by the set of points $\{v_0, \ldots, v_\ell\}$), along with all the faces of such simplices.

CWCness. An nD cell complex K is said to be *continuously well-composed* (CWC) if $|\partial K|$ is an $(n-1)$-manifold, that is, each point of $|\partial K|$ has a neighborhood homeomorphic to $\mathbb{R}^{n-1}$ into $|\partial K|$.

Let $\mathcal{S}$ be a set of ℓ-cells of K, $\ell \in [\![1, n]\!]$. We say that two ℓ-cells σ and σ' are *face-connected in $\mathcal{S}$* if there exists a path $\pi(\sigma, \sigma') = (\sigma_1 = \sigma, \sigma_2 \ldots, \sigma_{m-1}, \sigma_m = \sigma')$ of ℓ-cells of $\mathcal{S}$ such that for any $i \in [\![1, m-1]\!]$, σ_i and σ_{i+1} share exactly one $(\ell - 1)$-cell of K. The set $\mathcal{S}$ is *face-connected* if any two ℓ-cells σ and σ' in $\mathcal{S}$ are face-connected in $\mathcal{S}$.

wWCness. An nD cell complex K is *weakly well-composed (wWC)* if for any 0-cell μ in K, the set of n-cells incident to μ, denoted by $\mathcal{A}_K^{(n)}(\mu)$, is face-connected.

A *canonical size-4 n-cube* is a size-4 n-cube centered at a point in $\mathbb{Z}^n$ having their $(n-1)$-faces parallel to the coordinate hyperplanes.

Given an nD picture I, the nD cell complex whose n-cells are the canonical size-4 n-cubes centered at each point in F_I is denoted by $Q(I)$.

We say that $J = (\mathbb{Z}^n, F_J)$ *encodes* $Q(I)$ if F_J is the set of barycenters of the cells in $Q(I)$[5]. We say that $p \in F_J$ *encodes* $\sigma \in Q(I)$ if p is the barycenter of σ. In that case, we denote σ as $\sigma_{Q(I)}(p)$.

Notation 1 *Let $N, M \in \mathbb{Z}$ such that $0 \leq N < M$. Let $p = (x_1, \ldots, x_n) \in \mathbb{Z}^n$. Then $N_M(p)$ denotes the set of indices $\{i \in [\![1, n]\!] \ : \ x_i \equiv N \mod M\}$.*

Remark 1. $2\mathbb{Z}^n$ can be decomposed into the disjoint sets

$$\mathcal{E}_\ell := \{p \in 2\mathbb{Z}^n : \mathrm{card}(0_4(p)) \text{ is } \ell\}$$

where $\mathrm{card}(S)$ is the number of elements in the set S. For example $\mathcal{E}_n = 4\mathbb{Z}^n$ and $\mathcal{E}_0 = (2\mathbb{Z} \setminus 4\mathbb{Z})^n$. Besides, $\mathbb{Z}^n \setminus 2\mathbb{Z}^n$ can be decomposed into the disjoint sets:

$$\mathcal{O}_\ell := \{p \in \mathbb{Z}^n \setminus 2\mathbb{Z}^n \ : \ \mathrm{card}(0_2(p)) \text{ is } \ell\}.$$

Notice that ℓ-cells of the cubical complex $Q(I)$ are encoded by points in $\mathcal{E}_\ell$. Intuitively, sets $\mathcal{O}_\ell$ will serve as support to build new cells, in the repairing process, around critical configurations.

Proposition 2 ([3]). *The set of points of F_J encoding the faces of a cell $\sigma_{Q(I)}(p)$ is:*

$$\mathcal{D}_{F_J}(p) := \mathcal{D}_{F_J}^+(p) \setminus \{p\} \quad \text{where } \mathcal{D}_{F_J}^+(p) = \left\{ p + \sum_{j \in 0_4(p)} \lambda_j \, e^j \ : \ \lambda_j \in \{0, \pm 2\} \right\}.$$

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[5] Observe that $F_J \subset 2\mathbb{Z}^n$

The set of points encoding the i-faces of $\sigma_{Q(I)}(p)$ will be denoted by $\mathcal{D}^i_{F_J}(p)$ and is, in fact, $\mathcal{D}_{F_J}(p) \cap \mathcal{E}_i$. In particular, the set of points encoding the vertices of $\sigma_{Q(I)}(p)$ is

$$\mathcal{D}^0_{F_J}(p) = \left\{ p + \sum_{j \in 0_4(p)} \lambda_j\, e^j \, : \, \lambda_j \in \{\pm 2\} \right\}.$$

Proposition 3 ([3]). *The set of points of F_J encoding the cells incident to a cell $\sigma_{Q(I)}(p)$ is:*

$$\mathcal{A}_{F_J}(p) := \mathcal{A}^+_{F_J}(p) \backslash \{p\} \; where \; \mathcal{A}^+_{F_J}(p) = \left\{ p + \sum_{j \in 2_4(p)} \lambda_j\, e^j : \lambda_j \in \{0, \pm 2\} \right\} \cap F_J.$$

The set of points encoding the i-cells incident to $\sigma_{Q(I)}(p)$ will be denoted by $\mathcal{A}^i_{F_J}(p)$. In particular,

$$\mathcal{A}^n_{F_J}(p) := F_J \cap \left\{ p + \sum_{j \in 2_4(p)} \lambda_j\, e^j \, : \, \lambda_j \in \{\pm 2\} \right\}.$$

Proposition 4 ([3]). *If $I = (\mathbb{Z}^n, F_I)$ is DWC, then $Q(I)$ is wWC.*

Cell complexes over nD pictures. A *cell complex over an nD picture I* is an nD cell complex, denoted by $K(I)$, such that there exists a deformation retraction from $K(I)$ onto $Q(I)$.

In [3] we detailed a procedure to "locally repair" the cubical complex $Q(I)$ when $I = (\mathbb{Z}^n, F_I)$ is not DWC to obtain a simplicial complex $P_S(I)$ such that $|P_S(I)|$ is homotopy equivalent to $|Q(I)|$ and $P_S(I)$ is a wWC simplicial complex. The principal steps are summarized below:

First, Procedure 1 from [3] is used to compute the set R of (critical) points of F_J representing the critical cells in $Q(I)$. Basically, we look for critical configurations and for the cells incident to vertices involved in such critical configurations. These cells are the candidates to be repaired in order to obtain the wWC simplicial complex $P_S(I)$. We reproduce below the procedure so that the paper is self-contained.

Once the critical points are computed, extra points are added around them, defining the set F_L that will contain all the vertices of the new construction $P_S(I)$:

$$F_L := F_J \cup \bigcup_{p \in R} S(p),$$

where $\; S(p) := \left\{ p + \sum_{j \in 2_4(p)} \lambda_j\, e^j \, : \, \lambda_j \in \{0, \pm 1\} \right\}.$ These points are classified into sets $\mathcal{C}_\ell$, for $\ell \in [\![0, n]\!]$, defined as follows:

$$\mathcal{C}_n := (\mathcal{E}_n \cap F_L) \cup R \text{ and } \mathcal{C}_\ell := ((\mathcal{E}_\ell \setminus R) \cup \mathcal{O}_\ell) \cap F_L \; \text{ for } \ell \in [\![0, n-1]\!],$$

where $\mathcal{E}_\ell$ and $\mathcal{O}_\ell$ were defined in Remark 1.

Procedure 1: Obtaining the critical points R in F_J (Procedure 1 in [3]).

Input: $I = (\mathbb{Z}^n, F_I)$ and $J = (\mathbb{Z}^n, F_J)$.
Output: The set R of critical points in F_J.
$V := \emptyset$; $R := \emptyset$;
for $B \in \mathcal{B}(4\mathbb{Z}^n)$ *of dimension* $k \in [\![2, n]\!]$ *and* $p \in B$ **do**
$\quad$ $p' := \operatorname{antag}_B(p)$;
$\quad$ **if** $(F_I \cap B = \{p, p'\}$ *or* $B \setminus F_I = \{p, p'\})$ **then**
$\quad\quad$ $V := V \cup \mathcal{D}_{F_J}^0 \left(\frac{p+p'}{2} \right)$
$\quad$ **end**
end
for $q \in F_J$ *such that* $\mathcal{D}_{F_J}^0(q) \cap V \neq \emptyset$ **do**
$\quad$ $R := R \cup \{q\}$
end

Finally, F_L is the set of vertices used to compute $P_S(I)$ successively applying the cone join operation $(*)$. For this construction, new sets $\mathcal{D}_{F_L}(p)$, for each $p \in F_L$, are defined.

Definition 1 ([3]). *For $p \in F_L$, define the set $\mathcal{D}_{F_L}(p) := \mathcal{D}_{F_L}^+(p) \setminus \{p\}$ where:*

- *If $p \in \mathcal{C}_0$ then $\mathcal{D}_{F_L}^+(p) := \{p\}$;*
- *If $p \in \mathcal{E}_\ell \setminus R$, for $\ell \in [\![1, n]\!]$, then $\mathcal{D}_{F_L}^+(p) := \mathcal{D}_{F_J}^+(p)$;*
- *If $p \in \mathcal{E}_\ell \cap R$, for $\ell \in [\![1, n]\!]$, then*

$$\mathcal{D}_{F_L}^+(p) := S(p) \sqcup (\mathcal{D}_{F_J}(p) \setminus R) \sqcup \bigsqcup_{r \in \mathcal{D}_{F_J}(p) \cap R} (S(r) \cap \mathcal{N}(p));$$

- *If $p \in \mathcal{O}_\ell$, for $\ell \in [\![1, n-1]\!]$, then $\exists! \, q \in R$ s.t. $p \in S(q)$. We have:*

$$\mathcal{D}_{F_L}^+(p) := (S(q) \cap \mathcal{N}^+(p)) \sqcup (\mathcal{D}_{F_J}(q) \setminus R) \sqcup \bigsqcup_{r \in \mathcal{D}_{F_J}(q) \cap R} (S(r) \cap \mathcal{N}(p));$$

Where $\mathcal{N}^+(p) := \left\{ p + \sum_{j \in 0_2(p)} \lambda_j e^j \; : \; \lambda_j \in \{0, \pm 1\} \right\}$ and $\mathcal{N}(p) := \mathcal{N}^+(p) \setminus \{p\}$.

Procedure 2: Obtaining the simplicial complex $P_S(I)$ (Procedure 6 of [3]).

Input: The set F_L.
Output: The simplicial complex $P_S(I)$.
$P_S(I) := \{\langle p \rangle : p \in \mathcal{C}_0\}$;
for $\ell \in [\![1, n]\!]$ **do**
$\quad$ **for** $p \in \mathcal{C}_\ell$ **do**
$\quad\quad$ let $K_{\mathcal{D}_{F_L}}(p)$ be the set of simplices whose vertices lie in $\mathcal{D}_{F_L}(p)$;
$\quad\quad$ $P_S(I) := P_S(I) \cup (p * K_{\mathcal{D}_{F_L}}(p))$
$\quad$ **end**
end

Proposition 5 ([3]). *The simplicial complex $P_S(I)$ is always wWC.*

3   The complement of an nD digital picture I

The *complement* of an nD picture $I = (\mathbb{Z}^n, F_I)$ is the pair $\bar{I} = (\mathbb{Z}^n, 4\mathbb{Z}^n \setminus F_I)$. Observe that $\bar{I}$ is not an nD picture (as defined in this paper) since $4\mathbb{Z}^n \setminus F_I$ is not finite.

The nD cell complex whose n-cells are the canonical size-4 n-cubes centered at each point in $4\mathbb{Z}^n \setminus F_I$ is denoted by $Q(\bar{I})$.

DWCness is *self-dual* in the sense that an nD picture $I = (\mathbb{Z}^n, F_I)$ is DWC iff its complement $\bar{I} = (\mathbb{Z}^n, 4\mathbb{Z}^n \setminus F_I)$ is DWC. The following proposition is a property of DWCness (see Theorem 1 in [4]) adapted to nD pictures.

Proposition 6 ([4]). *When $I = (\mathbb{Z}^n, F_I)$ is a DWC nD picture, then for any block $B \in \mathcal{B}(4\mathbb{Z}^n)$, and for any $p, q \in B \setminus F_I$ which are antagonists in B, there exists a $2n$-path joining p and q in $B \setminus F_I$.*

Then we obtain the following result.

Proposition 7. *If $I = (\mathbb{Z}^n, F_I)$ is DWC then, for any block $B \in \mathcal{B}(4\mathbb{Z}^n)$ and for any two points p, q in $B \setminus F_I$, there exists a $2n$-path in $B \setminus F_I$ from p to q.*

Proof. Similarly to the proof of Prop. 5 in [3], let $B \in \mathcal{B}(4\mathbb{Z}^n)$ be a block such that $B \setminus F_I$ has at least two points. For any two points $p, q \in B \setminus F_I$, there exists a block $B' \subseteq B$ such that $q = \text{antag}_{B'}(p)$. Then by Proposition 6, there exists a $2n$-path joining p and q in $B' \setminus F_I \subseteq B \setminus F_I$. $\square$

Proposition 8. *Let $J' = (\mathbb{Z}^n, F'_J)$ encoding $Q(\bar{I})$. Then:*

- $F_J \cup F'_J = 2\mathbb{Z}^n$
- $(\mathbb{Z}^n, F_J \cap F'_J)$ *encodes* $\partial Q(I)$.

Proof. First, by construction, $F_J \cup F'_J \subset 2\mathbb{Z}^n$. Let us assume that a point $p \in 2\mathbb{Z}^n$ belongs to $\mathcal{E}_\ell$. Consider $p' = p + \sum_{j \in 2_4(p)} 2e^j$. Then, either $p' \in F_I$ or $p' \in 4\mathbb{Z}^n \setminus F_I$, and hence, p encodes either, an ℓ-face of an n-cube of $Q(I)$ or an ℓ-face of an n-cube of $Q(\bar{I})$. That is, $p \in F_J \cup F'_J$.

Let $p = (x_1, \ldots, x_n) \in \mathcal{E}_{n-1}$ such that p encodes an $(n-1)$-cell of $\partial Q(I)$. Then, without loss of generality, we can assume that $x_i \equiv 0 \mod 4$ for $i \in [\![1, n-1]\!]$ and $x_n \equiv 2 \mod 4$. Consider the two points in $4\mathbb{Z}^n$ $q = p + 2e^n$ and $q' = p - 2e^n$. Necessarily, one of them lies in F_I and the other one in $4\mathbb{Z}^n \setminus F_I$, so $p \in F_J \cap F'_J$. If p encodes an ℓ-cell of $\partial Q(I)$, then, p is a face of an $(n-1)$-cell of $\partial Q(I)$ and the previous argument applies to conclude that $p \in F_J \cap F'_J$. Conversely, if $p \in F_J \cap F'_J$ (assume that $p \in \mathcal{E}_\ell$), then p encodes an ℓ-face of an $n-1$-cell in $Q(I)$ which is a face of both an n-cube in $Q(I)$ and an n-cube in $Q(\bar{I})$ (otherwise, it could not happen that $p \in F_J \cap F'_J$). Then, p encodes a cell in $\partial Q(I)$. Notice that the same argument proves that p encodes a cell in $\partial Q(\bar{I})$. $\square$

Proposition 9. *If I is DWC then $Q(\bar{I})$ is wWC.*

Proof. When I is DWC, $\bar{I}$ is DWC by definition of DWCness. Then $Q(\bar{I})$ is wWC by Proposition 4. $\square$

4 The simplicial complex $P_S(\bar{I})$

Similar to the work made in [3], we now detail a procedure to "locally repair" the cubical complex $Q(\bar{I})$ when $I = (\mathbb{Z}^n, F_I)$ is not DWC obtaining a simplicial complex $P_S(\bar{I})$ which is always wWC.

First, Procedure 1 is used to compute the set of points R' representing the critical cells in $Q(\bar{I})$, considering as input $I = (\mathbb{Z}^n, F_I)$ and $J' = (\mathbb{Z}^n, F'_J)$.
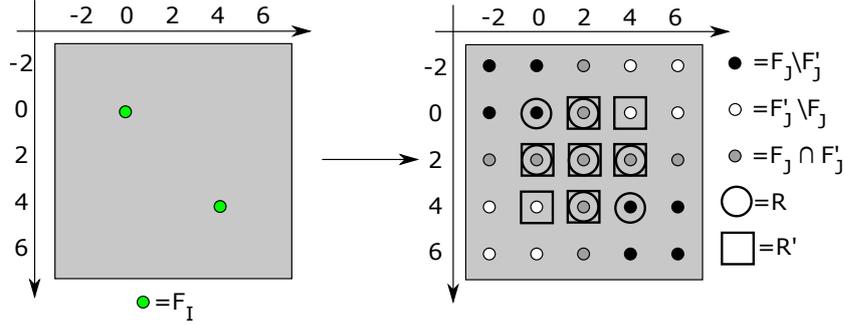
Fig. 2. The difference between R and R'.

Remark 2. In general, $R \cap R' \neq \emptyset, R, R'$ (see Figure 2).

Second, once the critical points R' are computed, the points of F'_J which do not belong to R' are preserved, the points of R' are replaced by a set of points using an operator $S'(.)$ introduced below. This part of the process is the one that differs from that in [3].

 – If $p \in R' \setminus R$ (that is, $\sigma_{Q(\bar{I})}(p)$ is not in $\partial Q(I)$) then $S'(p) := S(p)$.
 – If $p \in R' \cap R$ (that is, $\sigma_{Q(\bar{I})}(p)$ is in $\partial Q(I)$) then

$$S'(p) := \left\{ p + \sum_{j \in 2_4(p)} \lambda_j\, e^j \; : \; \lambda_j \in \{0, \pm 1\} \text{ and } p + \sum_{j \in 2_4(p)} 2\lambda_j\, e^j \in R' \setminus R \right\}.$$
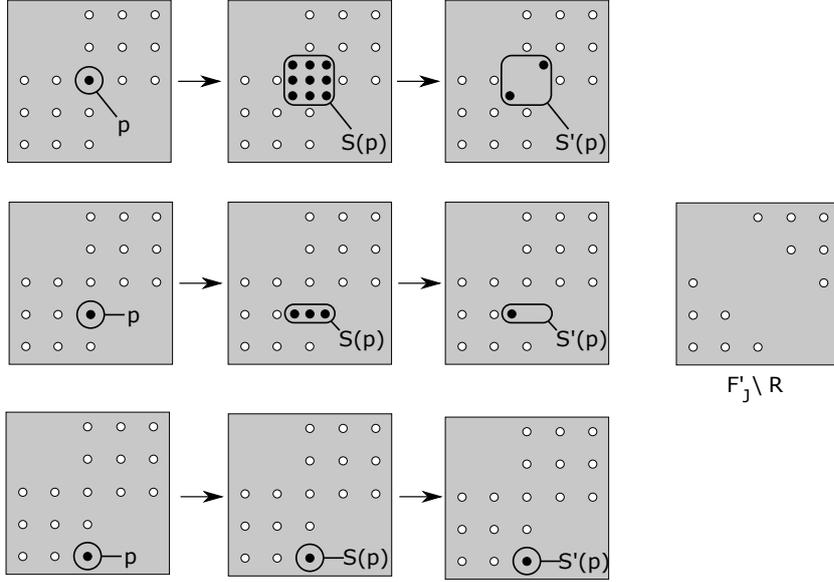
Figure 3 shows small examples of sets $S'(p)$. Then, the set F'_L is defined as:

$$F'_L := (F'_J \setminus R') \cup \bigcup_{p \in R'} S'(p)$$

Observe that not all the points in F'_J belong to F'_L, contrary to what happened with F_L.

Remark 3. As it happened with F_L in [3], the set F'_L can be decomposed into the disjoint sets:

$$\mathcal{C}'_n := (\mathcal{E}_n \cap F'_L) \cup (R' \setminus R)$$
$$\mathcal{C}'_\ell := ((\mathcal{E}_\ell \setminus R') \cup \mathcal{O}_\ell) \cap F'_L \text{ for } \ell \in [\![0, n-1]\!].$$

Fig. 3. Computation of $S'(p)$. From F'_J depicted in the upper left part of the figure by the black and white points, we deduce $S(p)$, and then $S'(p)$ which is the subset of $S(p)$ satisfying that they can be written $p + \sum_{j \in 2_4(p)} 2\lambda_j\, e^j \in F'_J \setminus R$, with $\lambda_j \in \{0, \pm1\}$.

Third, to compute $P_S(\bar{I})$, we cannot directly use Procedure 6 of [3] since F'_L has an infinite number of points. Now, the idea is to compute $P_S(\bar{I})$ locally around points in $\mathcal{C}'_n$:

- For each p in $(\mathcal{E}_n \cap F'_L) \setminus R'$, apply Proc. 2 with $\mathcal{D}^+_{F'_L}(p)^6$ as input. This process will produce a simplicial subdivision of the canonical 4-size n-cube.
- For the whole set $R' \setminus R$, apply Proc. 2 with

$$\bigcup_{p \in R' \setminus R} \mathcal{D}^+_{F'_L}(p)$$

as input (a finite set), what will produce the rest of the simplices in $P_S(\bar{I})$.

A complete procedure used to compute the simplicial complex $P_S(\bar{I})$ over an nD picture I which consists in two pixels sharing a vertex, is showed in Figure 4.

Theorem 2. $P_S(\bar{I})$ is wWC.

Proof. The proof is the same as the one provided for Theorem 44 in [3] where we proved that for any 0-cell μ in $P_S(I)$, the set of n-cells incident to μ, denoted by $\mathcal{A}^{(n)}_{P_S(I)}(\mu)$, is face-connected. The same happens for any 0-cell μ' in $P_S(\bar{I})$. See an example in Figure 5.   □

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[6] Definition of $\mathcal{D}^+_{F'_L}(p)$ can be obtained from Def. 1 replacing F_L by F'_L and R by R'.
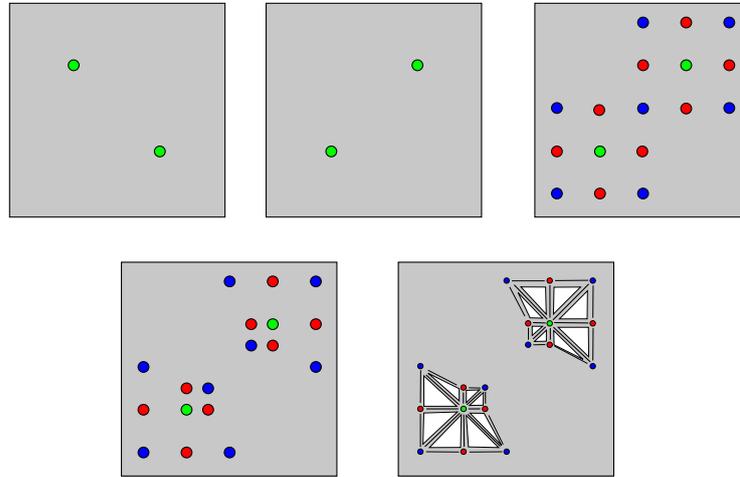
Fig. 4. Computation of $P_S(\bar{I})$: In the raster scan order, we start from I, we deduce $\bar{I}$, we compute F'_J encoding the cells of $Q(\bar{I})$, we construct the set F'_L and finally obtain the simplicial complex $P_S(\bar{I})$ built on the set of vertices F'_L.
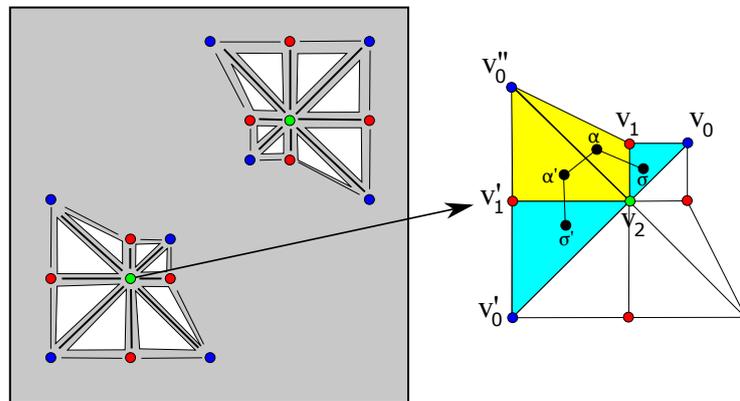
Fig. 5. How to compute the path joining two cells, $\sigma := \langle v_0, v_1, v_2 \rangle$ and $\sigma' := \langle v'_0, v'_1, v_2 \rangle$, both incident to $\langle v_2 \rangle$. Since we are in the case $i \neq i'$, we compute two new cells α and α' which "join" σ and σ': the final path joining σ and σ' is $\pi = \langle \sigma, \alpha, \alpha', \sigma' \rangle$.

Theorem 3. *The following properties hold:*

a) $|P_S(I)| \cup |P_S(\bar{I})| = \mathbb{R}^n$.
b) $P_S(I) \cap P_S(\bar{I}) = \partial P_S(I) = \partial P_S(\bar{I})$.

Proof. a) $\underline{|P_S(I)| \cup |P_S(\bar{I})| = \mathbb{R}^n}$:

For a given point $p \in \mathbb{R}^n$, consider the closest point $v_n(p) \in 4\mathbb{Z}^n = \mathcal{E}_n$ under L^1-norm. Notice that there may be more than one point if p is shared by several n-cubes; in that case, take any of them. Then p lies inside the 4-size n-cube $c_{v_n(p)}$ centered at $v_n(p)$.

- If $v_n(p) \notin R \cup R'$, then p lies inside a simplicial subdivision of the n-cube encoded by $v_n(p)$ (that is, there exist $v_0, \ldots, v_{n-1}$, with $v_k \in \mathcal{C}_k$ for all $k \in [\![0, n-1]\!]$ or $v_k \in \mathcal{C}'_k$ for all $k \in [\![0, n-1]\!]$, such that p lies inside $\sigma = \langle v_0, \ldots, v_{n-1}, v_n(p) \rangle$ for $\sigma \in P_S(I)$ or $\sigma \in P_S(\bar{I})$).

- Else, if $v_n(p) \in R$, then $|c_{v_n(p)}|$ is a subspace of $|X|$ for

$$X = \bigcup_{q \in \mathcal{C}_n \cap \mathcal{D}^+_{F_J}(v_n(p))} q * K_{\mathcal{D}_{F_L}}(q).$$

Then p lies in $|q * K_{\mathcal{D}_{F_J}}(q)|$ for some $q \in \mathcal{C}_n \cap \mathcal{D}^+_{F_J}(v_n(p))$.

- Else, $v_n(p) \in R' \setminus R$. Then $|c_{v_n(p)}|$ is a subspace of $|X'|$ for

$$X' = \bigcup_{q \in \mathcal{C}_n \cap \mathcal{D}^+_{F'_J}(v_n(p))} q * K_{\mathcal{D}_{F_L}}(q) \ \cup \bigcup_{q' \in \mathcal{C}'_n \cap \mathcal{D}^+_{F'_J}(v_n(p))} q' * K_{\mathcal{D}_{F'_L}}(q').$$

Then p lies in $|q * K_{\mathcal{D}_{F_L}}(q)|$ for some $q \in \mathcal{C}_n \cap \mathcal{D}^+_{F'_J}(v_n(p))$ or in $|q' * K_{\mathcal{D}_{F'_L}}(q')|$ for some $q' \in \mathcal{C}'_n \cap \mathcal{D}^+_{F'_J}(v_n(p))$.

Therefore, p lies in $|P_S(I)|$ or in $|P_S(\bar{I})|$.

b) $\underline{P_S(I) \cap P_S(\bar{I}) = \partial P_S(I) = \partial P_S(\bar{I})}$:

Let σ be a simplex in $\partial P_S(I)$. Let us prove that $\sigma \in P_S(\bar{I})$. By definition, σ belongs to an $(n-1)$-cell μ of $P_S(I)$ that is face of exactly one n-cell $\mu' = \langle v_0, \ldots, v_n \rangle$ of $P_S(I)$. We have: $v_n \in \mathcal{C}_n = (\mathcal{E}_n \cap F_L) \cup R$ by construction and $\mu = \langle v_0, \ldots, v_{n-1} \rangle$, with $v_{n-1} \in \mathcal{C}_{n-1}$, otherwise $\mu \notin \partial P_S(I)$.

When $v_n \in \mathcal{E}_n \cap F_L$, then v_n is the barycenter of a 4-size n-cube of $P_S(I)$ and there exists $v'_n \in \mathcal{E}_n \setminus F_L$ that is the barycenter of a 4-size n-cube of $P_S(\bar{I})$ such that $\langle v_0, \ldots, v_{n-1}, v'_n \rangle$ is an n-cell of $P_S(\bar{I})$.

When $v_n \in R$ then $v_{n-1} \in S(v_n)$. Since $\mu \in \partial P_S(I)$ then $v_{n-1} \in S'(v_n) \cap \mathcal{C}'_{n-1}$. Therefore, for $\lambda_j \in \{0, \pm 1\}$,

$$v_{n-1} = v_n + \sum_{j \in 2_4(v_n)} \lambda_j e^j \text{ and then } v'_n = v_n + \sum_{j \in 2_4(v_n)} 2\lambda_j e^j \in R' \setminus R.$$

Then μ is an $(n-1)$-face of $\langle v_0, \ldots, v_{n-1}, v'_n \rangle \in P_S(\bar{I})$, so $\mu \in P_S(\bar{I})$ and, hence $\sigma \in P_S(\bar{I})$.

The proof for $\partial P_S(\bar{I})$ is analogous.

If $\sigma \in P_S(I) \cap P_S(\bar{I})$, then all the vertices of σ lie necessarily in $\partial P_S(I) = \partial P_S(\bar{I})$.

$\square$

5   Conclusions

In this paper we provide a method to locally repair the cubical complex canonically associated to the complement of a given nD picture I obtaining a simplicial complex $P_S(\bar{I})$. We prove that $P_S(\bar{I})$ decomposes the complement space of the continuous analog of $P_S(I)$ and is weakly well-composed. As future works, we plan to study the combinatorial structure of $\partial P_S(I) = \partial P_S(\bar{I})$ and prove that it is a combinatorial $(n-1)$-manifold.

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