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How to Make n-D Plain Maps defined on Discrete Surfaces Alexandrov-Well-Composed in a Self-dual Way

Nicolas Boutry · Thierry Géraud · Laurent Najman

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Abstract In 2013, Najman and Géraud proved that by working on a *well-composed* discrete representation of a gray-level image, we can compute what is called its *tree of shapes*, a hierarchical representation of the shapes in this image. This way, we can proceed to *morphological filtering* and to *image segmentation*. However, the authors did not provide such a representation for the non-cubical case. We propose in this paper a way to compute a well-composed representation of any gray-level image defined on a *discrete surface*, which is a more general framework than the usual cubical grid. Furthermore, the proposed representation is *self-dual* in the sense that it treats bright and dark components in the image the same way. This paper can be seen as an extension to gray-level images of the works of Daragon *et al.* on discrete surfaces.

Keywords well-composedness, discrete surfaces, digital topology, Alexandrov spaces, frontier orders, cross-section topology, tree of shapes, mathematical morphology

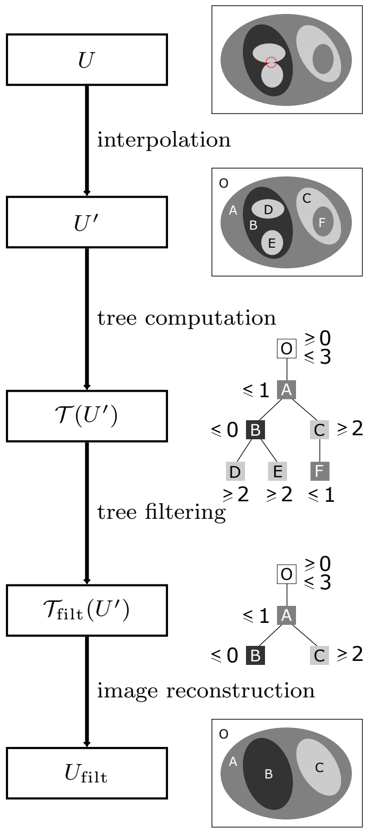

Fig. 1: How to filter an image using the *tree of shapes*: we start from a gray-level image U with possibly some pinches (encircled in red) in its boundaries, we interpolate it into a new similar image U' but without pinches, then we compute its tree of shapes, we filter it, and we construct the new image based on the filtered tree.

1 Introduction

The aim of this paper is to provide a representation that allows filtering gray-level images defined on *discrete surfaces* [25] (formally defined in Subsection 2.2), a combinatorial analog of pseudo-surfaces. Figure 1 illustrates this methodology: we start from a given gray-level image U defined on a subset of a discrete surface. Then we compute its *representation* U' (roughly speaking an image which is visually similar to the initial image and which is *well-composed*, see Subsection 3.3 for details). This representation is what we call a *plain map*, a particular kind of discrete function with some continuity properties (see Subsection 4.2). This computation is made in a *self-dual* way (see Subsection 2.9) so that we treat dark and bright components the same way (no assumption on the contrast of the image is required). After that, since the new representation U' is well-composed, we can compute its *tree of shapes* $\mathcal{T}(U')$ (formally defined in Subsection 4.3), a hierarchical representation of the shapes in the image. This enables us to make some filtering in the *shape space* [45] (we remove some particular nodes in the tree), and we obtain a new tree $\mathcal{T}_{\text{filt}}(U')$. Based on this new tree, we can construct the new image U_{filt} which corresponds to the filtered signal. By this procedure, we can do *image filtering* [45] or *image segmentation* [17] based on *mathematical morphology* [42,43].

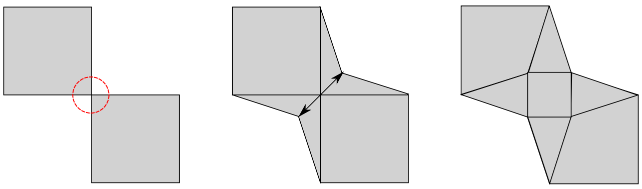

Fig. 2: How González-Díaz *et al.* dilate an initial cubical complex whose boundary contains a “pinch” (a location where a curve crosses itself) to obtain a new (polyhedral) complex whose boundary is a simple closed curve. This method preserves homotopy.

Nicolas Boutry
EPITA Research and Development Laboratory (LRDE)
14-16 rue Voltaire, FR-94276 Le Kremlin-Bicêtre, France
E-mail: nicolas.boutry@lrde.epita.fr

Thierry Géraud
EPITA Research and Development Laboratory (LRDE)
14-16 rue Voltaire, FR-94276 Le Kremlin-Bicêtre, France
E-mail: thierry.geraud@lrde.epita.fr

Laurent Najman
Université Paris-Est, LIGM, Équipe A3SI, ESIEE
E-mail: l.najman@esiee.fr

However, to be able to expose the new representation U' that we introduce in this paper, we need first to recall the methodologies used to perform “nice separations”. Separations are not ”nice” if they own ”pinches” and then they are not manifolds. The consequence is that topological issues can appear: the Jordan separation theorem can fail in the digitized world to provide one exterior and one interior, the algorithm [26] computing the tree of shapes can fail to provide a tree, and so on. Two approaches to compute nice separations are possible: either we want to separate sets of cells of dimension n, called n-*cells*, in $\mathbb{R}^n$ (*topological reparations*) with surfaces, or we want to separate sets of vertices, called 0-*cells*, in $\mathbb{R}^n$ (*point separations*), still with (discrete) surfaces. Note that these two approaches can be considered to be dual.

More exactly, we call *topological reparation* of a complex any method which transforms a given n-D cell complex with “pinches” in its boundary into a new complex which is *well-composed*, that is, whose boundary is made of simple closed $(n-1)$-surfaces according to Latecki [36]. Their aim is to restore manifoldness to the boundaries of digitized objects (boundaries of real objects are assumed to be manifolds in the continuous world). To the best of our knowledge, the only method existing today relating to topological reparation of complexes is that of González-Díaz *et al.* [14,27,28,29,30]. Starting from a cubical complex which is not well-composed (see the center of the red circle in Figure 2 on its left side), they want to obtain a new polyhedral complex which is homotopy-equivalent to the preceding one and which is well-composed. With this aim, they dilate the cubical complex at the location of the pinches (see the middle in Figure 2) to obtain the new complex (on the right side in Figure 2). However the n-D method [14] verifies only a weaker form of well-composedness, called *weak well-composedness*, and manifoldness of the boundary is not ensured. Other topological reparations exist and can be found in [13,20], but they concern only cubical or BCC grids.

Gray-level images too can be topologically repaired (see [12]): instead of repairing the boundary of a unique set, this method repairs at the same time all the *threshold sets* (the binarizations) of the image for any value λ (see Subsection 2.8 for details). This n-D topological reparation [12] has been observed to be fast in 2D in practice, but it works only with cubical grids and it does not preserve the topology of the initial image.

Extending operators from sets to gray-level images is called *cross-section topology* [7,8,9,40] and can be straightforward with simple morphological set operators (like dilation and erosion): we decompose a gray-

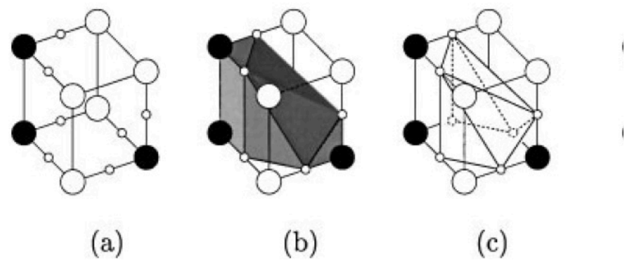

Fig. 3: How to separate white points from black points using the method of Lachaud and Montanvert [35] based on the convex hull.

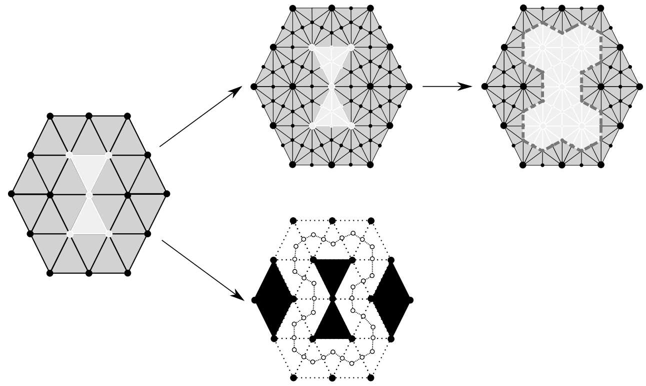

Fig. 4: Two equivalent computations of a separation. The first approach consists of a subdivision followed by a “thickening”; the dashed line is then the separation we are seeking. The second approach consists of computing the barycenters of the simplices whose at least one vertex is white and at least one vertex is black and joining these barycenters by edges.

level image into a stack of threshold sets, we treat the threshold sets separatly with the given set operator, and we build the new gray-level image from the new threshold sets. However, extending methods from sets to gray-level images can be more complex as in [12], or trickier as we are going to see in this paper.

Now, let us recall the most common approaches to computing *point separations*. To this aim, let us first recall the approach of Lachaud and Montanvert [35] which extends the well-known Marching Cubes algorithm [39] in n-D. More exactly, their approach consists of starting from a cubical grid where vertices are either black or white (see Figure 3 left) and looking locally (at each cube of the grid) how to draw the separation between these black points and the white ones. They first compute the *boundary points* (the little white spheres) at the centers of the edges simultaneously containing a white and a black vertex, which leads to Subfigure 3(a). Then, they compute the (solid) convex hull of the set made of the black and boundary points which leads to Subfigure 3(b). By keeping the boundary of this convex hull (see Subfigure 3(c)), they obtain the separation which was sought in the cube. Finally, by joining all the separations obtained for each cube in the 3D space, they obtain a 2-manifold separating black and white points. This method works in n-D and the separations thereby obtained are pseudo-manifolds.

Now, let us present the general approach of Daragon *et al.* [21,22] to proceed to point separation.

Let us imagine that we have some triangular mesh covering $\mathbb{R}^2$, where white vertices represent the *foreground* and black vertices represent the *background*. Two vertices which are connected by an edge are considered neighbors. The goal is to nicely separate the foreground from the background. Following the works of Daragon *et al.* [21,22], there are two methods to achieve this (see Figure 4):

– First method: We subdivide the triangles of the initial structure (on the left) into smaller triangles while preserving their color; we obtain the figure in the top middle of Figure 4. Then, we color all the little triangles which contain at least one white vertex in white; we obtain the figure at the top right of Figure 4. The resulting white part of the subdivided mesh is called the *derived neighborhood*, and its boundary (represented by a dashed gray curve) separates the two triangles (in the initial structure) from the rest of the mesh.

– Second method: Since we know which vertices belong to the foreground and which ones belong to the background, we are able to determine which triangles or edges consist only of white vertices, which ones consist only of black vertices, and the others. These last simplices constitute the so-called *frontier order* (see Subsection 2.5 for the formal definition), depicted at the bottom of Figure 4 by a simple closed curve. This simple closed curve represents the separation between the two initially white triangles and the rest of the mesh.

As stated by Daragon *et al.* [21,22], these two approaches are always equivalent; here we obtain the same simple closed curve in $\mathbb{R}^2$ satisfying the Jordan separation theorem ($\mathbb{R}^2$ is separated into two components, the interior which is bounded and the exterior which is unbounded). Details about the n-D case can be found in [1,31,34,38].

Now that we have seen how to compute separations, we will expose how to combine these methodologies to work with gray-level images, that is, how to modify the initial data so that the boundaries of the binarizations of the new image are nice separations. This way, we will be able to compute a tree of shapes for filtering and segmentation purposes.

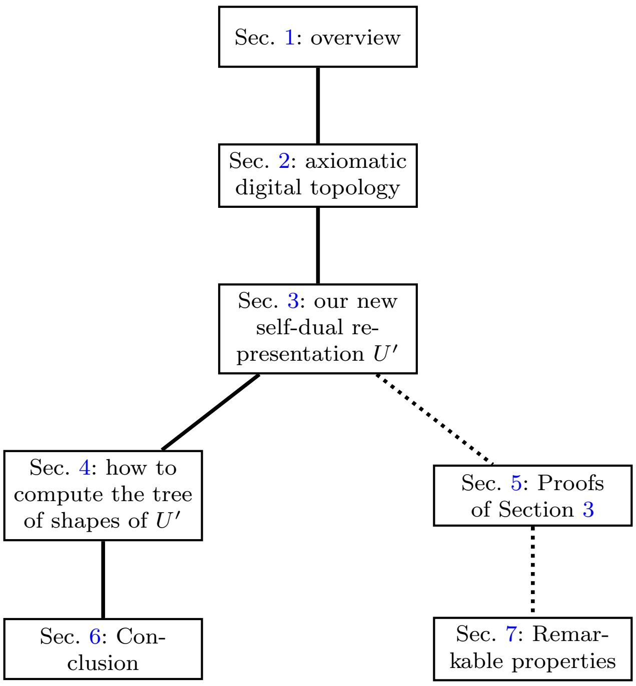

Fig. 5: The scheme of the whole paper. Note that Section 5 and Section 7 are optional for the intuitive understanding of the paper.

This paper is organized as follows: Section 2 recalls basics in the matter of *axiomatic digital topology*, Section 3 shows how we compute the new representation we propose in this paper and details its properties, Section 4 recalls how to compute the *tree of shapes*, Section 5 contains the proofs of properties stated in Section 3, Section 6 concludes the paper, and Section 7 contains some properties related to this paper and that we consider remarkable. In Figure 5, we depict the plan of the whole paper.

2 Axiomatic digital topology

In this section, we recall some theory about *axiomatic digital topology* [3] and *discrete topology* [2,6,10,23,41].

2.1 Posets

Let Y be a set of arbitrary elements. An *order relation* R on Y is a binary relation on Y which is reflexive, asymmetrical, and transitive. The *inverse* of an order relation R on Y is the binary relation R' on Y verifying $yR'x$ iff xRy for $x, y \in Y$. Given an order relation R on Y, we will note $R^{\Box}$ the binary relation defined such that, $\forall x, y \in Y$: $\left\{(x, y) \in R^{\Box}\right\}$ is equivalent to $\left\{(x, y) \in R \text{ and } x \neq y\right\}$. A set Y of arbitrary elements supplied with an order relation R on Y is denoted (Y, R) or $|Y|$ and is said to be a *partially ordered set* or simply a *poset*.

Let $|Y|$ be a partially ordered set. We will usually denote by α_Y the order relation associated to its domain Y, in such a way that $|Y| = (Y, \alpha_Y)$. Also, we will write β_Y the inverse of α_Y, and $\theta_Y = \alpha_Y \cup \beta_Y$.

When y is an element of Y, we define then the *combinatorial closure* of y in Y:

$$\alpha_Y(y) := \{p \in Y \;;\; pRy\},$$

the *combinatorial opening* of y in Y:

$$\beta_Y(y) := \{p \in Y \;;\; yRp\},$$

and the *combinatorial neighborhood* of y in Y:

$$\theta_Y(y) = \alpha_Y(y) \cup \beta_Y(y).$$

By extension, we will define for any X subset of Y:

$$\begin{cases} \alpha_Y(X) := \bigcup_{x \in X} \alpha_Y(x), \\ \beta_Y(X) := \bigcup_{x \in X} \beta_Y(x), \\ \theta_Y(X) := \bigcup_{x \in X} \theta_Y(x). \end{cases}$$

Let $|Y| = (Y, \alpha_Y)$ be a poset, and let X be a subset of Y. The *suborder* of $|Y|$ relative to X is the poset (X, α_X) with $\alpha_X := \alpha_Y \cap (X \times X)$. Then for any $x \in X$, $\alpha_X(x) = \alpha_Y(x) \cap X$, $\beta_X(x) = \beta_Y(x) \cap X$, and $\theta_X(x) = \theta_Y(x) \cap X$.

Let $(Y, \alpha_Y) = |Y|$ be a poset. Then, $|Y|$ is said to be *locally finite* if for any element $y \in Y$, the set $\theta_Y(y)$ is finite. Assuming that $|Y|$ is locally finite, the *rank* $\rho_Y(y)$ of an element y in $|Y|$ is 0 when $\alpha_Y^{\Box}(y) = \emptyset$; otherwise, it is equal to: $\max_{p \in \alpha_Y^{\Box}(y)}(\rho_Y(p)) + 1$. The *rank* of a poset $|Y|$ is denoted by $\rho(|Y|)$ and is equal to the maximum rank of its elements: $\rho(|Y|) := \max_{y \in Y}(\rho_Y(y))$. An element of Y such that $\rho_Y(y) = k$ is called *k-face* of Y; the set of k-faces of a poset $|Y|$ is denoted by Y_k. A face h of a poset $|Y|$ is said to be *maximal* when $\beta_Y(h) = \{h\}$.

Let $|Y|$ be a poset. A *path* [6] from $x \in Y$ to $y \in Y$ is a sequence $(p^0 = x, p^1, \ldots, p^{k-1}, p^k = y)$ of elements of Y such that for any $i \in [\![0, k-1]\!]$, $p^i \in \theta_Y^{\square}(p^{i+1})$. A poset $|Y|$ is said to be *path-connected* [6] if for any pair (x, y) of elements of Y, there exists a path from x to y. A *connected component* C of $|Y|$ is a subset of Y which is path-connected, and such that C is maximum for this property (in the inclusion sense).

2.2 Discrete n-surfaces

Let $|X| = (X, \alpha_X)$ be a partially ordered set. $|X|$ is said to be *countable* if its domain X is countable. A partially ordered set which is countable and locally finite is said to be a *CF-order*. Now, according to Evako *et al.* [25], let $|X| = (X, \alpha_X)$ be a CF-order, then $|X|$ is said to be a (-1)-surface if $X = \emptyset$, a 0-surface if X is made of two elements $x, y \in X$ s.t. $x \notin \theta_X(y)$, or an n-surface, $n \geq 1$, if $|X|$ is path-connected and for any $x \in X$, $|\theta_X^{\square}(x)|$ is an $(n-1)$-surface. Note that a well-known example of an n-surface is the Khalimsky Grid [25] of dimension n when $n \geq 1$.

2.3 Simplicial complexes

Let Λ be a set of arbitrary elements. Any set of $(n+1)$ elements of Λ, with $n \geq 0$, is called a *n-simplex*. Let C be a collection of simplices that is *closed under inclusion*[1] and such that the intersection of two simplices of C is either the empty set or a simplex of C; in this case, C is called a *simplicial complex*. The elements of this collection are then called *faces* of the simplicial complex C. The *support* of a simplicial complex C is denoted by Λ_C and is equal to $\bigcup_{s \in C} s$. Now, let C be a simplicial complex, and let K be a subset of C that is a simplicial complex, then K is said to be a *subcomplex* of C. Also, if every face of C included in Λ_K is also a face of K, then we say that K is a *full subcomplex* of C. A *vertex* of a simplicial complex C of support Λ_C is an element of C which can be written $\{p\}$ where $p \in \Lambda_C$. Note that simplicial complexes are naturally supplied with the order relation $\subseteq$ and in that sense, they are posets.

2.4 Subdivisions

Let $|X| = (X, \alpha_X)$ be a poset. A suborder $|S|$ of $|X|$ is said to be *totally ordered* when for any pair $(x, y) \in S \times S$, we have $x \in \alpha_S(y)$ or $y \in \alpha_S(x)$. Any subset c of X such that $|c| = (c, \alpha_c)$ is totally ordered is called a *chain* of $|X|$. Then, the set of all the chains of $|X|$ is denoted by X^1 and is called the *first subdivision* of $|X|$.

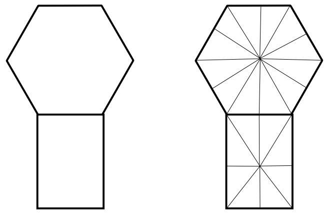

Fig. 6: From a cell complex (on the left side) representing a poset (the 2D cells and their faces are the elements of the poset and the incidence relation is the order relation between these cells) to its chains represented by triangles and their faces (on the right side).

The reason for which the set of the chains of a poset is called subdivision can be seen in Figure 6 where we see that chains can be interpreted geometrically as the simplices of the subdivided complex.

Note that X^1 is a simplicial complex, and that its support is equal to X. A particular property of the first subdivision is that when $|X|$ is a discrete n-surface, then X^1 is still a discrete n-surface according to Daragon [21].

Furthermore, we will denote by X^k, $k \geq 2$, the k^{th} *subdivision* of the poset X defined inductively as

$$X^k := (X^{k-1})^1.$$

2.5 Frontier orders

Let C be a simplicial complex. Let us decompose now Λ_C into the union of K the *foreground* and K' the *background*: $\Lambda_C = K \sqcup K'$ (where $\sqcup$ is the disjoint union operator). Then C can be decomposed into 3 disjoint parts: C_K, which is the set of the simplices contained in K; $C_{K'}$, which is the set of simplices contained in K'; and $C_{K/K'}$, which is the set of simplices contained neither in K nor in K'. Then $|C_{K/K'}| = (C_{K/K'}, \subseteq)$ is called the *frontier order* of K in Λ_C relative to C. Note that a frontier order is not a simplicial complex: since any vertex belongs either to K or to K', it does not belong to $C_{K/K'}$.

For example, let $C := \{\{a\}, \{b\}, \{a, b\}\}$ be a simplicial complex. The support Λ_C of C is equal to $\{a, b\}$. Let us assume that we have $K := \{a\}$ and $K' := \{b\}$ whose disjoint union is equal to Λ_C. The frontier order of K is then $\{\{a, b\}\}$, and its element $\{a, b\}$ is contained neither in K nor in K'.

[1] A set C of simplices is said to be *closed under inclusion* if for any element $h \in C$ and any element $h' \subseteq h$, then h' belongs to C.

Theorem 1 (Theorem 37 [21](p. 89)). *Let C be a simplicial complex which is a discrete n-surface, $n \geq 2$. Let K be a strict non-empty subset of Λ_C. The order $|C_{K/K'}|$ is a union of disjoint discrete $(n-1)$-surfaces.*

2.6 AWC Posets

Let us recall the definition of *combinatorial boundary* of Najman and Géraud [41]. Let X and Y be two posets such that X is a suborder of Y, then the *combinatorial boundary* or simply *boundary* $\mathrm{bd}(X,Y)$ of X in Y is defined as:

$$\mathrm{bd}(X,Y) := \alpha_Y(X) \cap \alpha_Y(Y \setminus X).$$

We can recall the definition of *well-composedness in the sense of Alexandrov*, initially introduced in [41], and later used in [11,13,15,16]: let X, Y be two non-empty posets such that $X \subseteq Y$, Y is an n-surface, $n \geq 1$, and X is equal to the combinatorial closure of the set of its n-faces. Then X is said to be *well-composed in the sense of Alexandrov (AWC)* if its boundary $\mathrm{bd}(X,Y)$ is a disjoint union of discrete $(n-1)$-surfaces or empty.

2.7 Dual cells

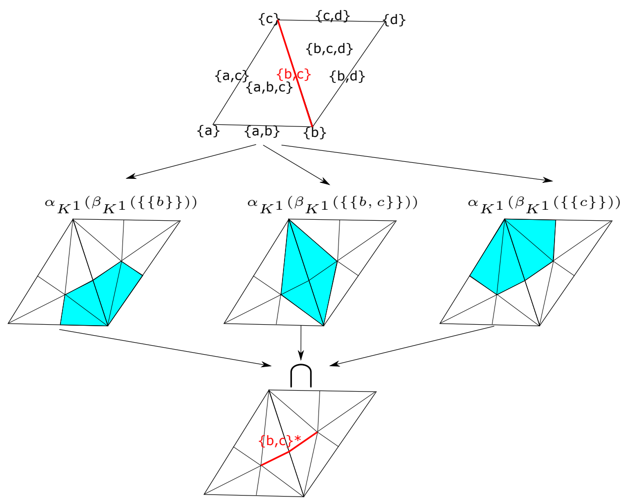

Fig. 7: How to compute dual cells. On the top, we show a simplicial complex K equal to the closure of two triangles whose common edge $\{b,c\}$ is depicted in red. In the middle, in blue we depict the closures of the opening of $\{\{b\}\}$, of the opening of $\{\{b,c\}\}$, and of the opening of $\{\{c\}\}$ in the subdivision K^1 of K. At the bottom, we compute the intersection of these last three sets, and we obtain the dual cell $\{b,c\}^*$.

Let us recall the combinatorial version of the definition of *dual cells* [3,10,32]. Let K be a poset, and K^1 its first subdivision. For any element $A \in K$, we define A^*, the *dual cell* of A relative to K, as:

$$A^{*,K} = \bigcap_{h \in \alpha_K(A)} \alpha_{K^1}(\beta_{K^1}(\{h\})).$$

Figure 7 shows how to compute the dual of an edge in a simplicial complex.

Finally, for any suborder L of a poset K,

$$L^{*,K} := \{A^{*,K} \ ; \ A \in L\}$$

is called the *dual* of L relative to K.

2.8 Span-valued maps

By $\mathbb{I}_\mathbb{R}$ let us denote the set of *closed intervals* of $\mathbb{R}$, that is, any set which can be written $[a,b]$ with $a, b \in \mathbb{R}$ and $a \leq b$. For a finite non-empty set S of values in $\mathbb{R}$, the *span operator* applied to S is denoted by $\mathrm{Span}(S) : S \to \mathbb{I}_\mathbb{R}$ and is equal to $[\min(S), \max(S)]$. We consider that the operators min, max, and thus Span, are recursive in the sense that $\min\{a, \{b,c\}\} = \min\{a,b,c\}$ and so on.

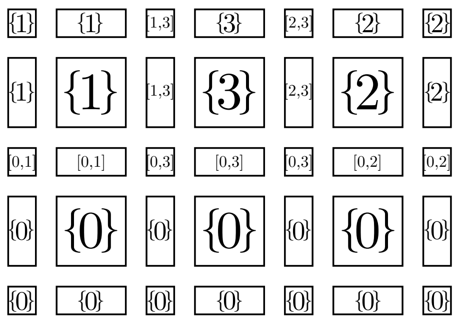

Fig. 8: An example of span-valued map on a 2D Khalimsky grid. Observe that this map is "flat" on the squares and that the values on the vertices and edges depend only on the values of U on the squares which contain them.

Let $|X|$ be a poset of rank $n \geq 0$. We refer to a *span-valued map* or *gray-level image* (defined on $|X|$) as any map $U : X \to \mathbb{I}_\mathbb{R}$ which verifies for each $h \in X_n$:

$$U(h) = \{v\} \text{ with } v \in \mathbb{R},$$

and for each $h \in X \setminus X_n$:

$$U(h) = \mathrm{Span}\{U(q) \ ; \ q \in \beta_X(h) \cap X_n\}.$$

An example of span-valued map is depicted in Figure 8.

Let $|X|$ be some poset; and let $U : X \to \mathbb{I}_{\mathbb{R}}$ be some span-valued map. For any $\lambda \in \mathbb{Z}$, we define the following sets as *threshold sets* [41] of U:

$$[U \rhd \lambda] = \{x \in X \; ; \; U(x) \rhd \lambda\},$$
$$[U \lhd \lambda] = \{x \in X \; ; \; U(x) \lhd \lambda\},$$

where $U(x) \rhd \lambda$ means that for any $y \in U(x)$, $y > \lambda$, and $U(x) \lhd \lambda$ means that for any $y \in U(x)$, $y < \lambda$.

The first set $[U \rhd \lambda]$ is called *strict upper* threshold set of threshold λ; the second set $[U \lhd \lambda]$ is called *strict lower* threshold set of threshold λ.

Let Y be a poset of rank $n \geq 2$. A span-valued map $U : Y \to \mathbb{I}_{\mathbb{R}}$ is said to be *well-composed in the sense of Alexandrov (AWC)* [41] if the boundary of each of its threshold sets is either made of a disjoint union of discrete $(n-1)$-surfaces or empty.

2.9 Self-duality

In mathematical morphology, we do not always know in advance if we want to treat bright components over dark components or dark components over bright components. Assuming that we work with some function $u : \mathcal{D} \to \mathbb{R}$, it can be then useful to work with *self-dual operators*: an operator Φ which maps u to a map $\Phi(u) : \mathcal{D} \to \mathbb{R} : x \to (\Phi(u))(x)$ is said to be *self-dual* if for any $x \in \mathcal{D}$,

$$-(\Phi(u))(x) = (\Phi(-u))(x).$$

For short, we will say that $-\Phi(u) = \Phi(-u)$.

An intuition of self-duality is given in Figure 9.

Now let us extend these tools from scalars to intervals. If a, b belong to $\mathbb{R}$ with $a < b$, then we call $[-b, -a]$ the *opposite* of $[a, b] \in \mathbb{I}_{\mathbb{R}}$. The same manner, assuming that a span-valued map $U : \mathcal{D} \to \mathbb{I}_{\mathbb{R}}$ is given, we define its *opposite* $-U : \mathcal{D} \to \mathbb{I}_{\mathbb{R}}$ to be such that for any $x \in \mathcal{D}$, $-U(x) = [-b, -a]$ when $U(x) = [a, b]$ with $a, b \in \mathbb{R}$ and $a \leq b$.

Let Φ be an operator from the space of span-valued maps to itself. We define $-\Phi$ to be the *opposite* of the operator Φ if for any span-valued map $U : \mathcal{D} \to \mathbb{I}_{\mathbb{R}}$ whose image by Φ is $\Phi(U) : \mathcal{D}' \to \mathbb{I}_{\mathbb{R}}$, we have for any $x \in \mathcal{D}'$,

$$(-\Phi(U))(x) := -(\Phi(U))(x).$$

Take care of the fact that $\mathcal{D}'$ and $\mathcal{D}$ can be different due to the modifications caused by the operator Φ.

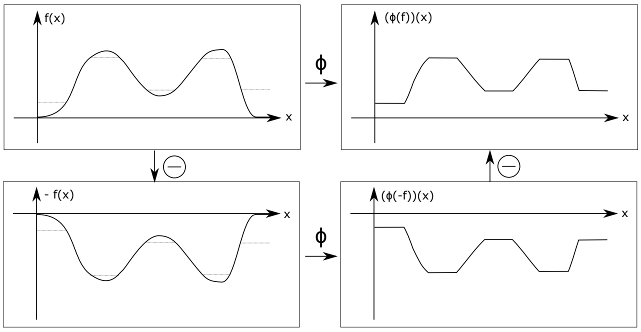

Fig. 9: Self-duality of a *grain filter* [18] Φ on a real function $f : \mathcal{D} \to \mathbb{R}$: bright and dark components which have an area lower than a given threshold α are removed from f (see the dotted lines). We obtain the same result if we compute the opposite of the filtering or the filtering of the opposite. In other words, bright components and dark components are treated the same way with self-dual filters.

We say that an operator Φ from the space of span-valued maps to itself is *self-dual* when for any span-valued map $U : \mathcal{D} \to \mathbb{I}_{\mathbb{R}}$, we obtain that $-\Phi(U) : \mathcal{D}' \to \mathbb{I}_{\mathbb{R}}$ is equal to $\Phi(-U) : \mathcal{D}' \to \mathbb{I}_{\mathbb{R}}$; that is, for any $U : \mathcal{D} \to \mathbb{I}_{\mathbb{R}}$ and for any $x \in \mathcal{D}'$,

$$(-\Phi(U))(x) = (\Phi(-U))(x).$$

3 A new self-dual representation

In this section, we present our definition of *cell-complexes*, and then we explain how we transform a poset into a cell-complex which is AWC. Then, we explain our method to make a gray-level image AWC.

3.1 Cell-complexes

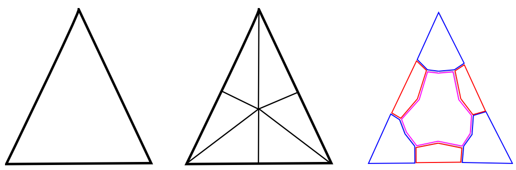

Fig. 10: Computation of a cell complex: from left to right, a simplicial complex K, its subdivision K^1, and its associated cell complex $\mathfrak{CC}(K)$.

Let us introduce our (informal) definition of *cell complexes*:

Definition 1. *Let $|K|$ be a given poset. Then, we call* cell complex associated to $|K|$ *the poset* $\mathfrak{CC}(K)$ *supplied with the relation* $\subseteq$ *made of the cells dual to the subdivisions of* K*. Furthermore, assuming that* $|X|$ *is a suborder of the poset* $|Y|$*, we call* cell complex associated to X relative to Y *the poset* $\mathfrak{CC}(X|Y)$ *supplied with* $\subseteq$ *made of the cells dual to the subdivisions of* X *relative to* Y*.*

The formal definition of our transform is given in Subsection 5.1 (see Definition 9).

3.2 On making posets AWC

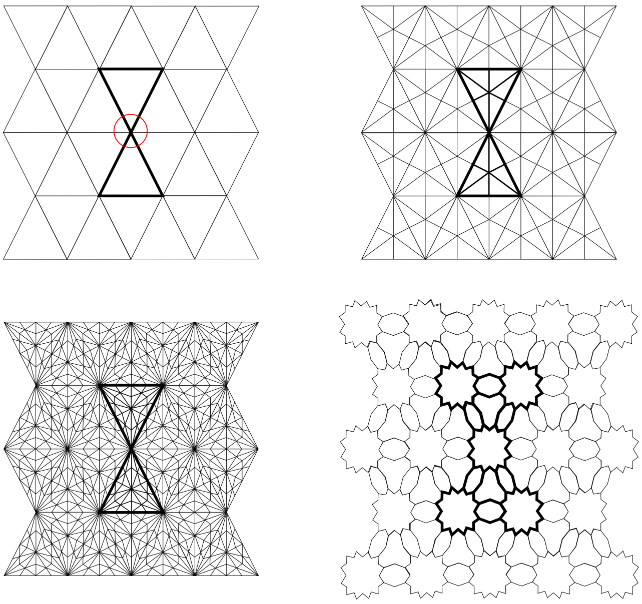

Fig. 11: Building the transform $\mathfrak{T}(X)$ of a suborder $|X|$ of a 2-surface $|Y|$: we start in the top left subfigure from a simplicial complex X (depicted by the thick segments in black) equal to the closure of a 2-simplex in a discrete 2-surface $|Y|$. Then, in the top right subfigure, we depict the subdivision X^1 of X and the subdivision Y^1 of Y. In the bottom left subfigure, we depict the second subdivision X^2 of X and the second subdivision Y^2 of Y which are both used to compute the dual cells of X^1 and Y^1. The bottom right subfigure shows the cell complex $\mathfrak{T}(X)$ (see the cells with thick borders) in $\mathfrak{CC}(Y)$.

Starting from a given poset $|X|$ which is a non-empty finite subset of a discrete n-surface $|Y|$, $n \geq 2$, we would like to obtain some new set which is, as much as possible, topologically equivalent to X but with no topological issues (no pinches in the boundary). For this reason we propose the following (informal) definition:

Definition 2. *Let $|X|$ be a suborder of the poset $|Y|$. Then the* transform *of X (relative to Y) is the cell complex resulting from the closure in $\mathfrak{CC}(Y)$ of $\mathfrak{CC}(X|Y)$. In other words, $\mathfrak{T}(X)$ is made of the cells of $\mathfrak{CC}(X|Y)$ plus its faces in $\mathfrak{CC}(Y)$.*

The formal definition of our transform is given in Subsection 5.1 (see Definition 11).

An example of such a transform is depicted in Figure 11.

We obtain the following properties, whose proofs are postponed to Section 5, page 22:

Theorem 2. *Let $|Y|$ be an n-surface, $n \geq 2$, and $|X|$ be a suborder of $|Y|$ which is equal to the combinatorial closure of its n-faces. Then $|\mathfrak{T}(X)|$ is well-composed in the sense of Alexandrov.*

This theorem claims that when the topological space $|Y|$ is a discrete surface, then the boundary of the transform $|\mathfrak{T}(X)|$ of any suborder $|X|$ of $|Y|$ is made of disjoint discrete surfaces. This property can be observed in Figure 11 where the initial poset $|X|$ has topological singularities at the middle (see the "pinch" encircled in red), and the boundary of its transform $|\mathfrak{T}(X)|$ is a simple closed curve (no "pinch"). In other words, *regularity*[2] is preserved from $|Y|$ to $\mathfrak{T}(X)$.

From now on, for sake of simplicity, we will simply write $\mathfrak{T}(X)$ instead of $|\mathfrak{T}(X)|$, that is, we will always assume that the transform of X is provided with its order relation $\subseteq$. The same applies for $\mathfrak{CC}(X|Y)$ and $\mathfrak{CC}(Y)$.

Also, we can prove that this transform preserves path-connectivity:

Proposition 1. *Let $|Y|$ be an n-surface, $n \geq 2$, and $|X|$ be a suborder of $|Y|$ which is equal to the combinatorial closure of its n-faces. Then, the transform $\mathfrak{T}(X)$ of X is path-connected iff X is path-connected. In other words, the mapping $X \to \mathfrak{T}(X)$* preserves path-connectivity*.*

The proof of this proposition has been postponed to Section 5, page 23, and preservation of path-connectivity can be observed in Figure 11.

3.3 Towards our self-dual representation

Now, let X, Y be two posets such that Y is an n-surface, $n \geq 2$, and X is a finite non-empty strict suborder of

[2] We say that a poset is *regular* (in the discrete sense) when it is a discrete surface or when its boundary is made of disjoint discrete surfaces.

Y that is the combinatorial closure of its n-faces:

$$X = \bigcup_{h \in X_n} \alpha_Y(h),$$

and let $U : X \to \mathbb{I}_\mathbb{R}$ be some given span-valued map. This map is assumed not to be AWC. Our aim is then to compute a new function U' obtained by a transform $\mathcal{I} : U \to U'$ such that:

1. $\mathcal{I}$ is *self-dual*: $\mathcal{I}(-U) = -\mathcal{I}(U)$; that is, it treats the same way dark and bright components. This property is useful when we do not know *a priori* which ones of the bright and dark components are the most important in the given signal;
2. U' is a *span-valued map*: the values on the n-faces must be degenerate sets[3] and the values of the k-faces must be equal to the span of the values on the n-faces which contain them;
3. U' is AWC: we want our representation to own "regularity" properties in the discrete sense so that we are able to compute its *tree of shapes*;
4. U' is *in-between*: this property means that the threshold sets of U' and U have the same boundaries where there is no pinch.

The map U' verifying this set of properties is then called *self-dual representation* of U.

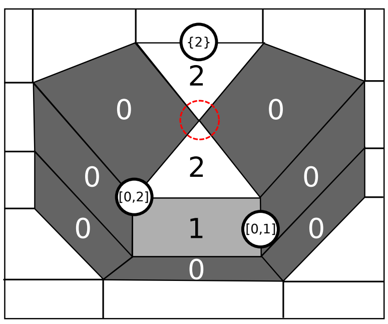

Fig. 12: The initial image $U : X \to \mathbb{I}_\mathbb{R}$, containing a pinch encircled in red: at this vertex, the boundary of $[U \rhd 1]$ crosses itself. Circles indicate the values of the span-valued map U on some of the k-faces of X ($k < n$).

Starting from a gray-level image U like in Figure 12 containing a pinch at its center, let us show step by step how we compute its self-dual representation.

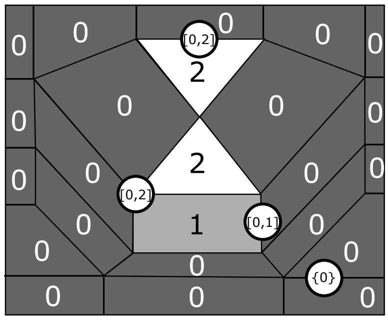

Fig. 13: We build an "extension" $U_{\text{USC}} : Y \to \mathbb{I}_\mathbb{R}$ of U to the whole discrete n-surface Y with the same value $\mathfrak{M} \in \mathbb{R}$ all over $Y \setminus X$ chosen in a self-dual way.

3.3.1 Step 1: from U to U_{USC}

Since by hypothesis X is a non-empty strict subset of Y, the boundary $\text{bd}(X, Y)$ of X in Y is non-empty. Then we can apply the *symmetric median operator*[4] to the restriction of U to the $(n-1)$-faces of the boundary to obtain:

$$\mathfrak{M} := \text{med}\{U(h)\ ;\ h \in (\text{bd}(X, Y))_{n-1}\},$$

where $(\text{bd}(X, Y))_{n-1}$ corresponds to the set of $(n-1)$-faces of $\text{bd}(X, Y)$.

This way, the operator med is applied on a set of degenerate sets. Indeed, when h is a $(n-1)$-face of $\text{bd}(X, Y)$, then $\beta^\Box_Y(h)$ is made of two n-faces (because $|Y|$ is an n-surface). Also, because $h \in \text{bd}(X, Y)$, $\beta^\Box_Y(h) = \{x, y\}$ with $x \in X_n$ and $y \in Y_n \setminus X_n$. Then, $\beta^\Box_X(h) = \{x\}$. This leads to $U(h) = \text{Span}\{U(x)\} = U(x)$ since $U(x)$ is degenerate. This means that $\mathfrak{M}$ is the median of a set of degenerate sets.

Note that $\mathfrak{M}$ is chosen in a self-dual way, since the symmetric median is a self-dual operator; this ensures self-duality of the complete procedure. In Figure 12,

$$\mathfrak{M} = \text{med}\,\{0, 0, 0, \underline{0}, \underline{0}, 0, 0, 2\} = 0.$$

Then, we define a new map $U_{\text{USC}} : Y \to \mathbb{I}_\mathbb{R}$, as depicted in Figure 13, as follows.

Definition 3. *Assuming that a span-valued map $U : X \to \mathbb{I}_\mathbb{R}$ is given on a suborder $|X|$ of a discrete n-surface $|Y|$ equal to the combinatorial closure of its n-faces, we define the map $U_{\text{USC}} : Y \to \mathbb{I}_\mathbb{R}$ such that*

[3] A set is said to be *degenerate* if it is a singleton.

[4] We recall that the *symmetric median operator* med applied to a list of an even number of elements of $\mathbb{R}$ returns the average of the two middle values of the sorted list; otherwise, when the list contains an odd number of elements of $\mathbb{R}$, it returns the middle value of the sorted list.

$\forall p \in Y$:

$$U_{\mathrm{USC}}(p) := \begin{cases} U(p) & \textit{when } p \in X \setminus \mathrm{bd}(X,Y), \\ \{\mathfrak{M}\} & \textit{when } p \in Y \setminus X, \\ \mathrm{Span}\{\mathfrak{M}, U(p)\} & \textit{when } p \in \mathrm{bd}(X,Y). \end{cases}$$

Note that this map satisfies some continuity property called *upper semi-continuity* and developed in Section 4.2; Property 8 detailing this fact is proven in Section 7 on page 25.

3.3.2 Step 2: from U_{USC} to $u^\flat$

Since we are going to introduce new "pixels between the pixels" (since we work with interpolations) in the final representation U', we have to flatten the map U_{USC} beforehand; otherwise, U' will not be flat on its n-faces. This step can be seen as a "smoothing" of U_{USC} into a single-valued map.

So let us choose some self-dual operator Op (like the symmetric median or the mean operators) which verifies:

$$\forall A \subset \mathbb{R}, \mathrm{Op}(A) \in \mathrm{Span}(A), \quad (P_{\mathrm{Op}})$$

where A is assumed to be finite.

Then, assuming that the map $U_{\mathrm{USC}} : Y \to \mathbb{I}_{\mathbb{R}}$ is known and computed as explained before, we can define the new function $u^\flat : Y \to \mathbb{R}$ in the following manner.

Definition 4. *Let us assume the hypotheses of Definition 3. For any $p \in Y_n$, $u^\flat(p) := v$ where v is the real value verifying $U_{\mathrm{USC}} = \{v\}$, and for any $k \in [\![1,n]\!]$ and any $p \in Y_{n-k}$,*

$$u^\flat(p) := \mathrm{Op}\{u^\flat(q);\ q \in (\beta_Y(p))_{n-k+1}\}.$$

In other words, $u^\flat$ can be computed using Algorithm 1.

Using this algorithm and choosing the symmetric median operator as operator Op, we obtain, then, a single-valued map $u^\flat$ computed in a self-dual way like depicted in Figure 14: here, we "flatten" U_{USC} into $u^\flat : Y \to \mathbb{R}$ using the symmetric median operator in Algorithm 1. The aim of this function is to value all the pixels by real values (to obtain later that the n-faces of the representation U' are valued by degenerate sets). Note however that the median operator (or the mean operator) of a set of integers is generally not an integer, which can be complicated to handle in practice.

Now, let us now define what we mean when we say that some single-valued map *preserves the boundaries*:

Algorithm 1: Computation of $u^\flat$.

ComputeUFlat(Y, U_{USC}, n) : $u^\flat$;
begin
  for $p \in Y_n$ **do**
    $u^\flat(p) \leftarrow v$ where $U_{\mathrm{USC}}(p) = \{v\}$
  for $k \leftarrow [\![1,n]\!]$ **do**
    for $p \in Y_{n-k}$ **do**
      $u^\flat(p) \leftarrow \mathrm{Op}\{u^\flat(q);\ q \in (\beta_Y(p))_{n-k+1}\}$
  Return $u^\flat$

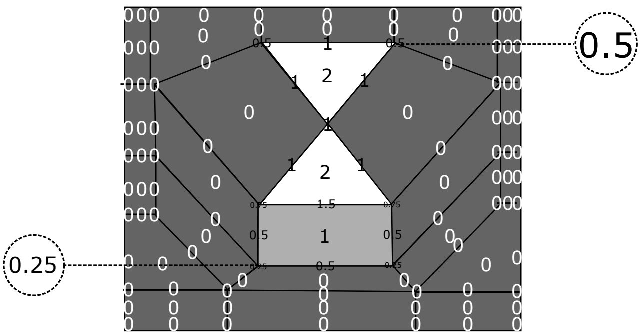

Fig. 14: Flattening of U_{USC} into $u^\flat : Y \to \mathbb{R}$ using the symmetric median operator.

Definition 5. *Let $\mathcal{D}$ be a discrete n-surface and let $F : \mathcal{D} \to \mathbb{I}_{\mathbb{R}}$ be a span-valued map. Then we say that $g : \mathcal{D} \to \mathbb{R}$ is a* single-valued in-between interpolation *of F if for any n-face p of $\mathcal{D}$, $g(p) \in F(p)$, and $\forall k \in [\![0, n-1]\!], \forall p \in \mathcal{D}_k$,*

$$g(p) \in \mathrm{Span}\left\{g(q)\ ;\ q \in (\beta_{\mathcal{D}}(p))_{k+1}\right\}.$$

In this sense, we say that g preserves the boundaries *of F.*

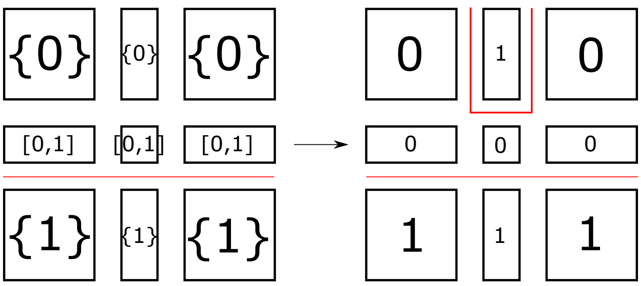

Fig. 15: An example of interpolation which is not in-between: violating the rule provided in Definition 5 can insert new components in the boundary of the threshold sets of the given image.

Figure 15 shows an example of interpolation which is not in-between: we start from a gray-level image $U_{\text{USC}} : \mathcal{D} \to \mathbb{I}_{\mathbb{R}}$ (on the left side) which is upper semi-continuous and we "flatten" it with some procedure preserving the values at the n-faces but which do not ensure in-betweeness. This way, we obtain a single-valued map $u_2 : \mathcal{D} \to \mathbb{R}$ (depicted on the right side). We can observe that we introduced some new components in the boundary of the gray-level image: $[U_{\text{USC}} \rhd \frac{1}{2}]$ has one component when $[u_2 > \frac{1}{2}]$[5] has two components. For this reason, we consider that verifying the relation provided in Definition 5 is a necessary condition to "preserve" the boundaries of the given gray-level image.

Then the following property is immediate:

Property 1. *For any operator satisfying (P_{Op}), the single-valued map $u^\flat : Y \to \mathbb{R}$ computed as described above is a single-valued in-between interpolation of U.*

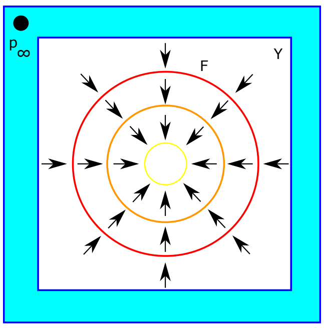

Fig. 16: How a front propagation algorithm works.

Another approach can be to use a *front propagation* or shortly *propagation* as detailed in [11,26] and depicted in Figure 16: the *front* is initialized at the light blue component surrounding the whole domain. Then, the front propagates into the depth of the image: the inner blue boundary propagates to the red boundary, which then propagates to the orange one, and then propagates to the yellow one, and finishes disappearing when the front covers the whole space Y.

Conjecture 1. *The single-valued map $u^\flat : Y \to \mathbb{R}$ computed using the front propagation algorithm described above is a self-dual single-valued in-between interpolation of U.*

An example of $u^\flat$ obtained with this front propagation (instead of the operator Op) is depicted in Figure 17: here, we "flatten" U_{USC} into $u^\flat : Y \to \mathbb{R}$ thanks

[5] We define the *threshold set* $[u > \lambda]$ for $\lambda \in \mathbb{R}$ and $u : \mathcal{D} \to \mathbb{R}$ as the set $\{x \in \mathcal{D} \,;\, u(x) > \lambda\}$.

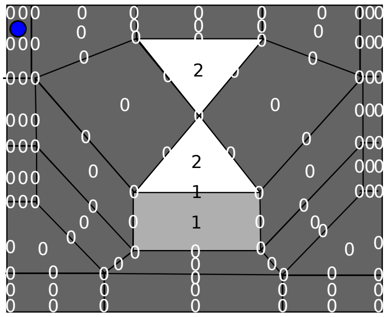

Fig. 17: Flattening of U_{USC} into $u^\flat : Y \to \mathbb{R}$ thanks to the front propagation algorithm.

to the front propagation algorithm depicted in Figure 16; the blue dot represents the "exterior" point p_∞ from which the propagation starts. Such a propagation does not need any floating-point arithmetic and thus is numerically easier to handle in practice: starting from an image U of values in $\mathbb{Z}$, we will obtain a median $\mathfrak{M} \in \mathbb{Z}/2$, which is easy to handle with a generic library [37].

At the opposite, using operators Op verifying (P_{Op}) like the symmetric median/mean operators will ensure smoother interpolations and then more regularity in the final image.

Since $u^\flat$ is neither a span-valued map nor AWC, it is not yet the result we are looking for, which justifies the following steps.

3.3.3 Step 3: from $u^\flat$ to $U_1^\flat$

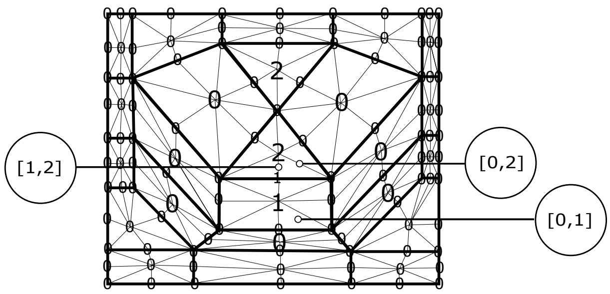

Fig. 18: From the in-between interpolation $u^\flat$ provided by the front propagation algorithm, we build a new map $U_1^\flat : Y^1 \to \mathbb{I}_{\mathbb{R}}$ defined on a subdivision Y^1 of Y. The aim of this map is to preserve the structure of $u^\flat$ while being easily convertible into a span-valued map.

Now let us define our intermediary map $U_1^\flat : Y^1 \to \mathbb{I}_{\mathbb{R}}$.

Definition 6. *Let us assume the hypotheses of Definition 3, and that we built $u^\flat$ following Definition 4 or by using the front propagation algorithm. From $u^\flat$, we deduce the new map $U_1^\flat : Y^1 \to \mathbb{I}_\mathbb{R}$ (see Figure 18) defined on Y^1 such that:*

$$\forall h^1 \in Y^1,\ U_1^\flat(h^1) := \text{Span}\{u^\flat(v)\ ;\ v \in h^1\}.$$

A last step remains then to obtain a span-valued map.

3.3.4 Step 4: from $U_1^\flat$ to U'

Now let us define our new representation $U' : \mathfrak{CC}(Y) \to \mathbb{I}_\mathbb{R}$.

Definition 7. *Now, assuming the hypotheses of Definition 6, let us define $U' : \mathfrak{CC}(Y) \to \mathbb{I}_\mathbb{R}$ such that:*

$$\forall h \in Y^1,\ U'(h^{*,Y^1}) := U_1^\flat(h).$$

We can then remark that this last function is a span-valued map (see Figure 19) by duality.

Indeed, it verifies that for any $p_n \in (\mathfrak{CC}(Y))_n$, there exist $v \in \mathbb{R}$ and $h_0 \in (Y^1)_0$ such that $h_0^{*,Y^1} = p_n$ and:

$$U'(p_n) = U_1^\flat(h_0) = \{v\},$$

and for any $p_k \in \mathfrak{CC}(Y) \setminus (\mathfrak{CC}(Y))_n$:

$$U'(p_k) = \text{Span}\left\{U'(p_n)\ ;\ p_n \in \left(\beta_{\mathfrak{CC}(Y)}(p_k)\right)_n\right\}.$$

This construction leads to the following result:

Theorem 3. *Let us assume the hypotheses of Definition 7, the span-valued map $U' : \mathfrak{CC}(Y) \to \mathbb{I}_\mathbb{R}$ is well-composed in the sense of Alexandrov for any $n \geq 2$.*

The proof of this theorem is postponed to Section 5, page 24.

This theorem means that thanks to the regularity of the domain $\mathfrak{CC}(Y)$ (due to the regularity of Y itself), the span-valued map U' defined on $\mathfrak{CC}(Y)$ is regular too since the boundaries of its threshold sets are discrete $(n-1)$-surfaces.

Now let us define the notion of *span-valued in-between interpolation*:

Definition 8. *For any span-valued map $F : \mathcal{D} \to \mathbb{I}_\mathbb{R}$ where $\mathcal{D}$ is a discrete n-surface, we say that the span-valued map $F' : \mathfrak{CC}(\mathcal{D}) \to \mathbb{I}_\mathbb{R}$ is a* span-valued in-between interpolation *of F if for any $p \in \mathcal{D}_n$,*

$$F'(\{p\}^{*,\mathcal{D}^1}) = F(p),$$

and for any $k \in [\![0, n-1]\!]$ and any $p \in \mathcal{D}_k$,

$$F'(\{p\}^{*,\mathcal{D}^1}) = \{v\},$$

with:

$$v \in \text{Span}\left\{F'(\{q\}^{*,\mathcal{D}^1})\ ;\ q \in (\beta_\mathcal{D}(p))_{k+1}\right\}.$$

Then we obtain the following property whose proof has been postponed to Section 5, page 24:

Property 2. *Let us assume the hypotheses of Definition 7, the span-valued map $U' : \mathfrak{CC}(Y) \to \mathbb{I}_\mathbb{R}$ is a span-valued in-between interpolation of U.*

Informally speaking, we can say that the boundaries are preserved from U to U'.

We can see in Figure 20 how in-between interpolations preserve boundaries (everywhere except at the pinches) and on the contrary how interpolations which are not in-between (see Figure 21) insert new extrema in the image and and somehow break the structure of the image.

Finally, let us observe that the complete procedure is self-dual (see Figure 22) since the valuation $\mathfrak{M} := \text{med}\{U(h)\ ;\ h \in (\text{bd}(X,Y))_{n-1}\}$ and the mappings $U \to U_{\text{USC}}$, $U_{\text{USC}} \to u^\flat$, $u^\flat \to U_1^\flat$, and $U_1^\flat \to U'$ are self-dual. This leads to an important result of this paper whose proof is immediate:

Property 3. *Let us assume the hypotheses of Definition 7. The span-valued map $U' : \mathfrak{CC}(Y) \to \mathbb{I}_\mathbb{R}$ is a self-dual representation of $U : X \to \mathbb{I}_\mathbb{R}$.*

Remark 1. *Let us define the* n-cells *of $\mathfrak{CC}(Y)$ as the dual cells of the vertices of Y^1. Then, we can observe that (the interior of) the closure in $\mathfrak{CC}(Y)$ of any finite set of n-cells of $\mathfrak{CC}(Y)$ is AWC (see Figure 19).*

Now let us define $U'|_{\mathfrak{T}(X)} : \mathfrak{T}(X) \to \mathbb{I}_\mathbb{R}$ as the restriction of U' to $\mathfrak{T}(X)$. Because for each $\lambda \in \mathbb{R}$ we have that $[U'|_{\mathfrak{T}(X)} \rhd \lambda]$ is equal to the interior of the closure of a set of n-cells of $\mathfrak{CC}(Y)$, then $[U'|_{\mathfrak{T}(X)} \rhd \lambda]$ is AWC, and then we obtain that $U'|_{\mathfrak{T}(X)}$ is AWC. This leads to the following conjecture:

Conjecture 2. *Let us assume the hypotheses of Definition 7. The image $U'|_{\mathfrak{T}(X)} : \mathfrak{T}(X) \to \mathbb{I}_\mathbb{R}$ is AWC and computed in a self-dual way. In that sense, $U'|_{\mathfrak{T}(X)}$ is a self-dual representation of U.*

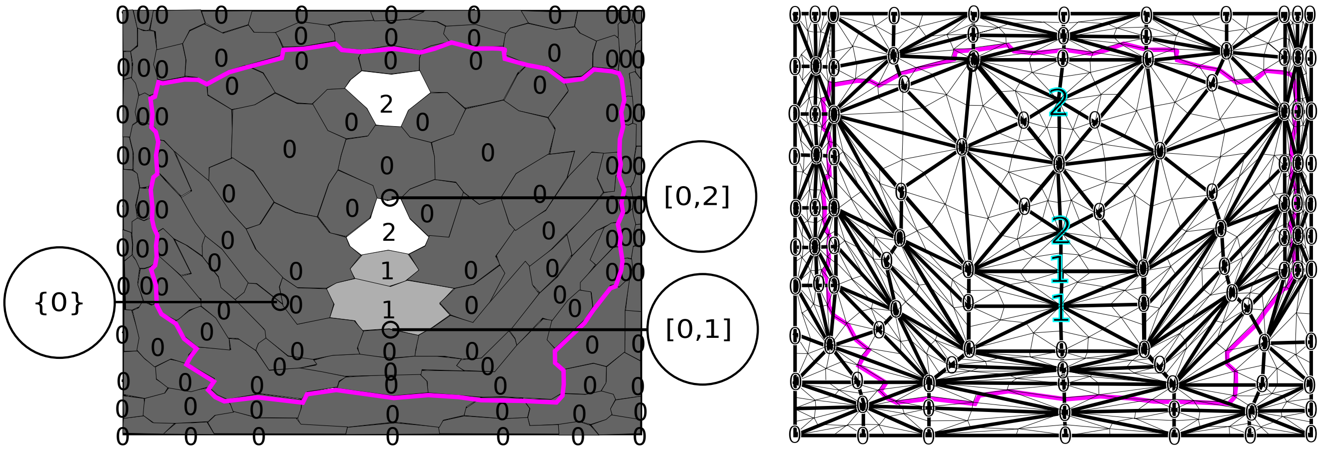

Fig. 19: Computation of the self-dual representation U' based on Figure 18: the initial pixels have been "eroded" to make possible the insertion of new pixels between them; this way, problematic pixels have been separated so that we do not have any more pinches in the boundaries. The delimitation of the domain $\mathfrak{T}(X)$ is depicted in pink. Note that the second subdivision of the domain of the initial image is depicted on the right side to show the link between the initial image and its final interpolation.

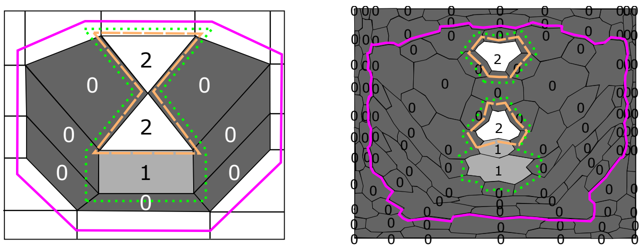

Fig. 20: Based on Figure 19, the in-between interpolation U' preserves the boundaries of the initial image U (except at the pinch): the orange dashed curves show $\mathrm{bd}([U \rhd 1], Y)$ and $\mathrm{bd}([U' \rhd 1], Y)$, the green dotted curves show $\mathrm{bd}([U \rhd 0], Y)$ and $\mathrm{bd}([U' \rhd 0], Y)$, and the pink curves show $\mathrm{bd}(X, Y)$ and $\mathrm{bd}(\mathfrak{T}(X), \mathfrak{C}\mathfrak{C}(Y))$ respectively on the left side and on the right side in the figure.

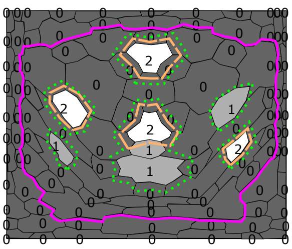

Fig. 21: A random interpolation of U which is not in-between: following the same procedure as the one used to compute U' except that we used an operator Op to compute $u^\flat$ which does not verify (P_{Op}), we obtain a span-valued map with new extrema compared to the initial map U. The legend is the same as in Figure 20.

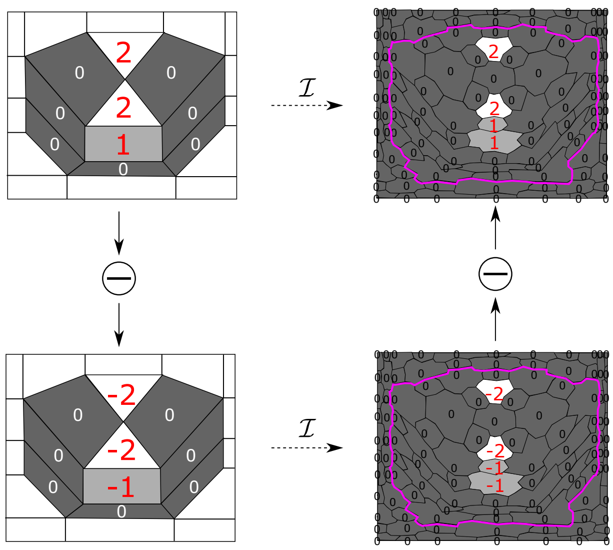

Fig. 22: Self-duality of the operator $\mathcal{I}$ which associates to a span-valued map U its representation U' (here, we use the front-propagation algorithm to compute $u^\flat$). The representation of the opposite of the initial image is equal to the opposite of the representation of the initial image.

Depending on the needs, it will then be preferable to use U' or $U'|_{\mathfrak{T}(X)}$. When the aim is to compute the tree of shapes, we need the domain of the AWC representation to be a discrete surface, then we can use U'. When the aim is just to obtain an AWC *plain map* (see Subsection 4.2), we can choose the representation $U'|_{\mathfrak{T}(X)}$.

4 The tree of shapes of our self-dual representation

After some reminders in matters of topology and *plain maps*, we recall under which conditions we can compute the tree of shapes of a gray-level image.

4.1 Topological spaces

Let X be a set of arbitrary elements, and let $\mathcal{U}$ be a set of subsets of X. We say that $\mathcal{U}$ is a *topology* on X if $\emptyset$ and X are elements of $\mathcal{U}$, if any union of elements of $\mathcal{U}$ are elements of $\mathcal{U}$, and if any finite intersection of elements of $\mathcal{U}$ is an element of $\mathcal{U}$. X supplied with $\mathcal{U}$ is denoted $(X,\mathcal{U})$ or, for short, X and is called a *topological space*. The elements of X are called *points* of X, and the elements of $\mathcal{U}$ are then called the *open sets* of X. Any complement of an open set in X is called a *closed set* of X. We say that a subset of X which contains an open set containing an element $x \in X$ is a *neighborhood* of x in X. A topological space is said to be *connected* if it is not the disjoint union of two non-empty open sets. Now, let X be a non-empty topological space and let x be an element of X. The maximum connected subset (in the inclusion sense) containing x is denoted by $\mathcal{CC}(X,x)$ and is called the *connected component* of X containing x. More generally, the maximum connected subsets of X are called the *connected components* of X and the set of these components is denoted by $\mathcal{CC}(X)$.

We say that a topological space $(X,\mathcal{U})$ verifies the $T0$ *axiom of separation* if, for any two different elements of X, at least one has a neighborhood not containing the other. A topological space which verifies the $T0$ axiom of separation is said to be a $T0$-*space* [3,4,33]. A topological space $(X,\mathcal{U})$ is said to be *discrete* [2] if the intersection of any family of open sets of X is open in X. A discrete $T0$-space is said to be an *Alexandrov space* [23].

As explained by Theorem 6.52 [3] (p. 28), a poset equipped with the topology made of its combinatorial openings is an Alexandrov space. Furthermore, according to Eckhardt *et al.* [23], connectedness is equivalent to path-connectedness in Alexandrov spaces.

Note that the definition of combinatorial boundary of a poset X suborder of another poset Y is equal to its *topological boundary* when X is equal to the combinatorial closure in Y of its n-cells. Recall that the topological boundary is defined as:

$$\partial X := \alpha_Y(X) \setminus \mathrm{Int}_Y(X),$$

where $\mathrm{Int}_Y(X) := \{x \in X \; ; \; \beta_Y(x) \subseteq X\}$ denotes the *topological interior* of X as a subset of the topological space Y. The proof of this equality is new and is postponed to Section 7, page 25.

4.2 Plain maps

Let us now recall some mathematical background coming from [5,41]. Let $(X,\mathcal{U})$ be an Alexandrov space and let $\mathbb{R}$ be equipped with its usual topology. An application $F : X \to \mathcal{P}(\mathbb{R})$ (equivalently written $F : X \rightsquigarrow \mathbb{R}$) is said to be a *set-valued map*. The *domain* of F is the set $\mathcal{D}(F)$ of points x of X such that $F(x) \neq \emptyset$.

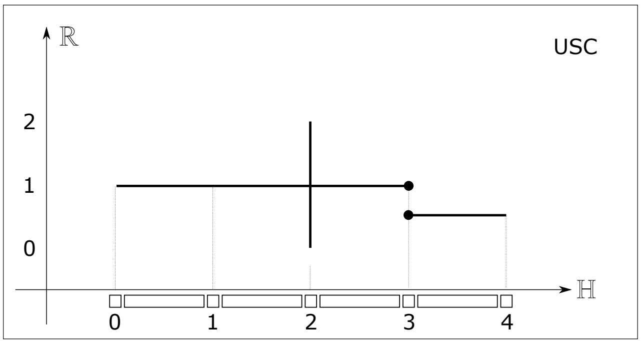

Fig. 23: An example of USC map $F : \mathcal{D}(F) \rightsquigarrow \mathbb{R}$ on the 1D Khalimsky grid $\mathbb{H}$: we have $F(0) = \{1\}$, $F(1) = \{1\}$, $F(2) = [0,2]$, $F(3) = \{\frac{1}{2},1\}$ and $F(4) = \{\frac{1}{2}\}$. We observe at the discontinuity $(x, F(x))$ that for any neighborhood $\mathcal{U}$ of $F(x)$, for any $x' \in \beta_X(x)$, we have $F(x') \subseteq \mathcal{U}$, which shows that F is USC.

The *neighborhood* of a set $A \subseteq \mathbb{R}$ is a subset of $\mathbb{R}$ which contains an open set in $\mathbb{R}$ containing A. A set-valued map $F : X \rightsquigarrow \mathbb{R}$ is said to be *upper semi-continuous (USC)* (see Figure 23) at $x \in \mathcal{D}(F)$, if for any neighborhood $\mathcal{U}$ of $F(x)$, $\forall x' \in \beta_X(x)$, $F(x') \subseteq \mathcal{U}$. A set-valued map is said to be *upper semi-continuous (USC)* if it is USC at each point $x \in \mathcal{D}(F)$.

A set-valued USC map $F : X \rightsquigarrow \mathbb{R}$ is said to be a *quasi-simple map* (see Figure 24) if for any $x \in \mathcal{D}(F)$, $F(x)$ is a closed connected set and furthermore, for any $x \in \mathcal{D}(F)$ such that $\{x\} = \beta_X(x)$, $F(x)$ is *degenerate*.

A quasi-simple map $F : X \rightsquigarrow \mathbb{R}$ is said to be a *simple map* (see Figure 25) if for any quasi-simple map $F_2 : X \rightsquigarrow \mathbb{R}$ such that $F(x) = F_2(x)$ when $\beta_Y(x) = \{x\}$ with $x \in \mathcal{D}(F)$, then for any $x \in \mathcal{D}(F)$, $F(x) \subseteq F_2(x)$.

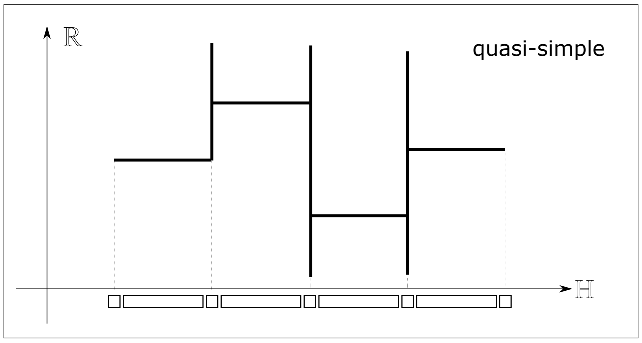

Fig. 24: An example of quasi-simple map on the 1D Khalimsky grid $\mathbb{H}$: we can observe that this map U is USC, that for any x in the domain of U, the set $U(x)$ is connected and closed, and that $U(x)$ is “flat” on the maximal faces.

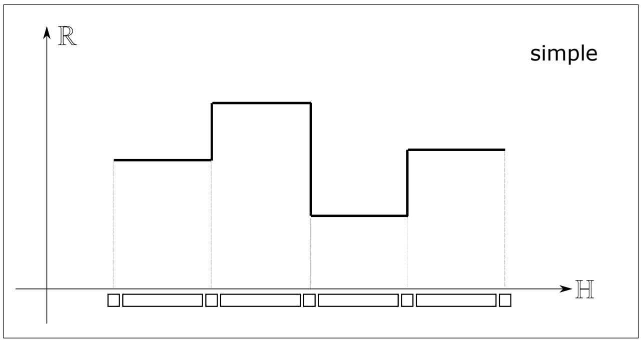

Fig. 25: An example of simple map on the 1D Khalimsky grid $\mathbb{H}$: this quasi-simple map U is the “intersection” of all quasi-simple maps whose values are equal to U of the maximal faces. In that sense, U is a simple map.

A simple map $F : X \rightsquigarrow \mathbb{R}$ is said to be *interval-valued*: for any $x \in \mathcal{D}(F)$, $F(x)$ is equal to (a, b)[6] for some $a, b \in \mathbb{R}$ with $a \leq b$. Furthermore, this map F is said to be *closed-valued* if for each $x \in \mathcal{D}(F)$, $F(x)$ is a closed interval. A set-valued map $F : X \rightsquigarrow \mathbb{R}$ is said to be a *plain map* if it is a closed-valued simple map.

We can prove that a span-valued map defined on an Alexandrov space is a plain map (the proof is left to the reader). Then, any span-valued map $U \to \mathbb{I}_{\mathbb{R}}$ defined on an Alexandrov space $|X|$ can be written $U : X \rightsquigarrow \mathbb{R}$.

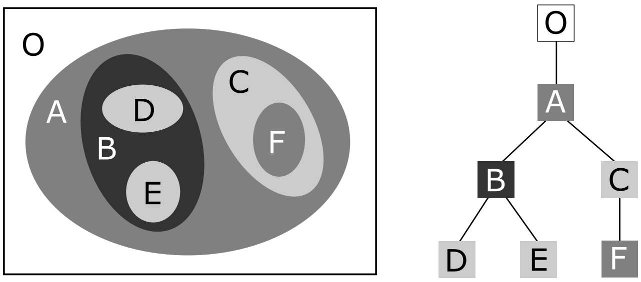

Fig. 26: An image and its tree of shapes: an intuitive way to understand how to build the tree of shapes is to observe that O contains A, A contains B and C, B contains D and E, and C contains F. All these observations lead directly to the tree of shapes of this image.

4.3 The tree of shapes

We have defined in Subsection 2.8 what threshold sets are for span-valued maps, and for plain maps since posets are Alexandrov spaces. Now let us see that based on the *saturation operator* [19], we can define *shapes* [19, 26, 41] and then, subject to some constraints, build the *tree of shapes* (see Figure 26).

The *saturation operator* or the *cavity fill-in operator*, denoted Sat, is defined for any subset S of a topological space $\mathcal{D}$ as:

$$\mathrm{Sat}(S) := \mathcal{D} \setminus \mathcal{CC}(\mathcal{D} \setminus S, p_\infty),$$

where p_∞ is some reference point [19]. Intuitively, this operator fills the component S relative to p_∞ which is then seen as the “exterior”; for this reason, in practice, we choose to locate p_∞ outside the domain of the initial image. For example, in Figure 26, p_∞ is chosen inside the white component O which “surrounds” the gray-level components A to F. A simple example of saturation in Figure 26 would be that $\mathrm{Sat}(B) = B \cup D \cup E$.

Now, assuming that some plain map $U : \mathcal{D} \rightsquigarrow \mathbb{R}$ is given, we can define the *upper shapes* of U:

$$\mathcal{S}_{\triangleright}(U) := \{\mathrm{Sat}(\Gamma)\ ;\ \Gamma \in \mathcal{CC}([U \triangleright \lambda]),\ \lambda \in \mathbb{Z}\},$$

and the *lower shapes* of U:

$$\mathcal{S}_{\triangleleft}(U) := \{\mathrm{Sat}(\Gamma)\ ;\ \Gamma \in \mathcal{CC}([U \triangleleft \lambda]),\ \lambda \in \mathbb{Z}\}.$$

Then we are going to see that these sets of shapes can be grouped together to make the tree of shapes but under some conditions.

[6] An interval is denoted (a, b) when its is equal to $[a, b]$, $]a, b]$, $[a, b[$, or $]a, b[$.

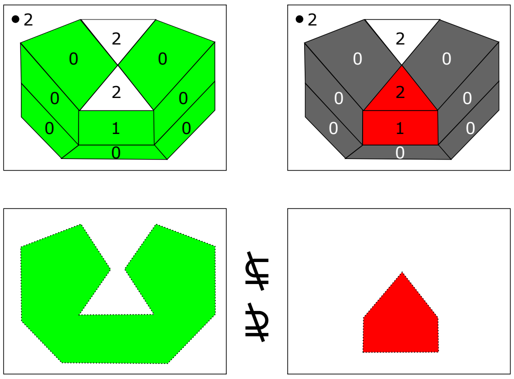

Fig. 27: Inconsistencies between a lower shape and an upper shape of a same image because of a pinch: the first shape coming from $[U \lhd 2]$ (in green) and the second shape coming from $[U \rhd 0]$ (in red) overlap but they are not nested. Note that the exterior has been set at 2, as indicated by the value next to the point p_∞ represented by a black dot. Remark that the set of two 2-simplices at 2 is not connected because at the pinch p joining them, $U(p) = [0, 2]$, and then p does not belong to $[U \rhd 0]$; this explains why only one of these two 2-simplices is in the red component.

When we compute the tree of shapes of an image which has pinches, then some shapes can overlap without being nested: for example, in Figure 27, the green shape corresponding to $[U \lhd 2]$ and one of the red shapes corresponding to $[U \rhd 0]$ overlap, but they are not nested, and so the set of shapes cannot be a tree (as it contains cycles).

On the contrary, when a plain map has some regularity properties like well-composedness in the sense of Alexandrov, we obtain the following theorem:

Theorem 4 (Theorem 21 [41])**.** *If $F : \mathcal{D} \rightsquigarrow \mathbb{R}$ is an AWC plain map defined on an unicoherent*[7] *domain $\mathcal{D}$, then the set $\mathfrak{S}(F) := S_\rhd(F) \cup S_\lhd(F)$ of shapes of F is a tree.*

Now, let us see that under constraints on the domain $\mathfrak{CC}(Y)$ of U', we can compute its tree of shapes.

4.4 The tree of shapes of our self-dual representation

Notations 1. *Since we have seen that our self-dual representation U' is an AWC plain map, we could compute directly the tree of shapes $\mathfrak{S}(U')$ of U' (assuming that $\mathfrak{CC}(Y)$ coming with its Alexandrov topology is unicoherent). However, we assumed that the domain of the initial map U was a strict subset of Y, and thus it is more logic to compute the restriction to $\mathfrak{T}(X)$ of the tree of the shapes of U':*

$$\mathfrak{S}|_{\mathfrak{T}(X)}(U) := \{S \cap \mathfrak{T}(X);\ S \in \mathfrak{S}(U')\}\,.$$

[7] Recall that a set X is said to be *unicoherent* if X is connected and for any two closed connected sets M and N such that $M \cup N = X$, then $M \cap N$ is connected.

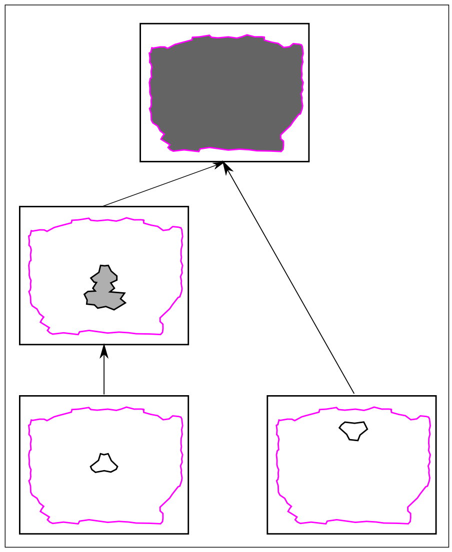

Fig. 28: The tree of shapes $\mathfrak{S}|_{\mathfrak{T}(X)}(U)$ based on Figure 19. Once all the shapes of U' have been computed, that is, for every threshold $\lambda \in \mathbb{R}$ and for the upper and lower threshold sets, we remove the redundant ones. At the end, four shapes remain: the dark gray component (in fact $\mathfrak{T}(X)$) coming from $[U' \rhd -1]$, the middle gray component (on the middle left) coming from $[U' \rhd 0]$ included in the dark gray one, a white component coming from $[U' \rhd 1]$ included in the middle gray one (on the left), and another white component coming from $[U' \rhd 1]$ included in the dark gray one (on the right). We finally obtain a tree (sometimes called *hierarchy*) of the shapes in U' representing U.

Note that we know $\mathfrak{S}|_{\mathfrak{T}(X)}(U)$ is a tree because we preserved the inclusion relationship between the nodes of $\mathfrak{S}(U')$. Indeed, when A and B are two components of $\mathfrak{S}(U')$, then we have three possible cases:

- $A \cap B = \emptyset$ and then $(A \cap \mathfrak{T}(X)) \cap (B \cap \mathfrak{T}(X)) = \emptyset$,
- or $A \subseteq B$ and $A \cap \mathfrak{T}(X) \subseteq B \cap \mathfrak{T}(X)$,
- or $B \subseteq A$ and $B \cap \mathfrak{T}(X) \subseteq A \cap \mathfrak{T}(X)$.

In other words, we have the following property:

Property 4. *Let us assume the hypotheses of Definition 7, of Theorem 4 and of Notation 1. When $\mathfrak{CC}(Y)$ is unicoherent, then the sets $\mathfrak{S}(U')$ and $\mathfrak{S}|_{\mathfrak{T}(X)}(U)$ are trees.*

In Figure 28 we depicted the computation of the tree of shapes of the representation seen in Figure 19.

From now on, we will call $\mathfrak{S}|_{\mathfrak{T}(X)}(U)$ the *tree of* U.

5 Proofs of the main results

Let us now expose our formal definitions of a cell complex and of our transform. Then, we prove new mathematical properties relative to posets, discrete surfaces, $\mathfrak{T}(X)$, and to our self-dual representation U'.

5.1 Formal definitions

Let us show how we define formally a cell complex associated with a suborder (see Figure 29); note that our definitions are purely combinatorial. Take care not to confuse our definition of cell-complexes with the *CW Complexes* of Whitehead [44].

Definition 9 (Cell-complexes)**.** *Let $|Z|, |T|$ be two posets such that $|Z|$ is a suborder of $|T|$. We define the* cell-complex *associated to Z relative to T as the dual of Z^1 relative to T^1, and we denote it $|\mathfrak{CC}(Z|T)| = (\mathfrak{CC}(Z|T), \subseteq)$. In other words, $|\mathfrak{CC}(Z|T)|$ is equal to:*

$$\left\{ A^{*,T^1} \; ; \; A \in Z^1 \right\},$$

supplied with $\subseteq$. When $Z = T$, we denote $|\mathfrak{CC}(Z|T)| = |\mathfrak{CC}(Z)|$.

Let us follow the construction of a cell complex on Figure 29. At the top in this figure, we start from a simplicial complex Y which is a set (supplied with an order relation) of faces which are sets of points of a set Λ_Y. Then we compute the subdivision Y^1 of Y, which is a set of chains of faces of elements of Λ_Y. Finally, we compute the dual cells of Y^1 to obtain $\mathfrak{CC}(Y)$. In other words, we compute a set of dual cells which are sets of chains of chains of faces which are sets of elements of Λ_Y.

Now let us assume that we equip cell complexes with the relation $\subseteq$:

Definition 10. *Let $|Y|$ be a poset. For any suborder $|X|$ of $|\mathfrak{CC}(Y)|$, we define:*

$$\alpha_{\mathfrak{CC}(Y)}(X) := \{h \in \mathfrak{CC}(Y) \; ; \; \exists h' \in X \text{ s.t. } h \subseteq h'\}.$$

Then we can define formally our *transform* of a suborder in a discrete surface:

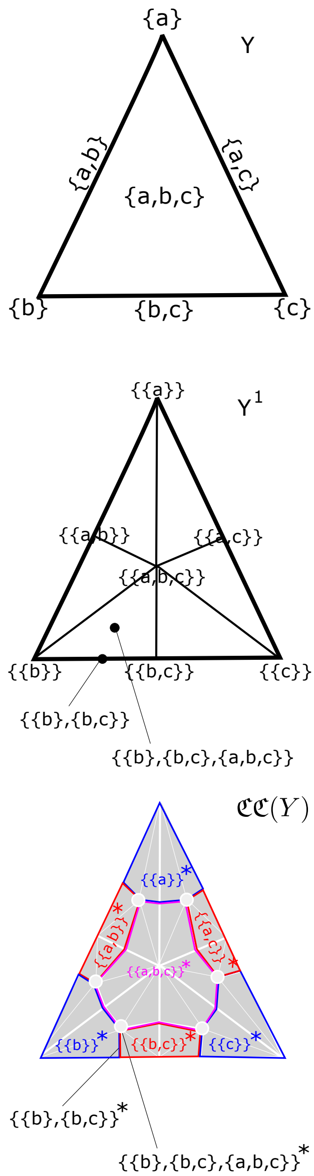

Fig. 29: Computation of a cell-complex: a simplicial complex Y, its first subdivision Y^1, and the computation of the dual of the first subdivision, that is, $\mathfrak{CC}(Y)$; the thin lines in the last figure correspond to Y^2.

Definition 11. *Let X be a non-empty finite set such that $|X|$ is a suborder of a discrete n-surface $|Y|$, $n \geq 2$. Let us assume now that X is equal to the combinatorial closure of its n-faces. We define the poset $|\mathfrak{T}(X)|$ by:*

$$\mathfrak{T}(X) := \alpha_{\mathfrak{CC}(Y)}(\mathfrak{CC}(X|Y)),$$

and we call it the transform *of X (relative to Y).*

5.2 New mathematical properties

Let us expose some new properties relative to posets and their subdivisions.

Notations 2. *Let $|Y| = (Y, R)$ be a poset and i be a positive integer. The expression $(a_0, \ldots, a_k)^{R,Y,i}$ represents the set $\{a_0, \ldots, a_k\}$ assuming that $a_0\ R^{\square}\ a_1$, ..., $a_{k-1}\ R^{\square}\ a_k$, and that $a_0, \ldots, a_k$ belong to the i^{th} subdivision of Y.*

Proposition 2. *Let Y be a poset. Then, for any element h of Y, we have the relation:*

$$\alpha_{Y^1}(\beta_{Y^1}(\{h\})) = [\theta_Y(h)]^1$$

Proof: The set $\beta_{Y^1}(\{h\})$ represents the chains of elements of Y containing h. Then, $\alpha_{Y^1}(\beta_{Y^1}(\{h\}))$ represents the suborders of the chains of elements of Y containing h. Besides, $\theta_Y(h)$ represents the set of elements of Y which are comparable to h. Then, $[\theta_Y(h)]^1$ is the set of chains of elements of Y comparable to h, that is, the set of elements that can be written:

$$(h_0, \ldots, h_k)^{\alpha_Y, Y, 0} \text{ with } h_i \in \theta_Y(h), \forall i \in [\![0, k]\!].$$

These two sets are then equal. $\square$

Proposition 3. *Let A, B be two suborders of a same poset $|Y|$. Then,*

$$A^1 \cap B^1 = [A \cap B]^1.$$

Proof: Let c be an element of $A^1 \cap B^1$. Then, $c = (h_0, \ldots, h_k)^{\alpha_Y, Y, 0}$ is both a chain of elements of A and a chain of elements of B, then each element h_i for $i \in [\![0, k]\!]$ belongs to $A \cap B$, then it is a chain of elements of $A \cap B$. Conversely, when c is a chain of elements of $A \cap B$, it is at the same time a chain of elements of A and a chain of elements of B. $\square$

Proposition 4. *Let A, B be two suborders of a same poset. Then,*

$$A^1 \subseteq B^1 \Leftrightarrow A \subseteq B.$$

Proof: Let A, B be two posets such that $A \subseteq B$, and let c be an element of A^1. Then, c is a chain of elements of A, and then a chain of elements of B. Then $c \in B^1$. Conversely, when A, B are two posets such that $A^1 \subseteq B^1$, then:

$$A = \Lambda_{A^1} \subseteq \Lambda_{B^1} = B,$$

then $A \subseteq B$. $\square$

Now, let us show that the intuition that the closure becomes an opening by duality is true.

Proposition 5. *Let C be a simplicial complex of rank $n \geq 0$. We assume that every duality for faces, sets, and complexes are relative to C. For any $a, b \in C$, then $a \in \alpha_C(b)$ iff $b^* \in \alpha_{C^*}(a^*)$ where $a^*, b^* \in C^*$ are the dual cells of a and b respectively. The same way, $a \in \beta_C(b)$ is equivalent to $b^* \in \beta_{C^*}(a^*)$, and $a \in \theta_C(b)$ is equivalent to $a^* \in \theta_{C^*}(b^*)$. In other words, for any $a \in C$, $(\alpha_C(a))^* = \beta_{C^*}(a^*)$, $(\beta_C(a))^* = \alpha_{C^*}(a^*)$, and $(\theta_C(a))^* = \theta_{C^*}(a^*)$.*

Proof: Let a, b be two elements of C. Then, $a^* \in \alpha_{C^*}(b^*)$ is equivalent to $a^* \subseteq b^*$, itself equivalent by Proposition 2 to:

$$\bigcap_{h \in \alpha_C(a)} [\theta_C(h)]^1 \subseteq \bigcap_{h \in \alpha_C(b)} [\theta_C(h)]^1,$$

which is equivalent by Proposition 3 to:

$$\left[\bigcap_{h \in \alpha_C(a)} \theta_C(h)\right]^1 \subseteq \left[\bigcap_{h \in \alpha_C(b)} \theta_C(h)\right]^1.$$

This is equivalent by Proposition 4 to:

$$\bigcap_{h \in \alpha_C(a)} \theta_C(h) \subseteq \bigcap_{h \in \alpha_C(b)} \theta_C(h),$$

which is true iff:

$$\alpha_C(a) \supseteq \alpha_C(b),$$

that is, $b \in \alpha_C(a)$.

With similar reasoning, we obtain that $a \in \beta_C(b)$ is equivalent to $b^* \in \beta_{C^*}(a^*)$, from which we conclude that $a \in \theta_C(b)$ is equivalent to $a^* \in \theta_{C^*}(b^*)$.

Finally, for any $a \in C$:

$$(\alpha_C(a))^* = \bigcup_{b \in \alpha_C(a)} \{b^*\} = \bigcup_{b^* \in \beta_{C^*}(a^*)} \{b^*\} = \beta_{C^*}(a^*),$$

and the other results are obtained the same way. $\square$

Corollary 1. *Let X be a poset. Then its dual X^* is path-connected iff X is path-connected.*

Proof: This follows directly from Proposition 5. $\square$

Now, let us show that the dual of a frontier order of a support of a simplicial complex X is the boundary of the dual of X.

Proposition 6. *Let C be a simplicial complex and let X be a finite full subcomplex of C such that $\emptyset \subsetneq \Lambda_X \subsetneq \Lambda_C$. Then, the dual of the frontier order of Λ_X in Λ_C relative to C is equal to the boundary $\mathrm{bd}(X^*, C^*)$ of X^* in C^* (see Figure 30).*

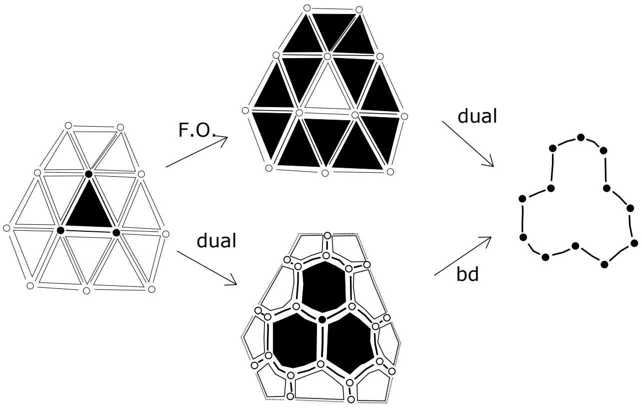

Fig. 30: The dual of the frontier order is equal to the boundary of the dual. We start from a set of points Λ_X (the 3 points in black in the left subfigure), a set Λ_C (the 12 black and white points in the same subfigure), and a simplicial complex C (this same figure in totality). Then, we obtain the frontier order of Λ_X in Λ_C relative to C in the top-middle subfigure in black. Then, we compute its dual and we obtain the simple closed curve (in the right subfigure). Restarting from the set X (in black in the left subfigure), we compute its dual and we obtain the black poset made of 3 hexagons, 3 edges, and one vertex (see the middle-bottom figure). Finally, we compute its boundary and we obtain the same simple closed curve as before.

Proof: Let z be an element of $\mathrm{bd}(X^*, C^*)$, then z belongs to $\alpha_{C^*}(X^*) \cap \alpha_{C^*}(C^* \setminus X^*)$, which means that there exist $u \in X^*$ and $v \in C^* \setminus X^*$ such that $z \in \alpha_{C^*}(u) \cap \alpha_{C^*}(v)$. Also, $u \in X^*$ implies that there exists $x_u \in X$ such that $x_u^* = u$ and $v \in C^* \setminus X^*$ implies that there exists $y_u \in C \setminus X$ such that $y_u^* = v$. This means that $z \in \alpha_{C^*}(x_u^*) \cap \alpha_{C^*}(y_u^*)$. Moreover, $z \in C^*$ implies that there exists some $d \in C$ such that $d^* = z$, then $d^* \in \alpha_{C^*}(x_u^*) \cap \alpha_{C^*}(y_u^*)$ implies that $x_u, y_u \in \alpha_C(d)$ by Proposition 5. Then d includes a simplex x_u which contains only vertices of X and an element y_u not containing only vertices of X, then d belongs to the frontier order of Λ_X. Since $d^* = z$, z belong to the dual of the frontier order of Λ_X.

Conversely, let us assume that z belongs to the dual of the frontier order of Λ_X. There exists y in the frontier order of Λ_X such that $z = y^*$, and such that y is a simplex of C containing vertices of Λ_X and of $\Lambda_C \setminus \Lambda_X$. Then there exists $v_+ \in X_0$ such that $v_+ \in \alpha_C(y)$ and $v_- \in (C \setminus X)_0$ such that $v_- \in \alpha_C(y)$. Then, by Proposition 5, $y^* \in \alpha_{C^*}(v_+^*) \cap \alpha_{C^*}(v_-^*)$ with $v_+^* \in X^*$ and $v_-^* \in (C \setminus X)^* = C^* \setminus X^*$. This leads to y^* belonging to $\mathrm{bd}(X^*, C^*)$, and then finally z belongs to the boundary of the dual of X. □

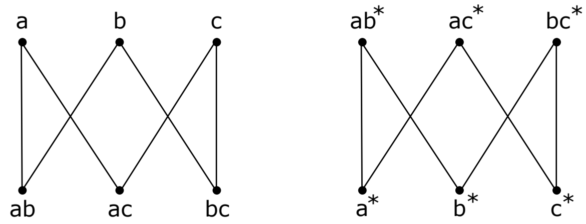

Fig. 31: As depicted on these Hasse diagrams of a discrete 1-surface (on the left side) and its dual (on the right side), the dual of a discrete 1-surface is still a discrete 1-surface.

Let us show that the dual of a discrete surface is also a discrete surface (see Figure 31).

Proposition 7. *Let C be a poset of rank $n \geq 0$ and let X be a non-empty CF-order subset of C. Then, X is a k-surface, $k \in [\![0, n]\!]$, implies that $X^{*,C}$ is a k-surface. The same manner, if X is a disjoint union of k-surfaces, then $X^{*,C}$ is also a disjoint union of k-surfaces.*

Proof: Note that in this proof, every dual is computed relatively to C. Let us proceed by induction. When $k = 0$, then $X = \{a, b\}$ with $a \notin \theta_C(b)$. Then, $X^* = \{a^*, b^*\}$ with $a^* \notin \theta_{C^*}(b^*)$ by Proposition 5, and then X^* is a 0-surface too. Let $k \in [\![1, n]\!]$ be an integer and let us assume that $n \geq 1$, and that the dual of any $(k-1)$-surface is a $(k-1)$-surface. Now let X be a k-surface, since $k \geq 1$, X is path-connected, and then X^* is path-connected too by Corollary 1. Furthermore, let z^* be an element of X^*, we obtain by Proposition 5 that $\theta^\Box_{X^*}(z^*) = (\theta^\Box_X(z))^*$ which is a $(k-1)$-surface by the induction hypothesis. Then, X^* is a k-surface. By induction, we have then that the dual of any k-surface is a k-surface. If X is not path-connected, we proceed component by component and we obtain the same way that the dual of a disjoint union of k-surfaces is a disjoint union of k-surfaces, since path-connectivity is preserved by duality by Corollary 1. □

5.3 Proofs of Section 3.2

Let us now prove that when $|X|$ is a suborder of a discrete n-surface $|Y|$ verifying some particular constraints, then $\mathfrak{T}(X)$ is well-composed in the sense of Alexandrov, but for that let us before announce some properties.

Property 5. *Note that a property of dual cells is that when A is an element of a discrete surface $|Y|$, then we have the following property:*

$$A^* = [\beta_Y(A)]^1.$$

Remark 2. *The direct consequence of Property 5 is the injectivity of the dual operator: for $A, B \in Y$ such that $A \neq B$, then $\beta_Y(A) \neq \beta_Y(B)$ and then $A^* \neq B^*$. The surjectivity of the dual operator is obtained by definition. In other words, the dual operator is bijective when applied on subsets of a discrete surface.*

Definition 12. *Let us assume that X is a finite nonempty strict subset of a discrete n-surface $|Y|$, that is equal to the combinatorial closure of its n-faces, let $\mathfrak{T}(X)$ be the transform of X, and let $\mathfrak{N}$ be the boundary of $\mathfrak{T}(X)$. Then we define the following sets:*

$$\xi_{\mathfrak{T}(X)} := \bigcup_{p \in X} \beta_{Y^1}(\{p\}),$$

and

$$\xi_{\mathfrak{N}} := \beta_{Y^1}(\xi_{\mathfrak{T}(X)}) \cap \beta_{Y^1}(Y^1 \setminus \xi_{\mathfrak{T}(X)}).$$

Since the definitions of the sets $\xi_{\mathfrak{T}(X)}$ and $\xi_{\mathfrak{N}}$ are not very intuitive, let us observe that $\xi_{\mathfrak{T}(X)}$ is in fact the subset of Y^1 whose dual is equal to $\mathfrak{T}(X)$ (see Figure 32), and that $\xi_{\mathfrak{N}}$ is the subset of Y^1 whose dual is the boundary $\mathfrak{N}$ of $\mathfrak{T}(X)$ (see Figure 33), as presented by the following property.

Property 6. *Assuming the hypotheses of Definition 12, the two following relations hold:*

$$\xi_{\mathfrak{T}(X)}^{*,Y^1} = \mathfrak{T}(X), \text{ and } \xi_{\mathfrak{N}}^{*,Y^1} = \mathfrak{N}.$$

Proof: Let us begin with the first equality.

Let us prove then that $\xi_{\mathfrak{T}(X)}^{*,Y^1} \subseteq \mathfrak{T}(X)$. Let p be an element of $\xi_{\mathfrak{T}(X)}^{*,Y^1}$, then there exists some $z \in \xi_{\mathfrak{T}(X)}$ verifying $z^{*,Y^1} = p$. Also, this element $z \in \xi_{\mathfrak{T}(X)}$ verifies that there exists some $h \in X$ verifying $z \in \beta_{Y^1}(\{h\})$, and then by Proposition 5, $z^{*,Y^1} \in \alpha_{\mathfrak{CC}(Y)}(\{h\}^{*,Y^1}) \subseteq \mathfrak{T}(X)$. This implies that $p = z^{*,Y^1} \in \mathfrak{T}(X)$.

Let us proceed to the converse inclusion:

$$\mathfrak{T}(X) \subseteq \xi_{\mathfrak{T}(X)}^{*,Y^1}.$$

$$\begin{aligned} z \in \mathfrak{T}(X) &\Rightarrow z \in \alpha_{\mathfrak{CC}(Y)}(\mathfrak{CC}(X|Y)),\\ &\Rightarrow \exists p \in \mathfrak{CC}(X|Y), z \in \alpha_{\mathfrak{CC}(Y)}(p),\\ &\Rightarrow \exists p \in \left\{A^{*,Y^1} \; ; \; A \in X^1\right\}, z \in \alpha_{\mathfrak{CC}(Y)}(p),\\ &\Rightarrow \exists A \in X^1 \text{ s.t. } z \in \alpha_{\mathfrak{CC}(Y)}(A^{*,Y^1}). \quad (R) \end{aligned}$$

Moreover, $z \in \alpha_{\mathfrak{CC}(Y)}(A^{*,Y^1}) \subseteq \mathfrak{CC}(Y)$ implies that there exists some $\mathcal{Z}^1 \in Y^1$ such that $z = (\mathcal{Z}^1)^{*,Y^1}$. Then, we obtain that:

$$\begin{aligned} &(R)\\ &\Rightarrow \exists A \in X^1, (\mathcal{Z}^1)^{*,Y^1} \in \alpha_{\mathfrak{CC}(Y)}(A^{*,Y^1}) \text{ s.t. } z = (\mathcal{Z}^1)^{*,Y^1},\\ &\Rightarrow \exists A \in X^1, (\mathcal{Z}^1)^{*,Y^1} \in \alpha_{(Y^1)^*}(A^{*,Y^1}) \text{ s.t. } z = (\mathcal{Z}^1)^{*,Y^1},\\ &\overset{(P5)}{\Longrightarrow} \exists A \in X^1, \mathcal{Z}^1 \in \beta_{Y^1}(A) \text{ s.t. } z = (\mathcal{Z}^1)^{*,Y^1},\\ &\Rightarrow \mathcal{Z}^1 \in \beta_{Y^1}(X^1) \text{ s.t. } z = (\mathcal{Z}^1)^{*,Y^1},\\ &\Rightarrow z \in \left(\beta_{Y^1}(X^1)\right)^{*,Y^1},\\ &\Rightarrow z \in (\cup_{x \in X^1} \beta_{Y^1}(x))^{*,Y^1}. \quad (R') \end{aligned}$$

Since X^1 is a simplicial complex, we have the remarkable equality:

$$\cup_{x \in X^1} \beta_{Y^1}(x) = \cup_{x \in (X^1)_0} \beta_{Y^1}(x).$$

Then, we can continue our reasoning:

$$\begin{aligned} (R') &\Rightarrow z \in \left(\cup_{x \in (X^1)_0} \beta_{Y^1}(x)\right)^{*,Y^1},\\ &\Rightarrow z \in \left(\cup_{\{p\} \in (X^1)_0} \beta_{Y^1}(\{p\})\right)^{*,Y^1},\\ &\Rightarrow z \in (\cup_{p \in X} \beta_{Y^1}(\{p\}))^{*,Y^1},\\ &\Rightarrow z \in \left(\xi_{\mathfrak{T}(X)}\right)^{*,Y^1}. \end{aligned}$$

We have proven $\xi_{\mathfrak{T}(X)}^{*,Y^1} = \mathfrak{T}(X)$.

Now let us prove that $\xi_{\mathfrak{N}}^{*,Y^1} = \mathfrak{N}$:

$$\begin{aligned} z \in \xi_{\mathfrak{N}}^{*,Y^1} &\Leftrightarrow z = \mathcal{Z}^{*,Y^1}\\ &\quad \text{s.t. } \mathcal{Z} \in \beta_{Y^1}(\xi_{\mathfrak{T}(X)}) \cap \beta_{Y^1}(Y^1 \setminus \xi_{\mathfrak{T}(X)}),\\ &\Leftrightarrow z = \mathcal{Z}^{*,Y^1}, \exists \mathcal{X} \in \xi_{\mathfrak{T}(X)}, \exists \mathcal{Y} \in Y^1 \setminus \xi_{\mathfrak{T}(X)},\\ &\quad \text{s.t. } \mathcal{Z} \in \beta_{Y^1}(\mathcal{X}) \cap \beta_{Y^1}(\mathcal{Y}),\\ &\Leftrightarrow z = \mathcal{Z}^{*,Y^1}, \exists \mathcal{X} \in \xi_{\mathfrak{T}(X)}, \exists \mathcal{Y} \in Y^1 \setminus \xi_{\mathfrak{T}(X)},\\ &\quad \text{s.t. } \mathcal{X}, \mathcal{Y} \in \alpha_{Y^1}(\mathcal{Z}),\\ &\overset{(P5)}{\Leftrightarrow} z = \mathcal{Z}^{*,Y^1}, \exists \mathcal{X} \in \xi_{\mathfrak{T}(X)}, \exists \mathcal{Y} \in Y^1 \setminus \xi_{\mathfrak{T}(X)},\\ &\quad \text{s.t. } \mathcal{X}^{*,Y^1}, \mathcal{Y}^{*,Y^1} \in \beta_{\mathfrak{CC}(Y)}(\mathcal{Z}^{*,Y^1}),\\ &\Leftrightarrow z = \mathcal{Z}^{*,Y^1}, \exists \mathcal{X} \in \xi_{\mathfrak{T}(X)}, \exists \mathcal{Y} \in Y^1 \setminus \xi_{\mathfrak{T}(X)},\\ &\quad \text{s.t. } \mathcal{Z}^{*,Y^1} \in \alpha_{\mathfrak{CC}(Y)}(\mathcal{X}^{*,Y^1}) \cap \alpha_{\mathfrak{CC}(Y)}(\mathcal{Y}^{*,Y^1}), \end{aligned}$$

which is equivalent to:

$$z \in \alpha_{\mathfrak{CC}(Y)}(\xi_{\mathfrak{T}(X)}^{*,Y^1}) \cap \alpha_{\mathfrak{CC}(Y)}((Y^1 \setminus \xi_{\mathfrak{T}(X)})^{*,Y^1}).$$

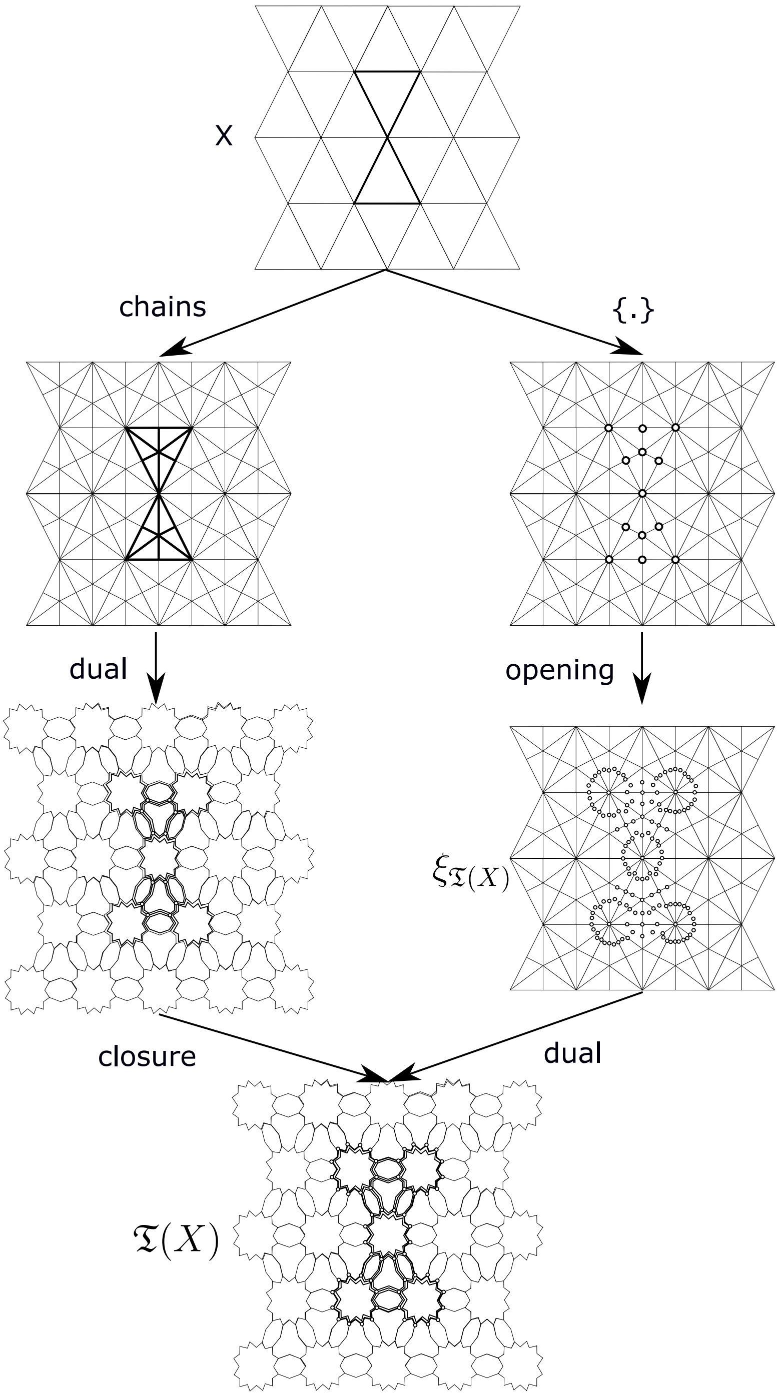

Fig. 32: Starting from the simplicial complex X resulting from the closure of two triangles sharing a vertex in a 2-surface $|Y|$, we compute on the left side the closure in $\mathfrak{CC}(Y)$ of the complex $\mathfrak{CC}(X|Y)$ and on the right side the dual of the set $\xi_{\mathfrak{T}(X)}$. Both lead to $\mathfrak{T}(X)$.

Now, by bijectivity of the duality (see Remark 2), we know that the complement of the dual is equal to the dual of the complement. Then,

$$(Y^1 \setminus \xi_{\mathfrak{T}(X)})^{*,Y^1} = \mathfrak{CC}(Y) \setminus \xi^{*,Y^1}_{\mathfrak{T}(X)},$$

and then we obtain that $z \in \xi^{*,Y^1}_{\mathfrak{N}}$ is equivalent to:

$$z \in \alpha_{\mathfrak{CC}(Y)}(\xi^{*,Y^1}_{\mathfrak{T}(X)}) \cap \alpha_{\mathfrak{CC}(Y)}(\mathfrak{CC}(Y) \setminus \xi^{*,Y^1}_{\mathfrak{T}(X)}),$$

equal by the first proven equality to:

$$z \in \alpha_{\mathfrak{CC}(Y)}(\mathfrak{T}(X)) \cap \alpha_{\mathfrak{CC}(Y)}(\mathfrak{CC}(Y) \setminus \mathfrak{T}(X)),$$

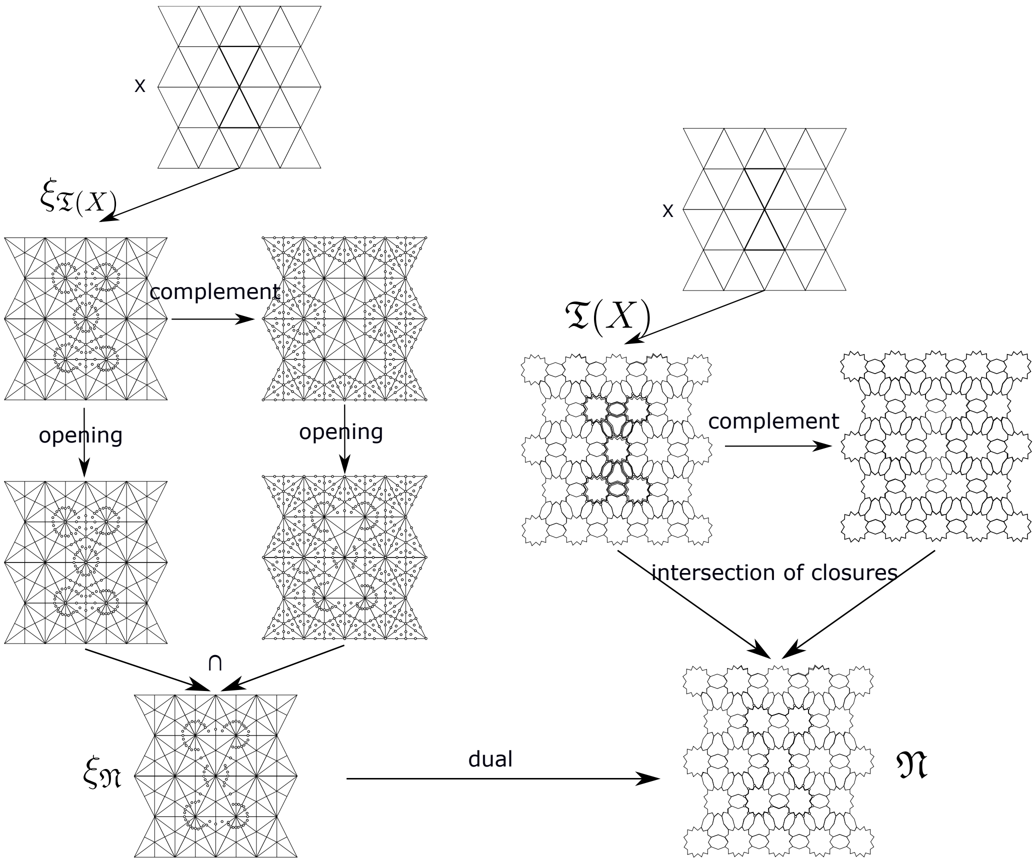

Fig. 33: Starting from the same posets as in Figure 32, we compute on the left side $\xi_{\mathfrak{N}}$ and on the right side the combinatorial boundary of $\mathfrak{T}(X)$. We finally obtain that $\xi_{\mathfrak{N}}^{*,Y^1} = \mathfrak{N}$.

which means that z belongs to the boundary $\mathfrak{N}$ of $\mathfrak{T}(X)$. The proof of the second equality is done. □

Lemma 1. *The boundary $\mathfrak{N}$ of $\mathfrak{T}(X)$ in $\mathfrak{CC}(Y)$ is equal to the dual of the frontier order of X in Y relative to Y^1.*

Proof: By Property 6, the dual $h^{*,Y^1} \in \mathfrak{CC}(Y)$ of $h \in Y^1$ belongs to $\mathfrak{N}$ iff $h \in \xi_{\mathfrak{N}}$, which is equivalent to $h \in \beta_{Y^1}(\xi_{\mathfrak{T}(X)})$ (P_a) and $h \in \beta_{Y^1}(Y^1 \setminus \xi_{\mathfrak{T}(X)})$ (P_b). However, (P_a) is equivalent to $h \in \bigcup_{p \in X} \beta_{Y^1}(\{p\})$ by transitivity of the operator β, which means that h belongs to the opening of a vertex $\{p\}$ of X^1. Also, (P_b) is equivalent to saying that there exists some $q \in Y^1 \setminus \xi_{\mathfrak{T}(X)}$ such that $h \in \beta_{Y^1}(q)$, which means that h belongs to the opening of an element q which depends only on elements of $Y \setminus X$. Since Y^1 is a simplicial complex, this last property is then equivalent to saying that h contains an element $q \in Y \setminus X$. In other words, h^{*,Y^1} belongs to the dual of the frontier order of X in Y relative to Y^1. □

Theorem 2. *Let $|Y|$ be an n-surface, $n \geq 2$, and $|X|$ be a suborder of $|Y|$ which is equal to the combinatorial closure of its n-faces. Then $|\mathfrak{T}(X)|$ is well-composed in the sense of Alexandrov.*

Proof: Let us first prove that $\mathfrak{T}(X)$ is equal to the closure of its n-faces. Because X^1 is a subdivision, it is a simplicial complex, and then it is closed under inclusion. The consequence is that for any $h \in X^1$, there exists $h_0 \in (X^1)_0$ such that $h_0 \in \alpha_{Y^1}(h)$. Using the dual operator, we obtain that for any $h \in (X^1)^{*,Y^1}$, there exists $h_n \in ((X^1)^{*,Y^1})_n$ such that $h \in \alpha_{\mathfrak{CC}(Y)}(h_n)$. In other words, any face of $\mathfrak{CC}(X|Y)$ is in the closure of $(\mathfrak{CC}(X|Y))_n$, that is,

$$\mathfrak{CC}(X|Y) \subseteq \alpha_{\mathfrak{CC}(Y)}\left((\mathfrak{CC}(X|Y))_n\right),$$

and then by applying $\alpha_{\mathfrak{CC}(Y)}$ on both sides, we obtain:

$$\mathfrak{T}(X) \subseteq \alpha_{\mathfrak{CC}(Y)}\left((\mathfrak{CC}(X|Y))_n\right).$$

Obviously,

$$\alpha_{\mathfrak{CC}(Y)}\left((\mathfrak{CC}(X|Y))_n\right) \subseteq \mathfrak{T}(X),$$

which means that we have

$$\mathfrak{T}(X) = \alpha_{\mathfrak{CC}(Y)}\left((\mathfrak{CC}(X|Y))_n\right).$$

Since $(\mathfrak{CC}(X|Y))_n = (\mathfrak{T}(X))_n$, then we obtain that $\mathfrak{T}(X)$ is equal to the closure of its n-faces.

Second, we can remark easily by Proposition 7 that $\mathfrak{CC}(Y)$ is a discrete n-surface since $|Y^1|$ is an n-surface.

Now, $\mathfrak{T}(X)$ is AWC iff its boundary $\mathfrak{N}$, subset of $\mathfrak{CC}(Y)$, is made of a disjoint union of discrete $(n-1)$-surfaces. However, by Lemma 1 $\mathfrak{N}$ is equal to the dual of the frontier order of X in Y relative to Y^1. Since $|Y^1|$ is an n-surface, then by Theorem 1 and Proposition 7, $\mathfrak{N}$ is a disjoint union of discrete $(n-1)$-surfaces, and then $\mathfrak{T}(X)$ is AWC. □

Proposition 1. *Let $|Y|$ be an n-surface, $n \geq 2$, and $|X|$ be a suborder of $|Y|$ which is equal to the combinatorial closure of its n-faces. Then, the transform $\mathfrak{T}(X)$ of X is path-connected iff X is path-connected. In other words, the mapping $X \to \mathfrak{T}(X)$ preserves path-connectivity.*

Proof: Since by Proposition 5, we can write:

$$\mathfrak{T}(X) = \left(\beta_{Y^1}(X^1)\right)^{*,Y^1},$$

then by Corollary 1, $\mathfrak{T}(X)$ is path-connected iff $\beta_{Y^1}(X^1)$ is path-connected.

Moreover, when X^1 is path-connected, $\beta_{Y^1}(X^1)$ is obviously path-connected. Conversely, let us assume that $\beta_{Y^1}(X^1)$ is path-connected. Then, for any pair of elements $(p,q) \in X^1 \times X^1$, there exists a path π_β of length $k \geq 0$ joining them in $\beta_{Y^1}(X^1)$. From this path, we can deduce a sequence of elements of X^1:

$$\text{Seq} := (\pi_\beta(i) \cap X)_{i \in [\![0,k]\!]}.$$

For any $i \in [\![0, k-1]\!]$, we know that either $\pi_\beta(i) \subseteq \pi_\beta(i+1)$ or $\pi_\beta(i+1) \subseteq \pi_\beta(i)$. Without loss of generality, let us assume that we have $\pi_\beta(i) \subseteq \pi_\beta(i+1)$, then $\pi_\beta(i) \cap X \subseteq \pi_\beta(i+1) \cap X$. Furthermore, since $\pi_\beta(i)$ and $\pi_\beta(i+1)$ belong to $\beta_{Y^1}(X^1)$, they verify $\pi_\beta(i) \cap X \neq \emptyset$ and $\pi_\beta(i+1) \cap X \neq \emptyset$: each one is "parent" of a chain of elements of X by construction, then their intersection with X is a chain of X, that is, an element of X^1. This way, Seq is a *path with possible successive duplicates*, that is, for any $i \in [\![0, k-1]\!]$,

$$\text{Seq}(i) \in \theta_{Y^1}(\text{Seq}(i+1)).$$

After having removed the successive duplicates in Seq, we obtain a new path π which joins p and q in X^1. Then X^1 is path-connected iff $\beta_{Y^1}(X^1)$ is path-connected.

Since X^1 is path-connected iff X is path-connected, the proof is done. □

5.4 Proofs of Subsection 3.3

Let us begin with the proof of the main property of $U_1^\flat$.

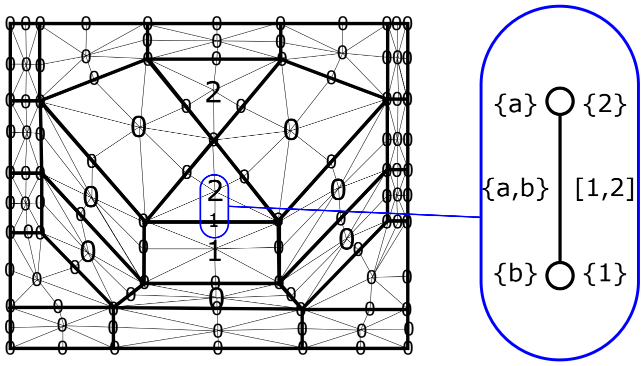

Fig. 34: The threshold sets of $U_1^\flat$ are full subcomplexes of Y^1: here, we observe that when we have $U_1^\flat(h) \rhd 0$ for some face $h \in Y^1$ like $h := \{a,b\}$, then for any face h' of h like $\{a\}$ and $\{b\}$, $U_1^\flat(h') \rhd 0$, which implies that $[U_1^\flat \rhd 0]$ is a simplicial complex. Conversely, when a face $h := \{a^1, \dots, a^k\} \in Y^1$ verifies $U_1^\flat(\{a^i\}) \rhd 0$ for any $i \in [\![1,k]\!]$, then $U_1^\flat(h) \rhd 0$, and then $[U_1^\flat \rhd 0]$ is full in Y^1.

Property 7. *Let us assume the hypotheses of Definition 6. Then for any $\lambda \in \mathbb{R}$, the threshold sets $[U_1^\flat \rhd \lambda]$ and $[U_1^\flat \lhd \lambda]$ are full subcomplexes of Y^1.*

Proof: Let us show first that for any $\lambda \in \mathbb{R}$, $[U_1^\flat \rhd \lambda]$ is a simplicial complex, and second that this same set is full in Y^1.

Let λ be a real value. For any face $h \in [U_1^\flat \rhd \lambda]$, $U_1^\flat(h) \rhd \lambda$, which means by definition of $U_1^\flat$ that we have for any $h' \in \alpha_{Y^1}(h)$:

$$\begin{aligned} U_1^\flat(h') &= \text{Span}\{u^\flat(v)\ ;\ v \in h'\}, \\ &\subseteq \text{Span}\{u^\flat(v)\ ;\ v \in h\}, \\ &\subseteq U_1^\flat(h), \end{aligned}$$

and then $U_1^\flat(h') \rhd \lambda$.

Now, we have to show that the intersection $p \cap q$ of two elements $p, q \in [U_1^\flat \rhd \lambda]$ verifying $p \cap q \neq \emptyset$ is an element of $[U_1^\flat \rhd \lambda]$. Since p and q belong to the simplicial complex Y^1, $r := p \cap q$ belongs to Y^1. Furthermore, we have $U_1^\flat(r) \subseteq U_1^\flat(p) \rhd \lambda$, and then $U_1^\flat(r) \rhd \lambda$, which means that r is a simplex of $[U_1^\flat \rhd \lambda]$. Then $[U_1^\flat \rhd \lambda]$ is a simplicial complex.

Now, let us show that $[U_1^\flat \rhd \lambda]$ is full in Y^1. Let $h := \{a^1, \dots, a^k\}$ be some face of Y^1, and let us assume that the vertices $\{a^1\}, \dots, \{a^k\}$ belong to $[U_1^\flat \rhd \lambda]$. Then it follows that for any $i \in [\![1,k]\!]$, $U_1^\flat(\{a^i\}) = \{u^\flat(a^i)\} \rhd \lambda$, and then $U_1^\flat(h) \rhd \lambda$. This means that $h \in [U_1^\flat \rhd \lambda]$ and then this set is full in Y^1.

The reasoning for the lower threshold sets is similar. The proof is done. □

Property 2. *Let us assume the hypotheses of Definition 7, the span-valued map $U' : \mathfrak{CC}(Y) \to \mathbb{I}_\mathbb{R}$ is a span-valued, in-between interpolation of U.*

Proof: For any $p \in Y_n$, we have:

$$U'(\{p\}^{*,Y^1}) = U_1^\flat(\{p\}) = \{u^\flat(p)\} = U(p).$$

Furthermore, by Property 1, $u^\flat$ verifies for any $k \in [\![0, n-1]\!]$ and any $p \in Y_k$:

$$u^\flat(p) \in \text{Span}\left\{u^\flat(q) \ ; \ q \in (\beta_Y(p))_{k+1}\right\},$$

then $U_1^\flat$ verifies:

$$\begin{aligned} U_1^\flat(\{p\}) &= \{u^\flat(p)\}, \\ &\subseteq \text{Span}\left\{u^\flat(q) \ ; \ q \in (\beta_Y(p))_{k+1}\right\}, \\ &\subseteq \text{Span}\left\{U_1^\flat(\{q\}) \ ; \ q \in (\beta_Y(p))_{k+1}\right\}. \end{aligned}$$

Moreover U' verifies:

$$\begin{aligned} U'(\{p\}^{*,Y^1}) &= U_1^\flat(\{p\}), \\ &\subseteq \text{Span}\left\{U_1^\flat(q) \ ; \ q \in (\beta_Y(p))_{k+1}\right\}, \\ &\subseteq \text{Span}\left\{U'(\{q\}^{*,Y^1}) \ ; \ q \in (\beta_Y(p))_{k+1}\right\}, \end{aligned}$$

which concludes the proof. □

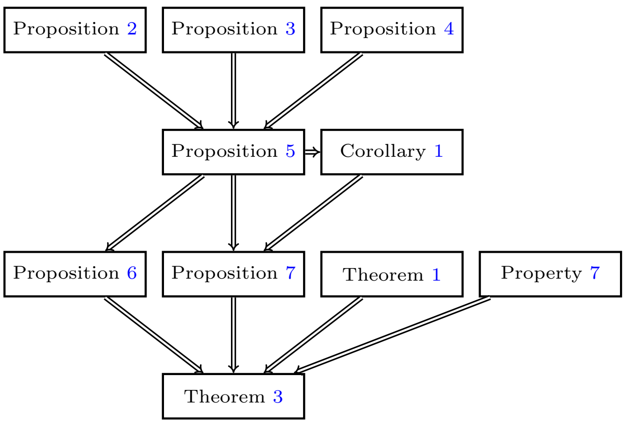

Fig. 35: The summary of the proof that U' is AWC.

Note that we summarized the proof of the following theorem in Figure 35.

Theorem 3. *Let us assume the hypotheses of Definition 7, the span-valued map $U' : \mathfrak{CC}(Y) \to \mathbb{I}_\mathbb{R}$ is well-composed in the sense of Alexandrov for any $n \geq 2$.*

Proof: Let λ be any real value. Let us prove that $[U' \rhd \lambda]$ is AWC. Note that the reasoning is exactly the same for $[U' \lhd \lambda]$. We want then to prove that $\text{bd}([U' \rhd \lambda], \mathfrak{CC}(Y))$ is made of a disjoint union of $(n-1)$-surfaces or empty. The set $[U' \rhd \lambda] \subseteq \mathfrak{CC}(Y)$ is the dual of $[U_1^\flat \rhd \lambda] \subseteq Y^1$ by construction of U', and then $\text{bd}([U' \rhd \lambda], \mathfrak{CC}(Y))$ is in fact the boundary of the dual of $[U_1^\flat \rhd \lambda]$. Since $[U_1^\flat \rhd \lambda]$ is a full subcomplex of Y^1 (see Property 7) and finite because Y^1 is finite, we can apply Proposition 6 to obtain that $\text{bd}([U' \rhd \lambda], \mathfrak{CC}(Y))$ is equal to the dual of the frontier order of $\Lambda_{[U_1^\flat \rhd \lambda]}$ in Λ_{Y^1} relative to Y^1. Note now that Y^1 is an n-surface. Then, three cases are possible:

- When the support of $[U_1^\flat \rhd \lambda]$ is empty, then $[U' \rhd \lambda]$ is empty, then its boundary is empty,
- When the support of $[U_1^\flat \rhd \lambda]$ is Y, then $[U' \rhd \lambda]$ is $\mathfrak{CC}(Y)$, then its boundary is empty,
- When the support of $[U_1^\flat \rhd \lambda]$ is a non-empty, strict subset of the support of Y^1, by Theorem 1 we obtain that the frontier order of $\Lambda_{[U_1^\flat \rhd \lambda]}$ in Λ_{Y^1} relative to Y^1 is a disjoint union of discrete $(n-1)$-surfaces, and then so does its dual $\text{bd}([U' \rhd \lambda], \mathfrak{CC}(Y))$ by Proposition 7.

Then $[U' \rhd \lambda]$ is AWC. □

6 Conclusion

In this paper, we showed how it is possible to make well-composed in the sense of Alexandrov a gray-level image defined on a subset of a discrete surface, assuming that the initial domain is a closure of a set of n-faces. This approach leads to a self-dual representation whose shapes make a hierarchy which can be considered as the tree of shapes of the initial image. Thanks to this hierarchy, we can proceed to image filtering and image segmentation.

Compared to our n-D topological reparation in [12] of gray-level images, the new method works on discrete surfaces and not only cubical grids, preserves the boundaries and is self-dual.

However, in our approach, we made the assumption that the initial domain was a closure of some set of n-faces. It would be interesting to investigate the dual approach: how can we "interpolate" a gray-level image defined on a subset of vertices of a discrete surface in such a way that the frontier orders that we obtain for each threshold make a tree together?

Also, since we proved that our approach leads to a representation which is "similar" to the initial gray-level image in the sense that boundaries are preserved wherever it is possible, an interesting question would be to measure a bound on the Hausdorff distance between the boundaries of the initial image and its representation.

Another relevant track could be to study how the *topological simplification* [24] of Edelsbrunner *et al.* is related to *morphological filtering* [45], since they are both based on the merging of some nodes corresponding to critical points to simplify an image.

Acknowledgment: We would like to acknowledge the time and effort devoted by the reviewers, which greatly improved the quality of our paper.

7 Appendix

In this section, we prove some remarkable properties specific to the framework of this paper.

Lemma 2. *Let X be a suborder of a poset Y of rank $n \geq 0$, with $X = \cup_{x \in X_n} \alpha_Y(x)$. Then, we have:*

$$\alpha_Y(Y \setminus X) \sqcup \mathrm{Int}_Y(X) = Y.$$

Proof: Let us first prove that the union is disjoint. Let us assume that there exists some $p \in \alpha_Y(Y \setminus X) \cap \mathrm{Int}(X)$. Since $p \in \alpha_Y(Y \setminus X)$, there exists some $z \in Y \setminus X$ such that $p \in \alpha_Y(z)$, that is, $\beta_Y(z) \subseteq \beta_Y(p)$. Besides, $Y \setminus X$ is open since X is closed, then $\beta_Y(z) \subseteq Y \setminus X$. However, $p \in \mathrm{Int}_Y(X)$ implies that $\beta_Y(p) \subseteq X$ and then:

$$\beta_Y(z) \subseteq \beta_Y(p) \subseteq X,$$

which contradicts $\beta_Y(z) \subseteq Y \setminus X$.

Let us now prove that the union is equal to Y. The fact that $\alpha_Y(Y \setminus X) \sqcup \mathrm{Int}_Y(X) \subseteq Y$ is obvious. Now, let us prove the converse inclusion. Let h be a face of Y. Two cases are possible:

- either $\beta_Y(h) \subseteq X$, then $h \in \mathrm{Int}_Y(X)$,
- or $\beta_Y(h) \not\subseteq X$, then $\beta_Y(h) \cap (Y \setminus X) \neq \emptyset$, and then there exists some $p \in \beta_Y(h) \cap (Y \setminus X)$; that is, $h \in \alpha_Y(p)$ and $p \in Y \setminus X$. In other words, $h \in \alpha_Y(Y \setminus X)$.

The proof is done. □

Proposition 8. *Let X be a suborder of a poset Y of rank $n \geq 0$, with $X = \cup_{x \in X_n} \alpha_Y(x)$. Then the topological boundary:*

$$\alpha_Y(X) \setminus \mathrm{Int}_Y(X)$$

of X in Y is equal to the combinatorial boundary:

$$\alpha_Y(X) \cap \alpha_Y(Y \setminus X)$$

of X in Y.

Proof: This proposition follows directly from Lemma 2. □

Property 8. *The set-valued map $U_{\mathrm{USC}} : Y \to \mathbb{I}_{\mathbb{R}}$ is upper semi-continuous.*

Proof: The fact that U_{USC} is USC relies on the fact that for any $z \in Y$ and for any $z' \in \beta_Y(z)$:

$$U_{\mathrm{USC}}(z') \subseteq U_{\mathrm{USC}}(z).$$

Indeed, let z be an element of Y and let z' be an element of $\beta_Y(z)$:

- when $z \in \mathrm{bd}(X, Y)$:

  $$U_{\mathrm{USC}}(z) = \mathrm{Span}\{\mathfrak{M}, \mathrm{Span}\{U(q)\ ;\ q \in \beta_Y(z) \cap X_n\}\},$$

  - when $z' \in \beta_Y(z) \cap \mathrm{bd}(X, Y)$, $U_{\mathrm{USC}}(z')$ is equal to:

    $$\mathrm{Span}\{\mathfrak{M}, \mathrm{Span}\{U(q)\ ;\ q \in \beta_Y(z') \cap X_n\}\},$$

    which is included in $U_{\mathrm{USC}}(z)$ since $\beta_Y(z') \subseteq \beta_Y(z)$,
  - when $z' \in \beta_Y(z)$ such that $z' \notin \mathrm{bd}(X, Y)$, either $z' \in X \setminus \mathrm{bd}(X, Y)$ (which is an open set), which implies $\beta_Y(z') \subseteq X$ and $U_{\mathrm{USC}}(z') = U(z') \subseteq U_{\mathrm{USC}}(z)$, or $z' \in Y \setminus X$ (which is an open set since it is equal to $Y \setminus \mathrm{bd}(X, Y)$), which implies $\beta_Y(z') \subseteq Y \setminus X$ and $U_{\mathrm{USC}}(z') = \{\mathfrak{M}\} = U(z') \subseteq U_{\mathrm{USC}}(z)$,
- when $z \in X \setminus \mathrm{bd}(X, Y)$, then $\beta_Y(z) \subseteq X$, which means that $z' \in \beta_Y(z)$ belongs to X, and then:

  $$U_{\mathrm{USC}}(z') = U(z') \subseteq U(z),$$
- when $z \in Y \setminus X$, then $\beta_Y(z) \subseteq Y \setminus X$, then for any $z' \in \beta_Y(z)$, $U_{\mathrm{USC}}(z') = \{\mathfrak{M}\} = U_{\mathrm{USC}}(z)$.

This concludes the proof. □

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