A Self-Dual and Digitally Well-Composed Representation of *n*D Digital Images

Nicolas Boutry[1,2] Thierry Géraud[1] Laurent Najman[2]

nicolas.boutry@lrde.epita.fr

EPITA Research and Development Laboratory, LRDE, France

Université Paris-Est, LIGM, Équipe A3SI, ESIEE, France

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Abstract

We propose here a new method to represent nD gray-valued images.

Mainly, this new representation:

- is *similar* to the initial image (in-between interpolation)
- treats bright and dark components in the same way (self-duality)
- get rid of the topological paradox of connectivities (well-composedness)
- is the first to work in nD (similar methods fail since $n \geq 3$)

Self-dual-operators applied on this representation are purely self-dual: the results are the same whatever the chosen connectivities.

Outline

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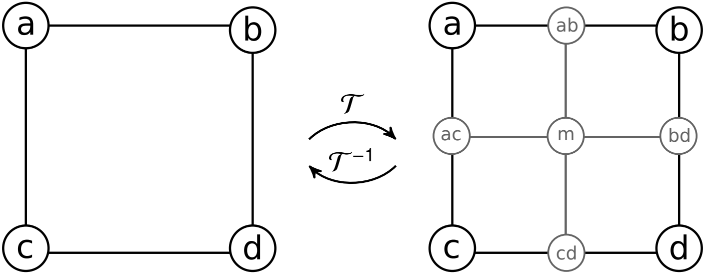
A Representation *Similar* to the Initial Image

u given, we compute the new representation $u' = \mathcal{T}(u)$ by interpolation.

The interpolation method is in-between (no new extrema):

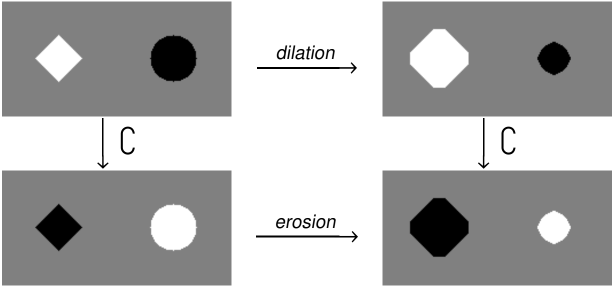


The new representation is *similar* to the initial image.

A Self-dual Representation (1)

Two operators Φ and Ψ are said dual iff $\complement \circ \Phi = \Psi \circ \complement$.

Dual operators are everywhere in MM: dilation/erosion, opening/closing, thickening/thinning,...

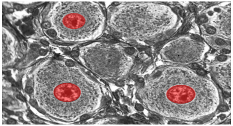


Φ treats the bright components as Ψ treats the dark components and vice-versa.

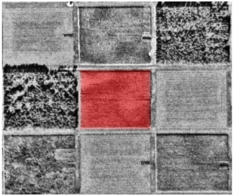
A Self-dual Representation (2)

Dual operators needs to know *a priori* the contrast of the “object”.

Dark nuclei over bright cytosols:



Crop fields with varying contrast:

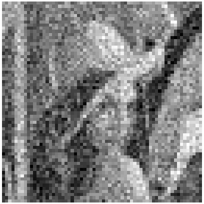


Dual operators fail when the constrast varies too much.

A Self-dual Representation (3)

An operator φ is said self-dual iff $\varphi \circ \complement = \complement \circ \varphi$.

Example (grain filtering):



u

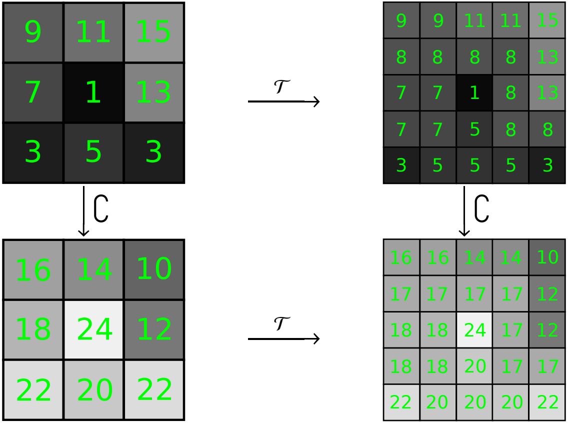
$\gamma(u)$

$\phi(u)$

$Center(u)$

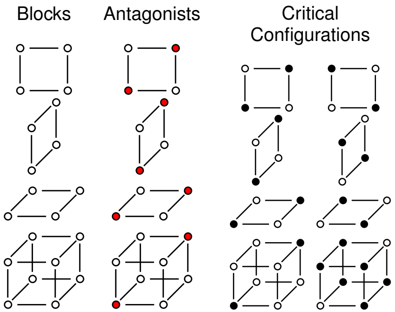
A self-dual operator treats symmetrically bright and dark components.

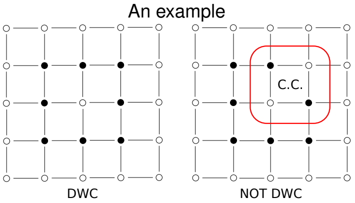
A Self-dual Representation (4)



Our transform $\mathcal{T}$ is self-dual: $\mathcal{T}(-u) = -\mathcal{T}(u)$.

A Digitally Well-Composed Representation (1)



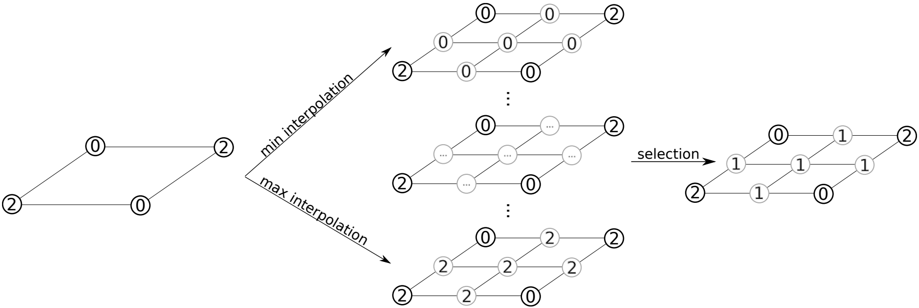


A nD set X is said DWC iff X and X^c does not contain any CC.

A nD image u is said DWC iff $\forall \lambda \in \mathbb{R}$, $\chi_\lambda(u)$ and $\chi_\lambda^c(u)$ are DWC.

A Digitally Well-Composed Representation (2)

Let $\mathcal{G}(u)$ be the collection of images *between* u_{min} and u_{max}.

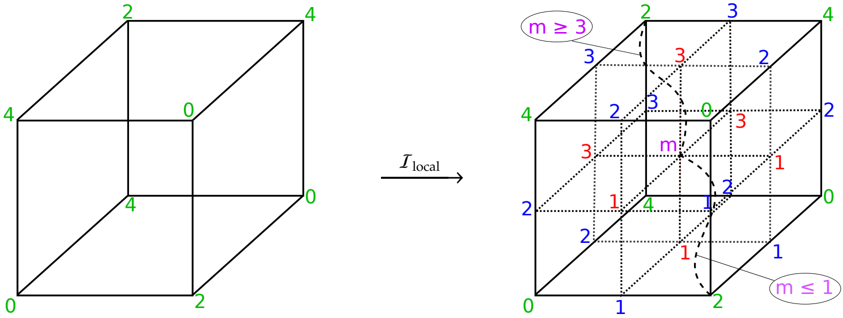


The selection process in our algorithm chooses in a self-dual way a DWC image which interpolates u (quasi-linear time).

Our representation is digitally well-composed.

A *n*D representation (1)

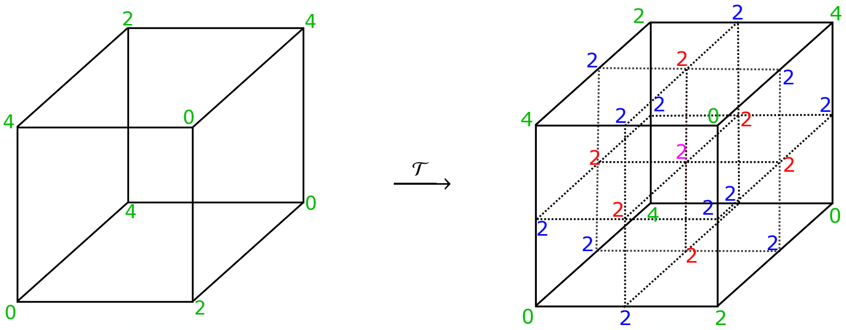
Interpolating this image into a DWC image in a self-dual way leads to an incompatible set of constraints (DGCI'2014):



<u>Local</u> interpolation methods fail in *n*D ($n \geq 3$).

A *n*D representation (2)

Using a front-propagation approach, we get rid of the local criterion.



> Our method succeeds to interpolate *n*D images.

Summary

The new representation:

1. is an interpolation
2. is in-between
3. is self-dual
4. is digitally well-composed
5. works in *n*D
6. is invariant by $\pi/2$ rotations and symmetries (not developed here)
7. is deterministic (not developed here)
8. is computed in quasi-linear time

Outline

Self-duality is impure

In digital topology, a self-dual operator φ uses dual connectivities (c_α, c_β) for upper/lower threshold sets.

We obtain the Self-Duality Equation by switching the connectivities:

$$\begin{array}{ccc} u & \xrightarrow{\varphi_{(\boxed{c_\alpha, c_\beta})}} & \varphi_{(c_\alpha,c_\beta)}(u) \\ \text{complementation} \downarrow & & \downarrow \text{complementation} \\ \complement u & \xrightarrow{\varphi_{(\boxed{c_\beta, c_\alpha})}} & \complement\, \varphi_{(c_\alpha,c_\beta)}(u) = \varphi_{(c_\beta,c_\alpha)}(\complement u) \end{array}$$

Self-duality is said impure.

DWCness to Pure Self-duality (1)

Property (DWC $\Rightarrow$ EWC)

Let u be a DWC image. Then the set of connected components of $\chi_\lambda(u)$ (resp. $\chi^c_\lambda(u)$) does not depend on the chosen connectivity (EWCness).

The connectivities of a DWC image are said equivalent.

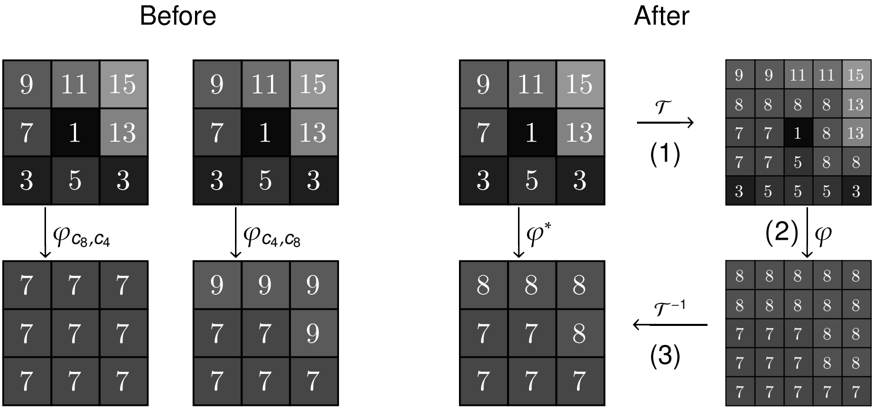
Consequence on the Self-Duality Equation:

$$\complement(\varphi_{(c_\alpha, c_\beta)}(u)) \overset{SD}{=} \varphi_{(c_\beta, c_\alpha)}(\complement(u)) \overset{DWC}{=} \varphi_{(c_\alpha, c_\beta)}(\complement(u))$$

DWCness + Self-Duality $\Rightarrow$ Pure Self-Duality

DWCness to Pure Self-duality (2)

With $\varphi = GF$ (grain filtering):



The results do not depend on the chosen connectivities (Unicity).

Outline

Contributions

Practical contributions:

- We provide the first method to compute self-dual digitally well-composed representations of *n*D images.
- We give an access to pure self-duality using any morphological self-dual operator.

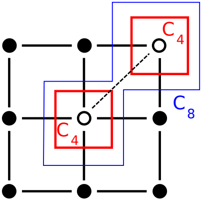
Theoretical contributions:

- A generalization of *n*D (digital) well-composedness (sets/functions)
- An extension of digitally well-composedness to interval-valued maps (not developed here)
- A characterization of digitally well-composed images (not developed here)
- A proof that DWCness implies the EWCness (in *n*D)

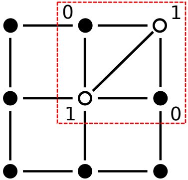
Well-Composedness: State-Of-The-Art

Thanks for your attention! ... Questions? :D

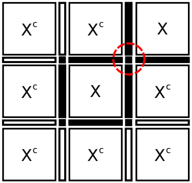
based on:	connectivities		C.C.'s		*n*-surfaces		manifolds
2D case:	**EWC**	$\Leftrightarrow$	DWC	$\Longleftrightarrow$	AWC	$\Longleftrightarrow$	CWC
3D case:	EWC	$\Leftarrow$	DWC	$\Longleftrightarrow$	AWC	$\Longleftrightarrow$	**CWC**
*n*D case:	$\boxed{\text{EWC}}$	$\Leftarrow$	**DWC**	$\Leftarrow ? \Rightarrow$	**AWC**	$\Leftarrow ? \Rightarrow$	**CWC**



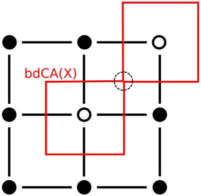
not EWC



not DWC



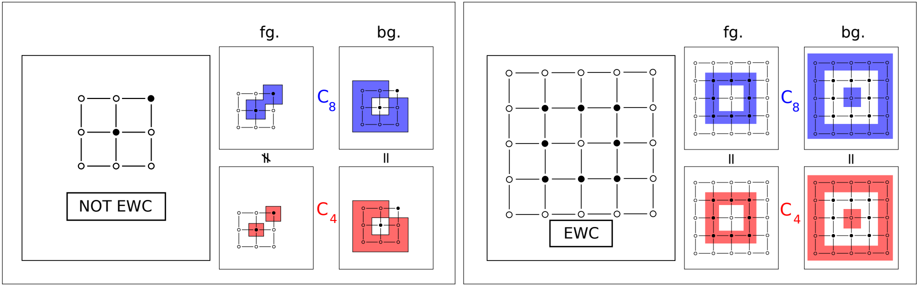
not AWC



not CWC

EWCness

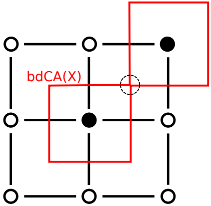
A set X is said EWC iff the set of $2n$-components of X (resp. X^c) is equal to the set of $(3^n - 1)$-components of X (resp. X^c).



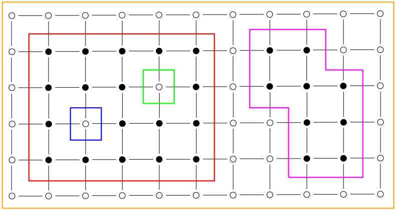
An image u is said EWC iff for any $\lambda \in \mathbb{R}$, $\chi_\lambda(u)$ and $\chi^c_\lambda(u)$ are EWC.

CWCness

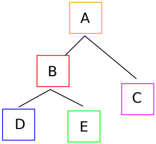
A set X is said C-well-composed or CWC iff the components of the boundary of its continuous analog are $(n-1)$-manifolds.



not CWC



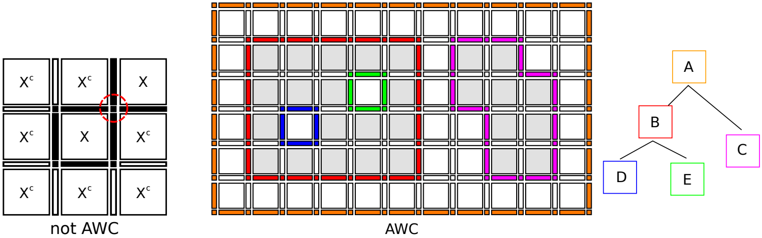
CWC



An image u is said CWC iff $\forall \lambda \in \mathbb{R}$, $\chi_\lambda(u)$ and $\chi^c_\lambda(u)$ are CWC.

AWCness

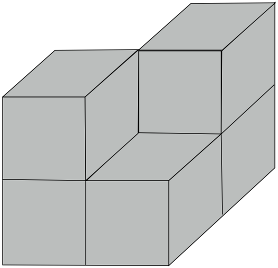
A set X is said Alexandrov-well-composed or AWC iff the components of its boundary are $(n-1)$-surfaces.



An image u is said AWC iff $\forall \lambda \in \mathbb{R}$, $\chi_\lambda(u)$ and $\chi^c_\lambda(u)$ are AWC.

EWC $\not\Rightarrow$ DWC

DWCness implies EWCness but the converse is not true:



Since $n \geq 3$, the equivalence *EWC* $\Leftrightarrow$ *DWC* does not hold anymore.

How the algorithm works

We start from this image:

9	11	15
7	1	13
3	5	3

How the algorithm works

We immerse it into this interval-valued image:

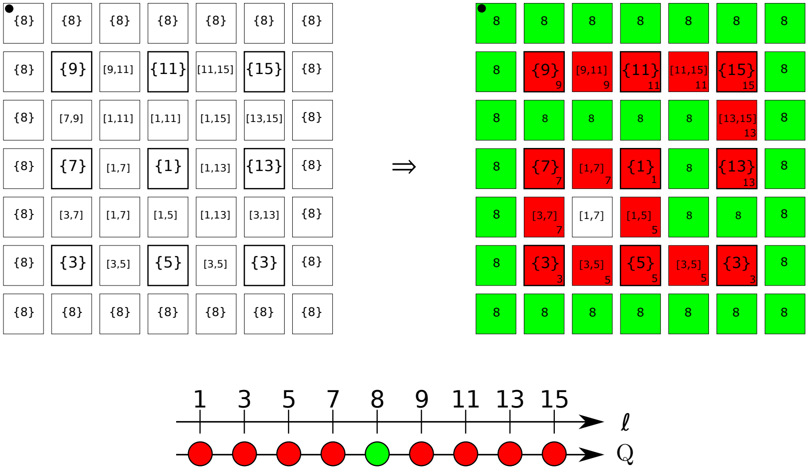
9	11	15
7	1	13
3	5	3

$\Rightarrow$

{8}	{8}	{8}	{8}	{8}	{8}	{8}
{8}	{9}	[9,11]	{11}	[11,15]	{15}	{8}
{8}	[7,9]	[1,11]	[1,11]	[1,15]	[13,15]	{8}
{8}	{7}	[1,7]	{1}	[1,13]	{13}	{8}
{8}	[3,7]	[1,7]	[1,5]	[1,13]	[3,13]	{8}
{8}	{3}	[3,5]	{5}	[3,5]	{3}	{8}
{8}	{8}	{8}	{8}	{8}	{8}	{8}

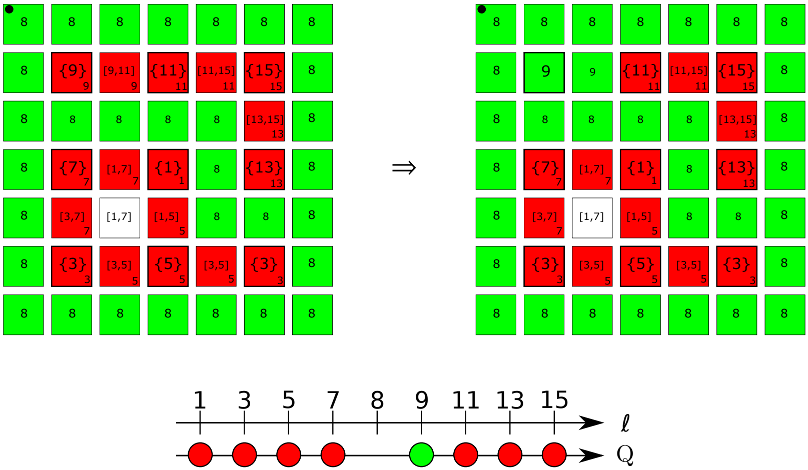
How the algorithm works

... and we start the propagation ($\ell = 8$):



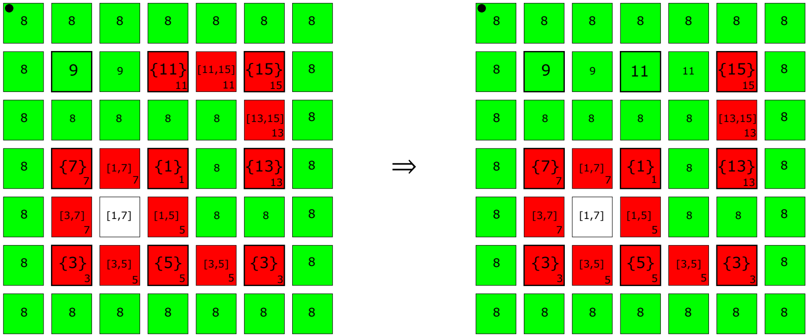
How the algorithm works

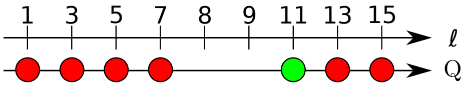
... and we continue until the hierarchical queue is empty ($\ell = 9$):



How the algorithm works

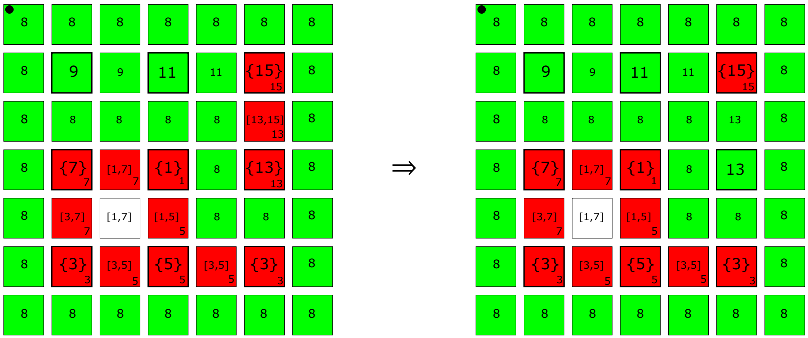
... and we continue until the hierarchical queue is empty ($\ell = 11$):

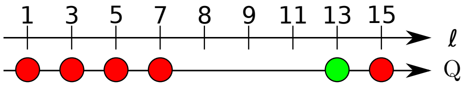




How the algorithm works

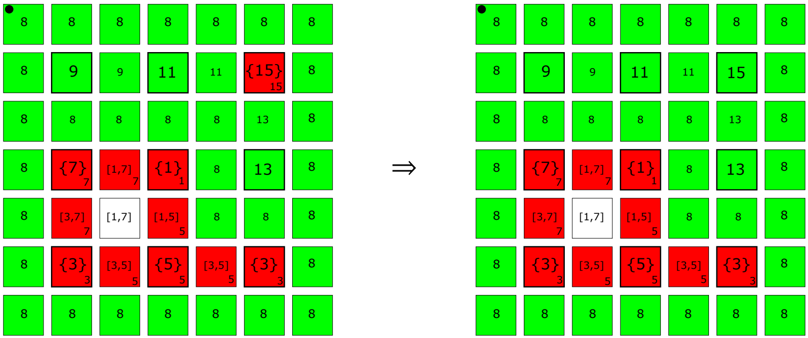
... and we continue until the hierarchical queue is empty ($\ell = 13$):

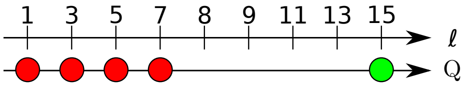




How the algorithm works

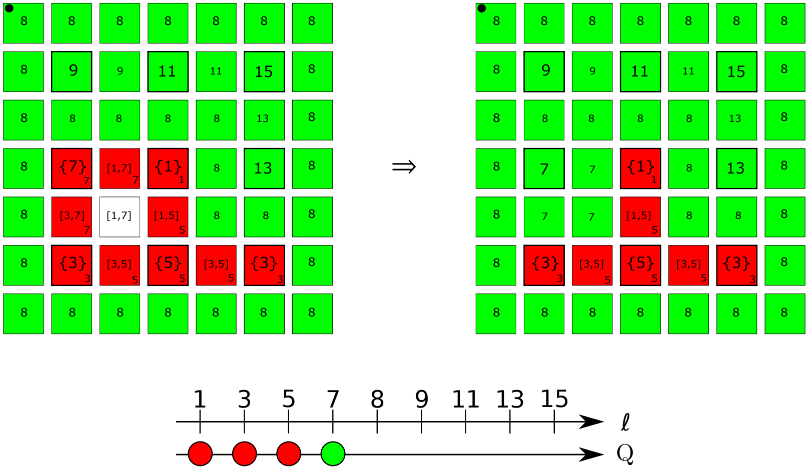
... and we continue until the hierarchical queue is empty ($\ell = 15$):





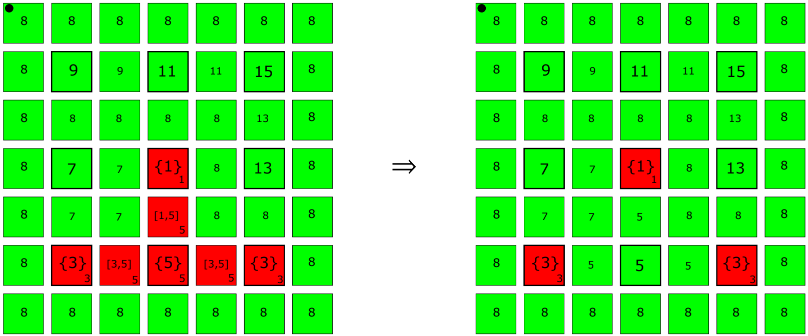
How the algorithm works

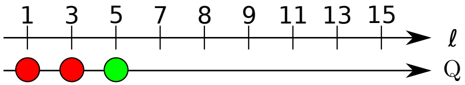
... and we continue until the hierarchical queue is empty ($\ell = 7$):



How the algorithm works

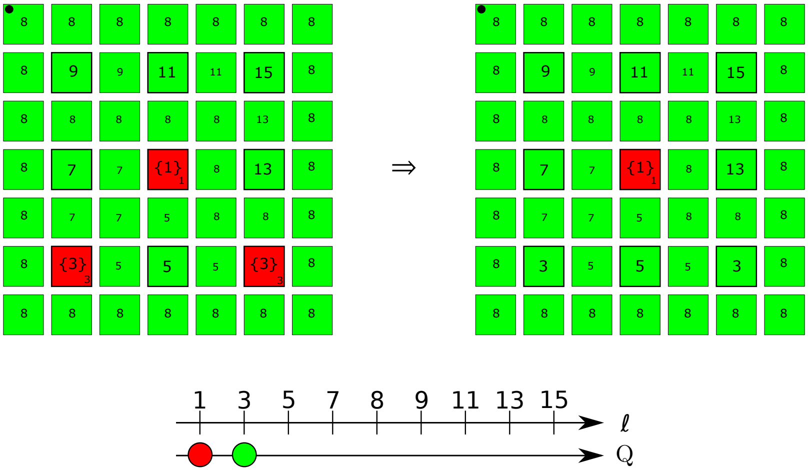
... and we continue until the hierarchical queue is empty ($\ell = 5$):





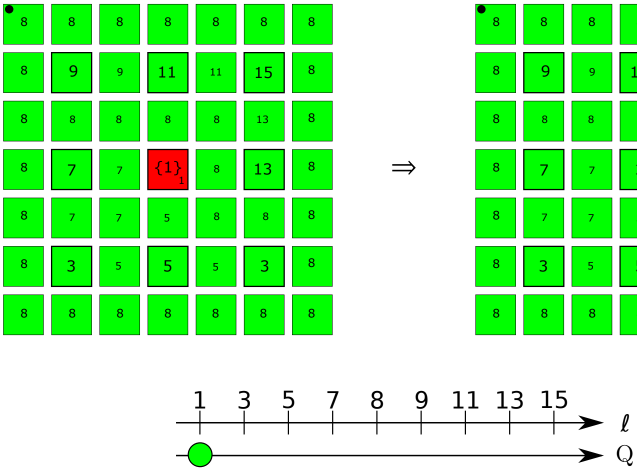
How the algorithm works

... and we continue until the hierarchical queue is empty ($\ell = 3$):



How the algorithm works

... and we continue until the hierarchical queue is empty ($\ell = 1$):



How the algorithm works

... and we obtain our new representation:

8	8	8	8	8	8	8
8	9	9	11	11	15	8
8	8	8	8	8	13	8
8	7	7	1	8	13	8
8	7	7	5	8	8	8
8	3	5	5	5	3	8
8	8	8	8	8	8	8

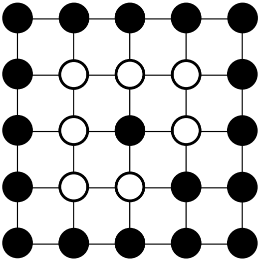
$\Longrightarrow$

9	9	11	11	15
8	8	8	8	13
7	7	1	8	13
7	7	5	8	8
3	5	5	5	3

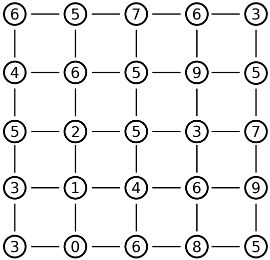
Underlying Graph(1)

Using 8-connectivity for foreground (black) and upper level sets, and 4-connectivity for background (white) and lower level sets:

Sets

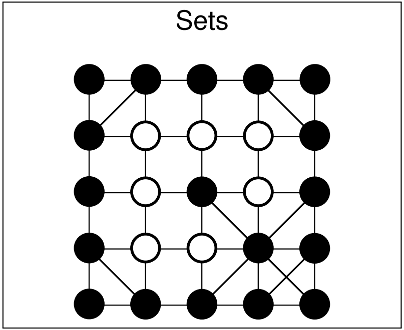


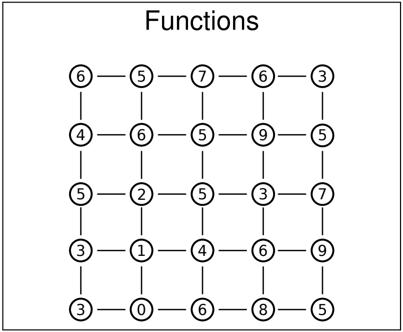
Functions



Underlying Graph(1)

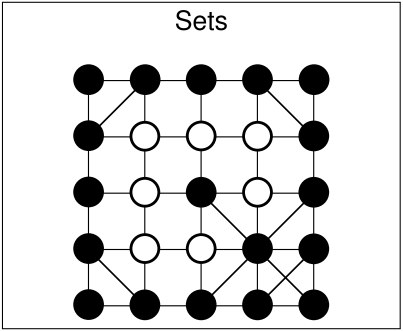
Using 8-connectivity for foreground (black) and upper level sets,
and 4-connectivity for background (white) and lower level sets:

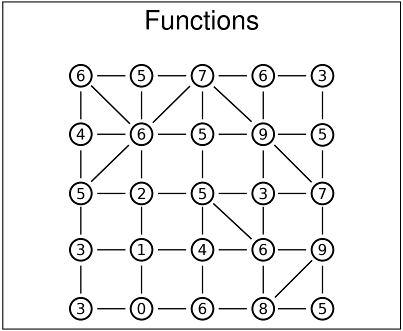




Underlying Graph(1)

Using 8-connectivity for foreground (black) and upper level sets, and 4-connectivity for background (white) and lower level sets:

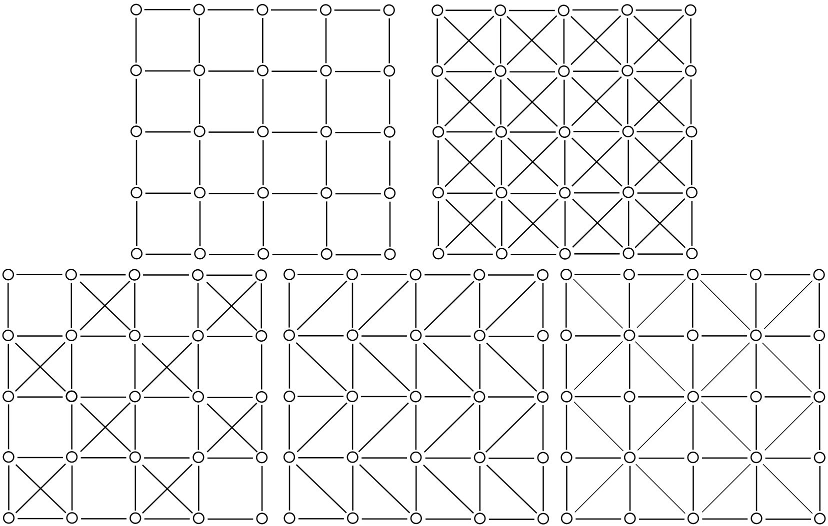




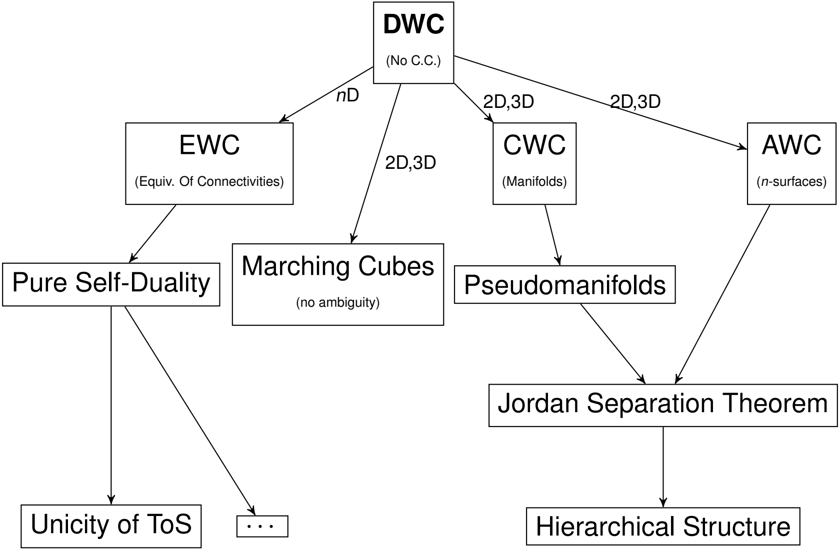
The underlying graph depends on the values of the image (not EWC).

Underlying Graph(2)

However, for EWC/DWC sets, all the connectivities are equivalent:



Summary


DWC
(No C.C.)
nD
2D,3D
2D,3D
2D,3D
EWC
(Equiv. Of Connectivities)
CWC
(Manifolds)
AWC
(n-surfaces)
Pure Self-Duality
Marching Cubes
(no ambiguity)
Pseudomanifolds
Jordan Separation Theorem
Unicity of ToS
...
Hierarchical Structure