Introducing the Dahu Pseudo-Distance

Que la montagne de pixels est belle. Jean Serrat.

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ISMM, Fontainebleau, France, May 2017

About image representations

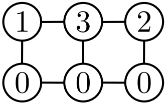
(mathematical morphology way of thinking)

topographical landscape

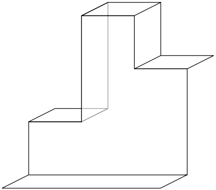
↑

1	3	2
0	0	0

a 2D array



a graph



a surface

↓

L.W. Najman and J. Cousty, "A graph-based mathematical morphology reader," *Pattern Recognition Letters*, vol. 47, pp. 3-17, Oct. 2014. [PDF]

The Minimum Barriere (MB) Distance

this talk is about a distance between points in gray-level images...

R. Strand, K.C. Ciesielski, F. Malmberg, and P.K. Saha, "The minimum barrier distance," *Computer Vision and Image Understanding*, vol. 117, pp. 429-437, 2013. [PDF]

K.C. Ciesielski, R. Strand, F. Malmberg, and P.K. Saha, "Efficient Algorithm for Finding the Exact Minimum Barrier Distance," *Computer Vision and Image Understanding*, vol. 123, pp. 53–64, 2014. [PDF]

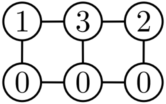
...and a variant of this distance, and its computation

The Minimum Barriere (MB) Distance

Barrier τ of a path π in an image u

Interval of gray-level values (dynamics of u) along a path:

$$\tau_u(\pi) \;=\; \max_{\pi_i \in \pi} u(\pi_i) \;-\; \min_{\pi_i \in \pi} u(\pi_i).$$

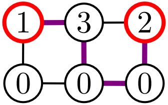


The Minimum Barriere (MB) Distance

Barrier τ of a path π in an image u

Interval of gray-level values (dynamics of u) along a path:

$$\tau_u(\pi) = \max_{\pi_i \in \pi} u(\pi_i) - \min_{\pi_i \in \pi} u(\pi_i).$$



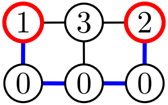
pink path values $= \langle 1, 3, 0, 0, 2 \rangle \rightsquigarrow$ interval $= [0, 3] \rightsquigarrow$ barrier $= 3$

The Minimum Barriere (MB) Distance

Barrier τ of a path π in an image u

Interval of gray-level values (dynamics of u) along a path:

$$\tau_u(\pi) \; = \; \max_{\pi_i \in \pi} \; u(\pi_i) \; - \; \min_{\pi_i \in \pi} \; u(\pi_i).$$



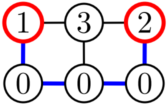
$$\text{blue path values} \; = \; \langle 1, 0, 0, 0, 2 \rangle \; \rightsquigarrow \; \text{interval} \; = \; [0, 2] \; \rightsquigarrow \; \text{barrier} \; = \; 2$$

The Minimum Barriere (MB) Distance

MB distance (MBD) between two points x and x'

MBD = minimum barrier (considering all paths) between these points:

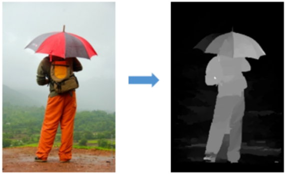
$$d_u^{\mathrm{MB}}(x, x') \;=\; \min_{\pi \in \Pi(x,x')} \tau_u(\pi).$$



$\rightsquigarrow$ distance $=$ minimum barrier $=$ 2

An important distance

- relying on function dynamics (so not a “classical” path-length distance)
- effective for segmentation tasks



J. Zhang, S. Sclaroff, Z. Lin, X. Shen, B. Price, and R. Mech, “Minimum barrier salient object detection at 80 FPS,” *in: Proc. of ICCV*, pp. 1404–1412, 2015. [PDF]

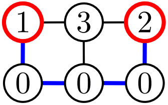
W.C. Tu, S. He, Q. Yang, and S.Y. Chien, “Real-time salient object detection with a minimum spanning tree,” *in: Proc. of IEEE CVPR*, pp. 2334–2342, 2016. [PDF]

J. Zhang, S. Sclaroff, “Exploiting Surroundedness for Saliency Detection: A Boolean Map Approach,” *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 38, num. 5, pp. 889–902, 2016. [PDF]

- and actually related to mathematical morphology...

The glitch!

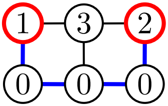
In the graph world:



the MB distance is **2**

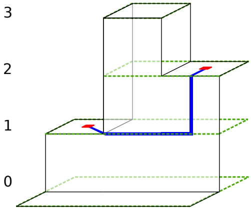
The glitch!

In the graph world:



the MB distance is **2**

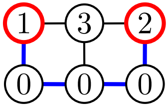
In the continuous world:



the MB distance should be **1**!

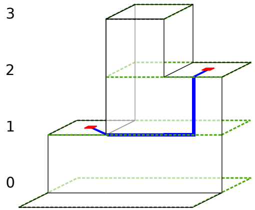
The glitch!

In the graph world:



the MB distance is **2**

In the continuous world:



the MB distance should be **1**!

$\Rightarrow$ **we need a new definition...**

The glitch!

to get a new definition...

a continuous representation of an image / surface is required...

A $\approx$ new representation!

Given a scalar image $u : \mathbb{Z}^n \to Y$, we use two tools:

- cubical complexes: $\mathbb{Z}^n$ is replaced by $\mathbb{H}^n$
- set-valued maps: Y is replaced by $\mathbb{I}_Y$

$\Rightarrow$ a *continuous*, yet *discrete*, representation of images

T. Géraud, E. Carlinet, S. Crozet, and L. Najman, "A quasi-linear algorithm to compute the tree of shapes of *n*-D images," *in: Proc. of ISMM*, LNCS, vol. 7883, pp. 98–110, Springer, 2013. [PDF]

L. Najman and T. Géraud, "Discrete set-valued continuity and interpolation," *in: Proc. of ISMM*, LNCS, vol. 7883, pp. 37–48, Springer, 2013. [PDF]

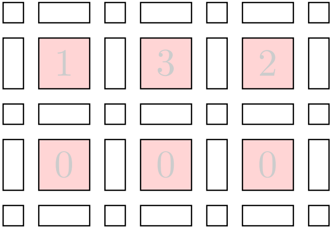
A both continuous and discrete representation

discrete point $x \in \mathbb{Z}^n$ $\rightsquigarrow$ n-face $h_x \in \mathbb{H}^n$

domain $\mathcal{D} \subset \mathbb{Z}^n$ $\rightsquigarrow$ $\mathcal{D}_H = cl(\{h_x;\ x \in \mathcal{D}\}) \subset \mathbb{H}^n$

1	3	2
0	0	0

from a scalar image u...



A both continuous and discrete representation

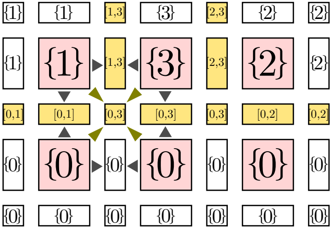
$$\text{discrete point } x \in \mathbb{Z}^n \quad \rightsquigarrow \quad n\text{-face } h_x \in \mathbb{H}^n$$

$$\text{domain } \mathcal{D} \subset \mathbb{Z}^n \quad \rightsquigarrow \quad \mathcal{D}_H = cl(\{h_x;\ x \in \mathcal{D}\}) \subset \mathbb{H}^n$$

$$\text{scalar image } u : \mathcal{D} \subset \mathbb{Z}^n \to Y \quad \rightsquigarrow \quad \text{interval-valued map } \widetilde{u} : \mathcal{D}_H \subset \mathbb{H}^n \to \mathbb{I}_Y$$

1	3	2
0	0	0

from a scalar image u...



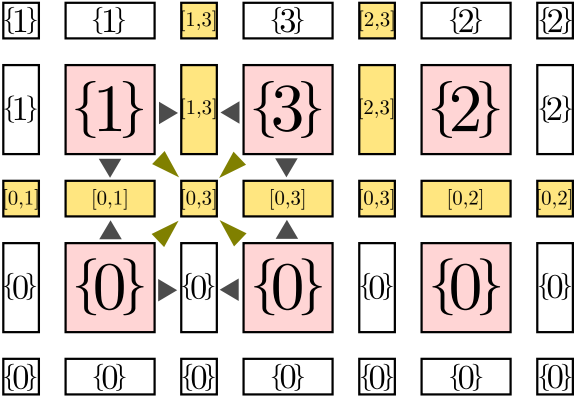
to an interval-valued image $\widetilde{u}$

We set:

$$\forall\, h \in \mathcal{D}_H,\ \ \widetilde{u}(h) \;=\; \operatorname{span}\{\, u(x);\ x \in \mathcal{D} \text{ and } h \subset h_x \,\}.$$

A both continuous and discrete representation

zoomed in:



$\widetilde{u}$

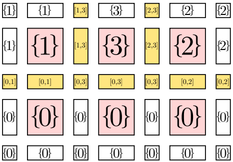
how huge!

A both continuous and discrete representation

1	3	2
0	0	0

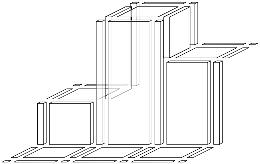
image u

$\rightsquigarrow$



set-valued image $\widetilde{u}$

=



$\widetilde{u}$ in 3D

A both continuous and discrete representation

1	3	2
0	0	0

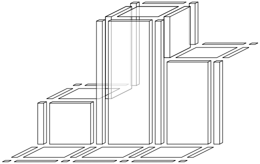
image u

$\rightsquigarrow$

{1}	{1}	[1,3]	{3}	[2,3]	{2}	{2}
{1}	{1}	[1,3]	{3}	[2,3]	{2}	{2}
[0,1]	[0,1]	[0,3]	[0,3]	[0,3]	[0,2]	[0,2]
{0}	{0}	{0}	{0}	{0}	{0}	{0}
{0}	{0}	{0}	{0}	{0}	{0}	{0}

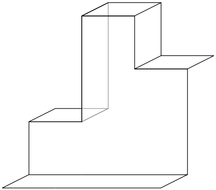
set-valued image $\widetilde{u}$

=



$\widetilde{u}$ in 3D

$\Updownarrow$



3D version of u in $\mathbb{R}^3$

continuity! $\Leftarrow$

A both continuous and discrete representation

we have a representation for the image surface

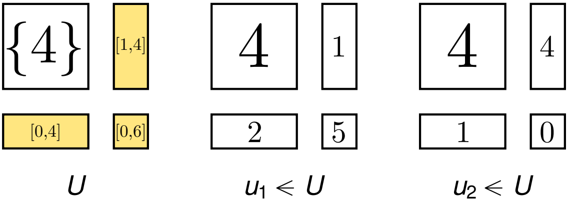
$\rightsquigarrow$ we want to express the "continuous" distance...

Notation

Inclusion

with u a scalar image, and U a set-valued image:

$$u \prec U \quad \Leftrightarrow \quad \forall\, x \in X,\;\; u(x) \in U(x)$$



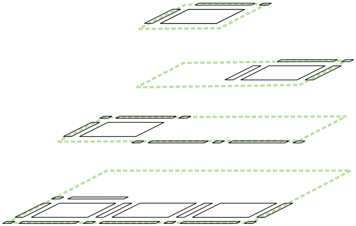
Searching for the continuous MB distance

{1}	{1}	[1,3]	{3}	[2,3]	{2}	{2}
{1}	{1}	[1,3]	{3}	[2,3]	{2}	{2}
[0,1]	[0,1]	[0,3]	[0,3]	[0,3]	[0,2]	[0,2]
{0}	{0}	{0}	{0}	{0}	{0}	{0}
{0}	{0}	{0}	{0}	{0}	{0}	{0}

interval-valued image $\widetilde{u}$

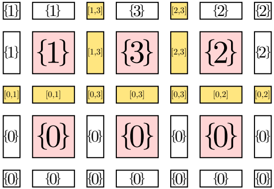
1	1	1	3	3	2	2
1	1	3	3	2	2	2
0	0	1	1	1	1	1
0	0	0	0	0	0	0
0	0	0	0	0	0	0

a scalar image $\overline{u} < \widetilde{u}$...

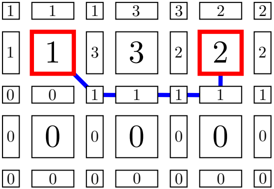


...and its 3D version

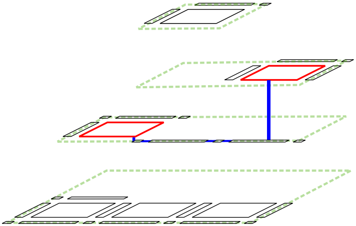
Searching for the continuous MB distance



interval-valued image $\widetilde{u}$

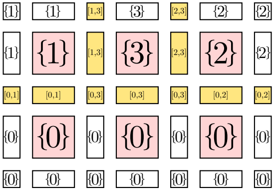


a minimal path in a $\overline{u} \lessdot \widetilde{u}$...

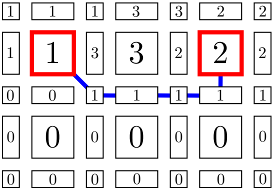


...and its 3D version

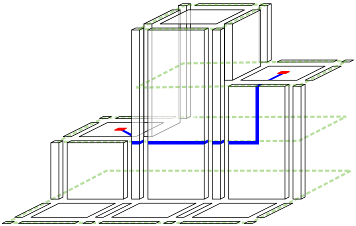
Searching for the continuous MB distance



interval-valued image $\widetilde{u}$



this path in this $\overline{u} \lessdot \widetilde{u}$...



...is a minimal path in $\widetilde{u}$

The Dahu distance

The Dahu distance:

$$D_u(x,x') \;=\; \min_{\overline{u} \leqslant \widetilde{u}} \; \underbrace{\min_{\pi \in \Pi(h_x, h_{x'})} \Big(\overbrace{\max_{\pi_i \in \pi} \overline{u}(\pi_i) \;-\; \min_{\pi_i \in \pi} \overline{u}(\pi_i)}^{\text{barrier } \tau_{\overline{u}}(\pi)} \Big)}_{\text{minimum barrier distance } d^{\mathrm{MB}}_{\overline{u}}(h_x, h_{x'})}$$

The Dahu distance

The Dahu distance:

$$D_u(x, x') \;=\; \min_{\overline{u} < \widetilde{u}} \; \underbrace{\min_{\pi \in \Pi(h_x, h_{x'})} \left(\overbrace{\max_{\pi_i \in \pi} \overline{u}(\pi_i) \;-\; \min_{\pi_i \in \pi} \overline{u}(\pi_i)}^{\text{barrier } \tau_{\overline{u}}(\pi)} \right)}_{\text{minimum barrier distance } d^{\mathrm{MB}}_{\overline{u}}(h_x, h_{x'})}$$

it looks like we have added an extra combinatorial complexity
w.r.t. the original MB distance...

Recap

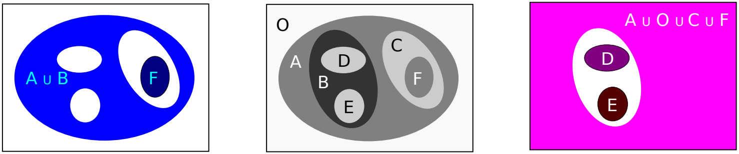
We have a combinatorial continuous-like def. of the MB distance...

...but it can be computed **exactly** and **efficiently** with:
the *morphological tree of shapes*!!!

The morphological tree of shapes (ToS)

With $\lambda \in Y$:

- lowel level sets: $[u < \lambda] \; = \; \{ \, x \in X; \; u(x) < \lambda \, \}$
- upper level sets: $[u \geq \lambda] \; = \; \{ \, x \in X; \; u(x) \geq \lambda \, \}$

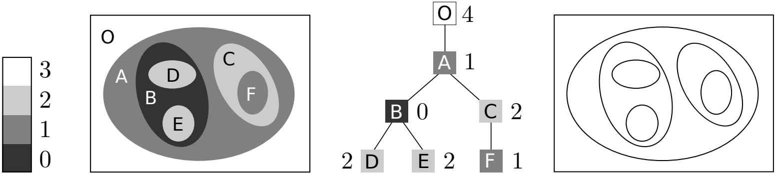


a lower level set — u — a upper level set

A couple of dual trees:

- min-tree: $\mathcal{T}_{\min}(u) \; = \; \{ \, \Gamma \in \mathcal{CC}(\, [u < \lambda] \,) \, \}_\lambda$
- max-tree: $\mathcal{T}_{\max}(u) \; = \; \{ \, \Gamma \in \mathcal{CC}(\, [u \geq \lambda] \,) \, \}_\lambda$

The morphological tree of shapes (ToS)



scale — image u — $\mathfrak{S}(u)$ — level lines of u

Tree of shapes:

$$\mathfrak{S}(u) \; = \; \{\, \mathrm{Sat}(\Gamma);\; \Gamma \in \mathcal{CC}([u < \lambda]) \cup \mathcal{CC}([u \geq \lambda]) \,\}_\lambda$$

A shape:

- an element $\mathcal{S} \in \mathfrak{S}(u)$
- a sub-tree in the representation above

Level lines: $\{\, \partial\Gamma;\; \Gamma \in \mathfrak{S}(u) \,\}$

The morphological tree of shapes (ToS)

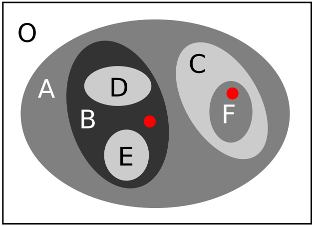
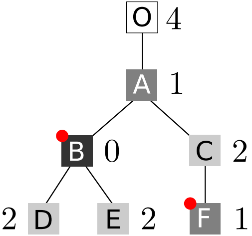


image u



its tree of shapes $\mathfrak{S}(u)$

Let us consider a couple of points of the image:
each point belongs to a particular ToS node

The morphological tree of shapes (ToS)

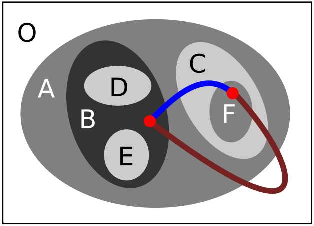
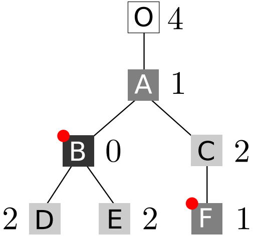


image u



its tree of shapes $\mathfrak{S}(u)$

finding a path between the red dots is straightforward:
all paths **have to** go through regions A and C...

The morphological tree of shapes (ToS)

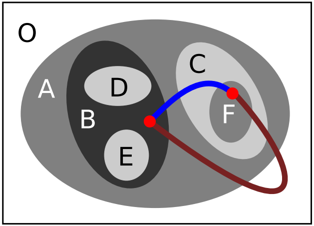
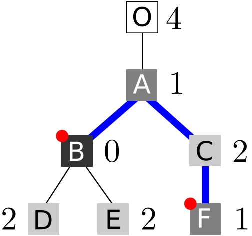


image u



its tree of shapes $\mathfrak{S}(u)$

$\leadsto$ a minimal path *in the image* only goes through
the minimal set of regions and it can be "**read**" on the ToS!

The morphological tree of shapes (ToS)

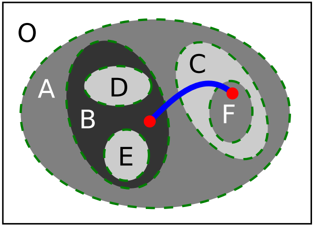
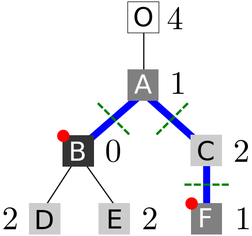


image u



its tree of shapes $\mathfrak{S}(u)$

and this minimal path crosses the image level lines
(so they have to be "well formed" $\leftarrow$ issue ignored in this talk!)

Mapping the Dahu distance on the tree

With

- t_x the node that corresponds to $x \in \mathbb{Z}^n$
- $\pi_{\mathfrak{S}(u)}(t_x, t_{x'})$ the path in $\mathfrak{S}(u)$ between the nodes t_x and $t_{x'}$
- $\mu_u(t)$ the corresponding gray level of node t in the image u

the definition of the Dahu distance becomes:

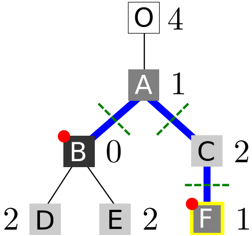
$$D_u(x, x') = \max_{t \in \pi_{\mathfrak{S}(u)}(t_x, t_{x'})} \mu_u(t) - \min_{t \in \pi_{\mathfrak{S}(u)}(t_x, t_{x'})} \mu_u(t)$$

The how-to:

1. pre-compute the ToS (...)
2. then get distances very efficiently for many couples (x, x').

Computing the Dahu distance on the tree

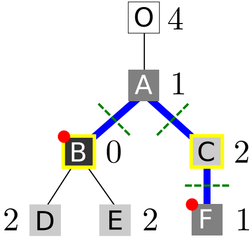
start from the deepest node



min = 1, max = 1

Computing the Dahu distance on the tree

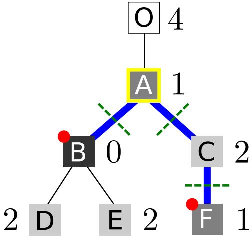
go up until to reach the same depth for both points



min = 0, max = 2

Computing the Dahu distance on the tree

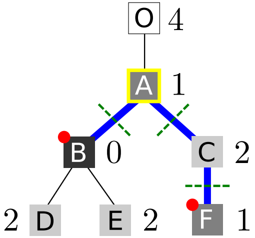
go up until to reach the same node (lca)



min = 0, max = 2

Computing the Dahu distance on the tree

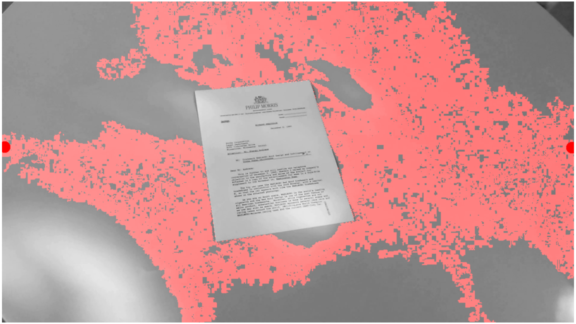
done!



min = 0, max = 2

$\rightsquigarrow$ distance = 2

A quick quiz



Red zone: region where every path between the red dots is minimal.

Quiz:

discuss / compare the different methods that compute the distance...

A quick test

For 1M couples of points (x, x') taken randomly:

lena size	256×256	512×512	1024×1024	...	pixels
average $\lvert\pi_{\mathfrak{S}(u)}(t_x, t_{x'})\rvert$	90	90	90	90	nodes
average $\lVert x' - x \rVert_1$	170	340	680	...	pixels

$\approx$ 1M Dahu distance computations per sec.

Conclusion / Take-home messages

Reminder:

- the MB distance is **great for computer vision**!

Conclusion / Take-home messages

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What we have done:

- introduce a new distance,
  that fits with a *continuous (yet discrete) representation* of images
- formalize it,
  and relate it to the morphological tree of shapes
- provide an efficient solution to compute distances.

Conclusion / Take-home messages

Reminder:

- the MB distance is **great for computer vision**!

What we have done:

- introduce a new distance,
  that fits with a *continuous (yet discrete) representation* of images
- formalize it,
  and relate it to the morphological tree of shapes
- provide an efficient solution to compute distances.

What we have skipped:

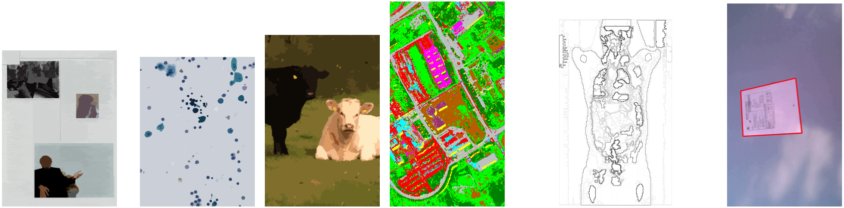
- actually many things...

A perspective:

- adapt the distance to color images ▶

Using the multivariate tree of shapes (MToS)...

grain-like filtering | shaping | simplification | classification | saliency | obj. detection



E. Carlinet and T. Géraud, "MToS: A tree of shapes for multivariate images," *IEEE Transactions on Image Processing*, vol. 24, num. 12, pp. 5330–5342, 2015. [PDF]

That's all folks!

Thanks for your attention.   Any questions?

Dahu descentius frontalis
(La Pointe Perce, 1895)

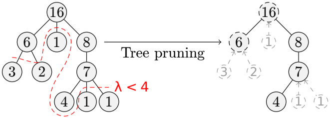
Dahu ascentius frontalis
(Le Charvin, 1901)

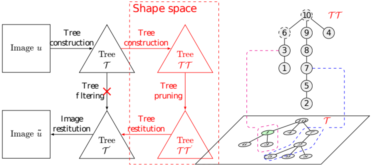
Dahu dextrogyre
(Col de la Colombire, 1904)

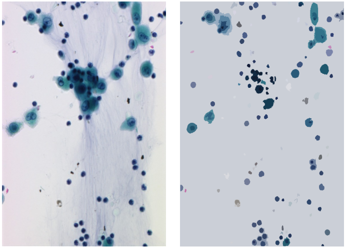
Young dahu lévogyre
(La Tournette, 1910)

the next slides are not part of the core presentation!

[BACKUP SLIDE] Apps based on the ToS





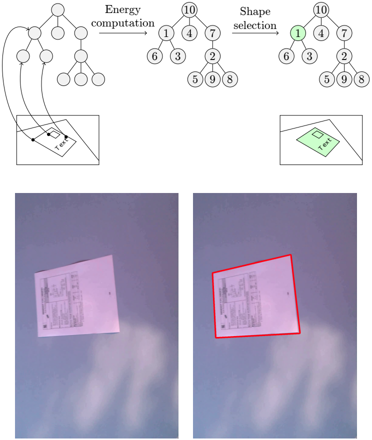


Grain filter.

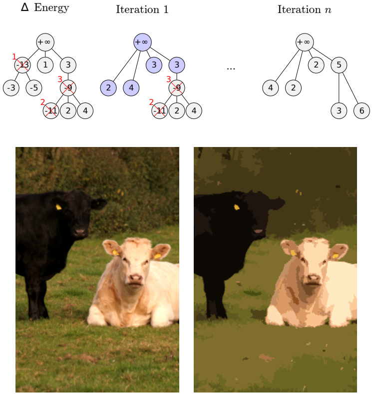
Shaping (filtering in shape space).

Y. Xu, T. Géraud, and L. Najman, "Connected filtering on tree-based shape-spaces," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 38, num. 6, pp. 1126–1140, 2016. [PDF]

Apps based on the ToS



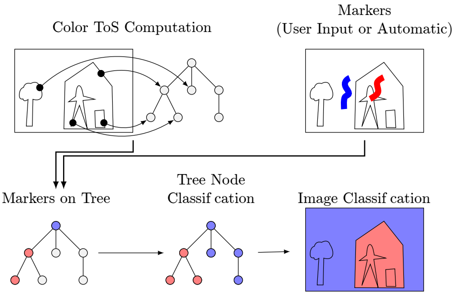
Object detection.

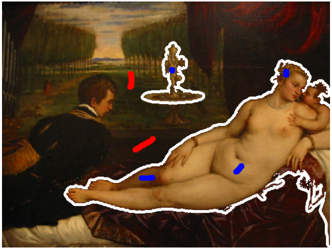


Simplification / segmentation.

Y. Xu, E. Carlinet, T. Géraud, and L. Najman, "Hierarchical segmentation using tree-based shape spaces," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 39, num. 3, pp. 457–469, 2017. [PDF]

Apps based on the ToS





Object picking from very few scribbles.

E. Carlinet and T. Géraud, "MToS: A tree of shapes for multivariate images," *IEEE Transactions on Image Processing*, vol. 24, num. 12, pp. 5330–5342, 2015. [PDF]

[BACKUP SLIDE] Cubical complex

The nD space of cubical complexes:

$$H^1_0 = \{\, \{a\};\, a \in \mathbb{Z} \,\} \qquad H^1_1 = \{\, \{a, a+1\};\, a \in \mathbb{Z} \,\}$$

$$\mathbb{H}^1 = H^1_0 \cup H^1_1 \qquad \mathbb{H}^n = \times_n H^1$$

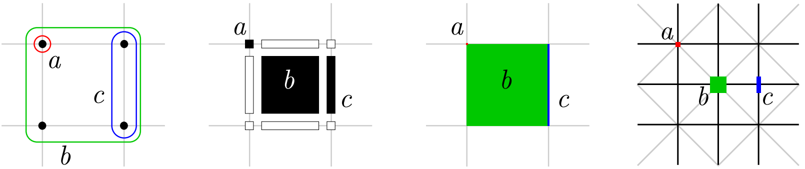
$h \in \mathbb{H}^n$: $\times$ product of d elements of H^1_1 and $n-d$ elements of H^1_0

- we have $h \subset \mathbb{Z}^n$
- h is a d-face
- d is the dimension of h

[BACKUP SLIDE] Cubical complex

Three faces of $\mathbb{H}^2$:

a	$= \{0\} \times \{1\}$	0-face	closed
b	$= \{0, 1\} \times \{0, 1\}$	2-face	open
c	$= \{1\} \times \{0, 1\}$	1-face	clopen



subsets of $\mathbb{Z}^2$

elements of the cellular complex

geometrical objects (parts of $\mathbb{R}^2$)

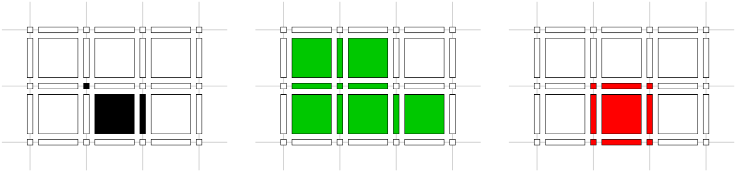
vertices of the Khalimsky grid

Cubical complex

With $h^{\uparrow} = \{ h' \in \mathbb{H}^n \mid h \subseteq h' \}$ and $h^{\downarrow} = \{ h' \in \mathbb{H}^n \mid h' \subseteq h \}$:

- $(\mathbb{H}^n, \subseteq)$
  is a poset,
- $\mathcal{U} = \{ U \subseteq \mathbb{H}^n \mid \forall h \in U,\ h^{\uparrow} \subseteq U \}$
  is a T0-Alexandroff topology on $\mathbb{H}^n$.

Topological operators:

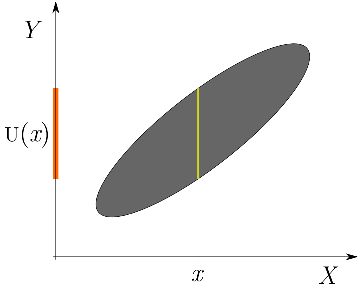


$E = \{ a, b, c \}$ star: $E^{\uparrow}$ closure: $E^{\downarrow}$

Set-valued analysis

A set-valued map $U : X \to \mathcal{P}(Y)$ is characterized by its graph:

$$\mathrm{Gra}(U) \;=\; \{\, (x,\, y) \in X \times Y \mid y \in U(x) \,\}.$$



Set-valued analysis

Continuity:

- when $U(x)$ is compact, U is USC at x if

  $\forall\, \varepsilon > 0,\ \exists\, \eta > 0$ such that $\forall\, x' \in B_X(x, \eta),\ \ U(x') \subset B_Y(U(x), \varepsilon)$.

- U is USC iif $\forall\, x \in X$, U is USC at x
- this is the "natural" extension of the *continuity* of a scalar function.

Inverse:

the *core* of $M \subset Y$ by U is $U^{\ominus}(M) \ = \ \{\ x \in X \mid U(x) \subset M\ \}$

A continuity characterization:

U is USC iff *the core of any open subset is open*.

Recap

We have a continuous-like definition of the MB distance
and it can be computed efficiently thanks to the tree of shapes

$\rightsquigarrow$ but we have to fix a digital topology issue
and to re-express the distance *on* the tree...

[BACKUP SLIDE] About digital topology

Digital topology implies:

- use of dual connectivities for object/background
- dual connectivities for lower/upper level sets $\Rightarrow$ the ToS exists

About digital topology

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Issues with *two* connectivities:

- it would be painful to consider paths [...]
- we would have some inconsistent results in distance computation [...]

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Digital topology implies:

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Issues with *two* connectivities:

- it would be painful to consider paths [...]
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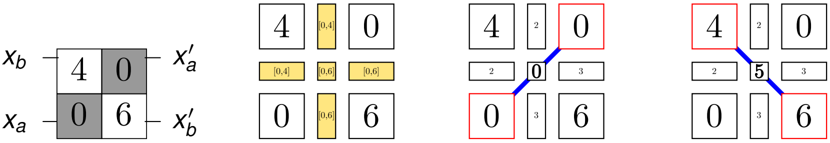
let's see that...

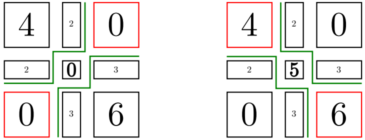
The issue with Digital Topology



u $\quad$ $\widetilde{u}$ $\quad$ $\overline{u}_a < \widetilde{u}$ $\quad$ $\overline{u}_b < \widetilde{u}$

$D_u^{\text{naive}}(x_a, x'_a) = 0$ $\quad$ $D_u^{\text{naive}}(x_b, x'_b) = 2$

this saddle case in 2D is a symptom of a discrete topology issue with $\widetilde{u}$



level lines $\lambda = 0.5$ $\quad$ level lines $\lambda = 3.5$

About digital topology

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Issues with *two* connectivities:

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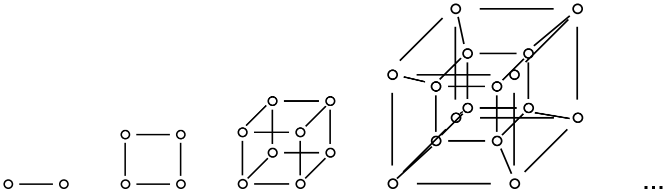
- we would have some inconsistent results in distance computation [...]

An important class of images: *digitally well-composed* (DWC) images

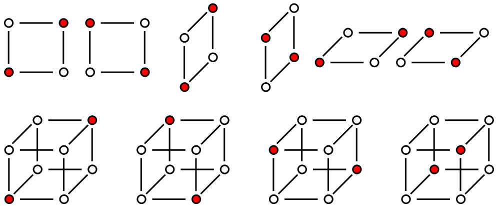
- connectivities are equivalent for all components of level sets
- boundaries of level sets do not have pinches
- if an image is DWC $\Rightarrow$ its ToS and the level lines are well defined

T. Géraud, E. Carlinet, S. Crozet, "Self-Duality and Discrete Topology: Links Between the Morphological Tree of Shapes and Well-Composed Gray-Level Images," *in: Proc. of ISMM*, LNCS, vol. 9082, pp. 573–584, Springer, 2015. [PDF]

*n*D blocks:

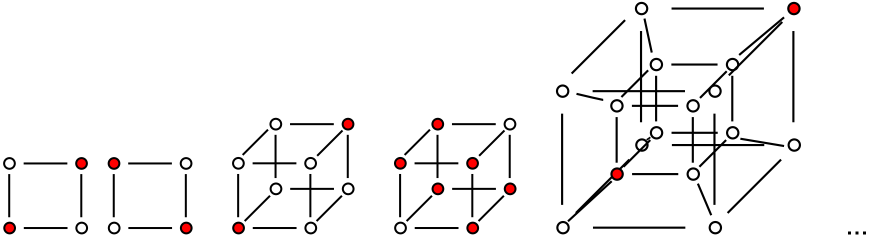


Antagonists in 3D:



DWC images

Critical configurations:



- A digital set $S \subset \mathbb{Z}^n$ is *digitally well-composed* (DWC) iff it does not contain any critical configuration
- A digital image $u : \mathbb{Z}^n \rightarrow Y$ is DWC iff its levels sets are DWC

[BACKUP SLIDE] About digital topology

An image can be made DWC by subdivision + interpolation:

- using the median operator in 2D,
- using a non-local process in *n*D.

N. Boutry, T. Géraud, and L. Najman, "How to make *n*D functions well-composed in a self-dual way," *in: Proc. of ISMM*, LNCS, vol. 9082, pp. 561–572, Springer, 2015. [PDF]

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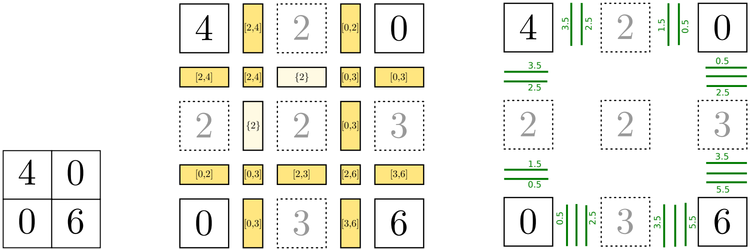

u $\widetilde{u}_{\text{med}}$ level lines of $\widetilde{u}_{\text{med}}$

what are the level lines?
(make the chunks connect...)

N. Boutry, T. Géraud, and L. Najman, "How to make *n*D functions well-composed in a self-dual way," *in: Proc. of ISMM*, LNCS, vol. 9082, pp. 561–572, Springer, 2015. [PDF]

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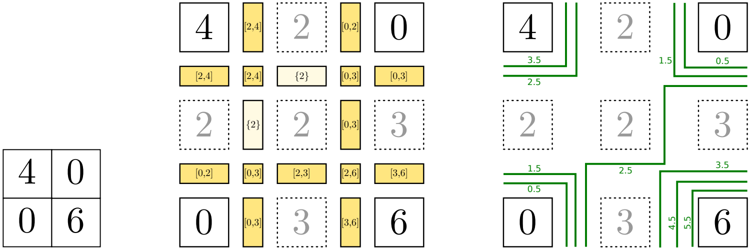


u $\qquad$ $\widetilde{u}_{\text{med}}$ $\qquad$ level lines of $\widetilde{u}_{\text{med}}$

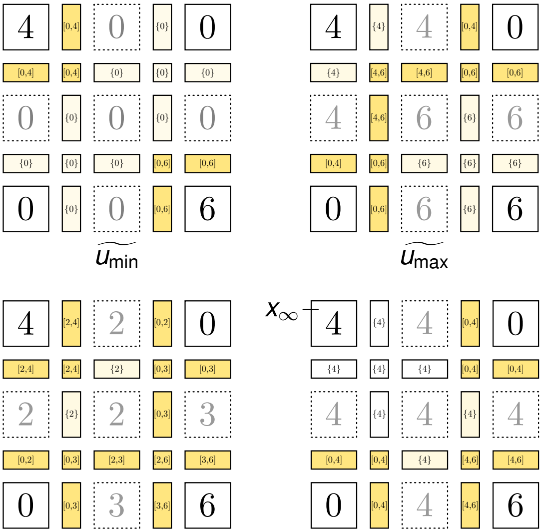
u_{med} is DWC $\Rightarrow$ there is only *one way* to arrange level lines (thus shapes) into an inclusion tree :-)

N. Boutry, T. Géraud, and L. Najman, "How to make *n*D functions well-composed in a self-dual way," *in: Proc. of ISMM*, LNCS, vol. 9082, pp. 561–572, Springer, 2015. [PDF]

[BACKUP SLIDE] Some well-composed representations

dual:

self-dual:



$\widetilde{u_{\min}}$

$\widetilde{u_{\max}}$

$\widetilde{u_{\text{med}}}$ (in 2D only)

$\widetilde{u}_\flat$ (works in n-D)

A flawless definition

NAIVE definition of the Dahu distance:

$$D_u(x, x') \;=\; \min_{\overline{u} \lessdot \widetilde{u}} \; d^{\mathrm{MB}}_{\overline{u}}(h_x, h_{x'})$$

A flawless definition

scalar image — DWC interpolated — interval-valued

$$(u : \mathbb{Z}^n \to Y) \xrightarrow{\text{step 1}} (u_\square : \left(\tfrac{\mathbb{Z}}{2}\right)^n \to Y') \xrightarrow{\text{step 2}} (\widetilde{u_\square} : \left(\tfrac{\mathbb{H}}{2}\right)^n \to \mathbb{I}_{Y'})$$

NAIVE definition of the Dahu distance:

$$D_u(x, x') \;=\; \min_{\overline{u} \lessdot \widetilde{u}} \; d^{\mathrm{MB}}_{\overline{u}}(h_x, h_{x'})$$

A flawless definition

scalar image		DWC interpolated		interval-valued
$(u : \mathbb{Z}^n \rightarrow Y)$	$\xrightarrow{\text{step 1}}$	$(u_\square : (\frac{\mathbb{Z}}{2})^n \rightarrow Y')$	$\xrightarrow{\text{step 2}}$	$(\widetilde{u_\square} : (\frac{\mathbb{H}}{2})^n \rightarrow \mathbb{I}_{Y'})$

NEW definition of the Dahu distance:

$$D_u(x, x') \;=\; \min_{\overline{u} \lessdot \widetilde{u_\square}} \; d^{\mathrm{MB}}_{\overline{u}}(h_x, h_{x'})$$

A flawless definition

scalar image		DWC interpolated		interval-valued
$(u : \mathbb{Z}^n \to Y)$	$\xrightarrow{\text{step 1}}$	$(u_\square : \left(\frac{\mathbb{Z}}{2}\right)^n \to Y')$	$\xrightarrow{\text{step 2}}$	$(\widetilde{u_\square} : \left(\frac{\mathbb{H}}{2}\right)^n \to \mathbb{I}_{Y'})$

NEW definition of the Dahu distance:

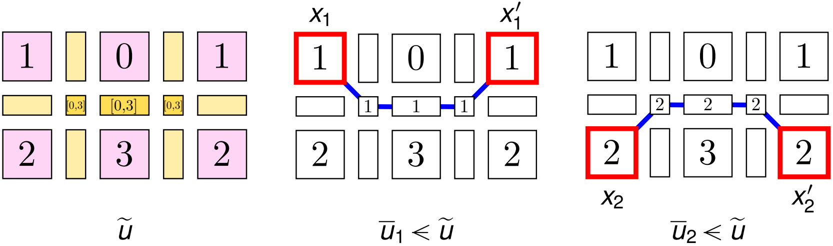
$$D_u(x, x') \; = \; \min_{\overline{u} \lessdot \widetilde{u_\square}} \; d^{\mathrm{MB}}_{\overline{u}}(h_x, h_{x'})$$

actually, the interpolation does not introduce a bias in the distance values;

it just makes their definition and computation sound and consistent :-)

[BACKUP SLIDE] This new combinatorial layer = a requirement

$$D_u(x, x') \;=\; \min_{\overline{u} \lessdot \widetilde{u}} d^{\mathrm{MB}}_{\overline{u}}(h_x, h_{x'})$$



We have:

$D_u(x_1, x'_1) \;=\; d^{\mathrm{MB}}_{\overline{u}_1}(h_{x_1}, h_{x'_1}) \;=\; 0$ and $D_u(x_2, x'_2) \;=\; d^{\mathrm{MB}}_{\overline{u}_2}(h_{x_2}, h_{x'_2}) \;=\; 0$

but: $\nexists\; \overline{u} \lessdot \widetilde{u},\;\; d^{\mathrm{MB}}_{\overline{u}}(h_{x_1}, h_{x'_1}) \;=\; d^{\mathrm{MB}}_{\overline{u}}(h_{x_2}, h_{x'_2}) \;=\; 0.$

so we do **not** have a unique $\overline{u} \lessdot \widetilde{u}$ that “works” for all different (x, x')