Digital topology and applications

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Outline of the talk

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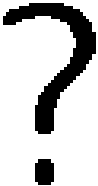
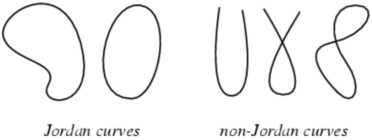
Images and topology I

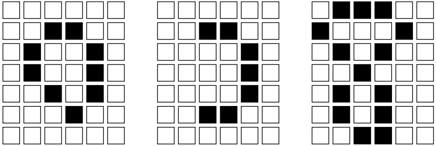


- Objectives: to identify, represent, measure, characterize, compare, index, simplify, localize, visualize objects and components in images
- Neighborhood, Connectedness, Manifold or Surface, Boundary, Topology invariants
- Topology for images = topology for $\mathbb{Z}^n$

Topologies for $\mathbb{Z}^n$ I



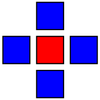
shape = subset of $\mathbb{R}^n$

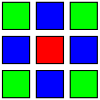


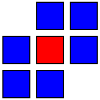
shape = subset of $\mathbb{Z}^n$

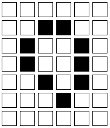
- Can we mimick standard topology in digital space ?
- Guide: Jordan property, sound definition of hypersurfaces
  1. graph approaches: adjacency graphs put on $\mathbb{Z}^n$ (n-cells)
  2. cellular approaches: cubical complex, abstract cellular complex, connected ordered topological space, orders (n-cells, . . . , 0-cells)
  3. intermediate approach: graph and arcs (n-cells, $n - 1$-cells)

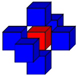
Adjacency graph

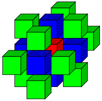

4-adjacency

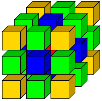

8-adjacency

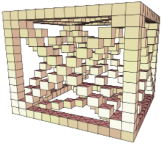

"6-adjacency"


connected ?


6-adjacency

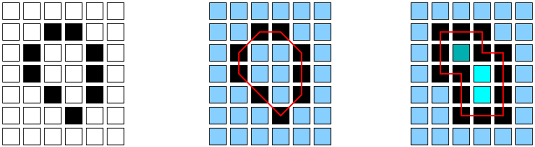

18-adjacency


26-adjacency


connected ?

- adjacency relations ρ: 4- and 8- in $\mathbb{Z}^2$, 6-, 18- and 26- in $\mathbb{Z}^3$, etc.
- connectedness relations in $X \subset \mathbb{Z}^n$ = transitive closure of ρ in X.
- ρ-components, ρ-pathes follow

Rosenfeld's paradox in $\mathbb{Z}^2$ I



simple 8-curve   one 8-comp.   three 4-comp.

- Digital analog of Jordan curve theorem
- Simple ρ-curve: any point has exactly two ρ-neighbors.

- A simple 4-curve may not separate $\mathbb{Z}^2$ in two 4-components
- A simple 8-curve may not separate $\mathbb{Z}^2$ in two 8-components

Rosenfeld's paradox in $\mathbb{Z}^2$ II

Theorem ([Rosenfeld])

- *A simple 4-curve (with more than 4 pixels) separates $\mathbb{Z}^2$ in two 8-components*
- *A simple 8-curve separates $\mathbb{Z}^2$ in two 4-components*

- Standard practice: choose one adjacency for the foreground (shape) and the other for the background.
- Note: local computations are enough to check that a curve is "Jordan"

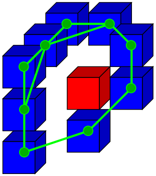
Do the same hold in 3D ? I

[Morgenthaler,Rosenfeld 81] [Malgouyres97]

Definition (Digital Surface)

$S \subset \mathbb{Z}^3$ is a surface iff S separates $\mathbb{Z}^3$ in two 6-connected components and every voxel of S is 6-adjacent to each component of $\mathbb{Z}^3 \setminus S$.

- Several local definitions that induces surfaces [Morgenthaler,Rosenfeld 81] [Malgouyres97]



$\forall u \in S$, the 26-neighbors of u in S constitute a 18-connected quasi-curve.

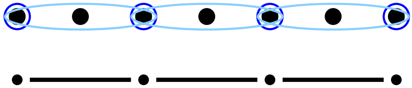
Do the same hold in 3D ? II

Theorem ([Malgouyres96])

There is no local characterization of surfaces in $\mathbb{Z}^3$.

- Note: local computations are not enough to check that a surface is "Jordan"

Khalimsky digital space I

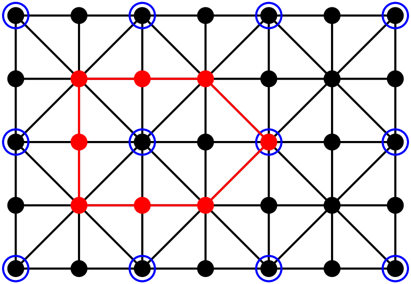


- Connected ordered topological space (COTS) [Khalimsky90]
- Even points of $\mathbb{Z}$ are closed, odd points are open. Aleksandrov topology.
- $\mathbb{Z}^n = \mathbb{Z} \times \ldots \times \mathbb{Z}$
- neighbors define an adjacency relation θ

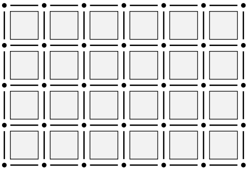
Khalimsky digital space II

Jordan property

Any simple θ-curve separates $\mathbb{Z}^2$ into two components.



Kovalevsky's cellular complex I



Remark [Kovalevsky89]

Any finite separable topological space is an abstract cellular complex

- Topologies for images are to be found in cellular complexes
- For $\mathbb{Z}^n$, complex = cellular grid, with induced topology.

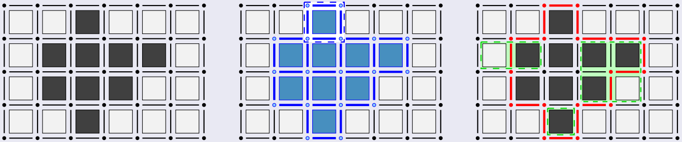
Kovalevsky's cellular complex II

- Identical to Khalimsky topology
- Neighborhood graph is enough iff its corresponding subcomplex is strongly connected
- Other cellular structures have better properties (hexagonal)

Surfaces in the cellular grid I

Definition (Surface as boundary of a shape)

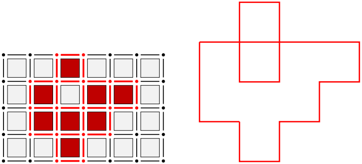
Let $\mathrm{Cl}(O)$ be the closure of a subset O of the cellular grid $\mathbb{C}^n$. The boundary of O is the subset of cells of $\mathrm{Cl}(O)$ whose star touches the complement of $\mathrm{Cl}(O)$ in $\mathbb{C}^n$.



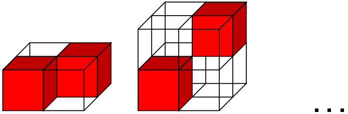
- if O is ω_n-connected, it is a strongly connected polyhedral $n-1$-complex.

Surfaces in the cellular grid II

- But boundaries may not be separating



Boundaries in well-composed pictures I



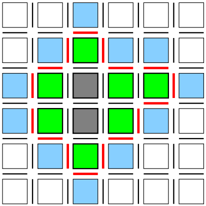
- Well-composed picture [Latecki97] : Picture without specific configurations

Theorem ([Latecki97])

Any boundary of a connected object in a well-composed picture is a combinatorial $n-1$-manifold

- but it is not a straightforward local process to make a picture well-composed

Intermediate approach of Herman I

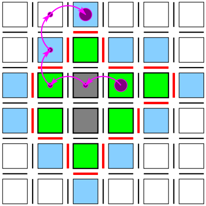


[Liu, Artzy, Frieder, Herman, Webster, Gordon, Udupa, Kong]

Definition

- Digital space is an adjacency graph (proto-adjacency ω_n)
- Surface element = *surfel* = arc $\in \omega_n$ = couple (u,v)
- Surface is a set of surfels

Jordan surfaces and Jordan pairs I



- immediate interior $II(S)$ $= \{u | (u, v) \in S\}$.
- immediate exterior $IE(S)$ $= \{v | (u, v) \in S\}$.

Definition (Jordan surface [Herman92])

$S \subset \omega_n \subset \mathbb{Z}^n \times \mathbb{Z}^n$ is a Jordan surface iff every ω_n-path from $II(S)$ to $IE(S)$ crosses S.

Jordan surfaces and Jordan pairs II

Definition (Strong Jordan pair)

Consider a subset X of $\mathbb{Z}^n$. A pair of adjacencies $\{\kappa, \lambda\}$ is a strong Jordan pair iff any boundary surface between a κ-component of X and a λ-component of X^c is Jordan.

- in 2D: $(8,4)$, $(4,8)$ are strong Jordan pairs for $(\mathbb{Z}^2, 4)$. $(4,4)$ is not.
- in 3D: $(26,6)$, $(6,26)$ are strong Jordan pairs for $(\mathbb{Z}^3, 6)$. $(6,6)$ is not.
- in nD: there exists such pairs [Herman92,Udupa94,Lachaud00]

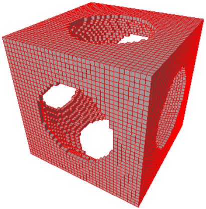
Jordan surfaces and Jordan pairs III

Summary

- boundaries of object are separating (and thin)
- a local topology may be defined on the surface
- theoretical framework extensible to many non regular digital spaces [Herman98]

Outline of the talk

Topology on digital surfaces ? I



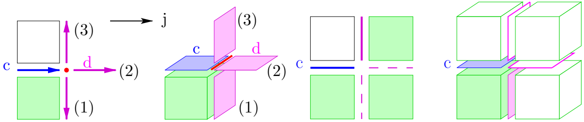
- For now, a surface is a set of surfels

Questions ?

Can we define local neighborhood relations so that

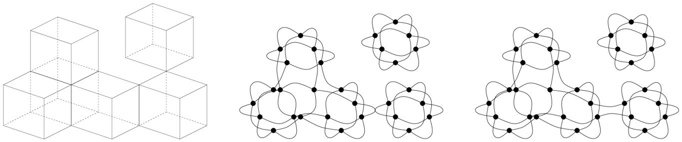
- a whole connected surface can be extracted by their tracking,
- Jordan property is satisfied

Bel adjacency in a picture I



- binary picture I: finite subset X of $\mathbb{Z}^n$
- boundary element or *bel* in I = surfel between X and X^c
- For each direction j ($n-1$ directions for each bel)
  - interior bel-adjacency from c (dir. j). d : first follower of c along j which is a bel
  - exterior bel-adjacency from c (dir. j). d : last follower of c along j which is a bel

Bel adjacency graph I



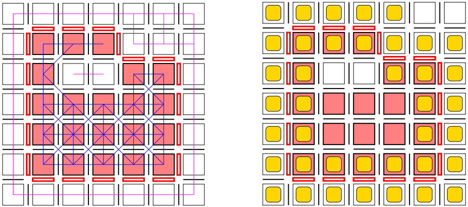
bels — interior graph — exterior graph

- For each direction, choose interior/exterior $\Rightarrow 2^{\frac{n(n-1)}{2}}$ bel-adjacencies

Theorem (3D [Herman,Webster83])

Let $O \subset X$ 6-connected, $Q \subset X^c$ 18-connected. c a bel. The all-interior bel-adjacency graph component containing c is the boundary surface between O and Q.

Bel adjacency graph II



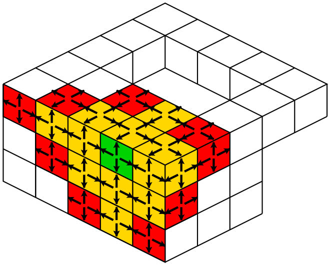
Theorem (nD, $n \geq 2$, [Udupa94])

Let $O \subset X$ $2n$-connected, $Q \subset X^c$ $2n^2$-connected. c a bel. The all-interior bel-adjacency graph component containing c is the boundary surface between O and Q.

- To extract a boundary component $\Rightarrow$ track it.

Tracking digital boundaries I

- boundary in parallepiped N^n
- number of bels is $V = O(N^{n-1})$
- degree of each vertex is $2n - 2$



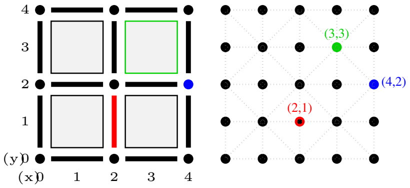
- breadth-first traversal of bel-adjacency graph
- each bel is visited $2n - 2$ times
- time complexity $\approx (2n - 2)V$

Tracking digital boundaries II

Lower bound on time complexity in 3D

- [Tutte56] Any 4-connected planar graph has a hamiltonian cycle
- lower bound is V in some case
- only $O(V)$ is known [Chiba89]

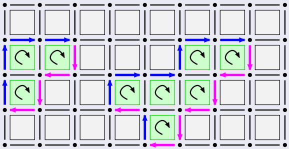
Cubical chain complex I



- isomorphism “grid” and “Khalimsky’s space”
- a cell is an element of $\mathbb{Z}^n$, parities = topology
- pixels, voxels, n-cells have odd parities

Cubical chain complex II

Construction of a chain complex

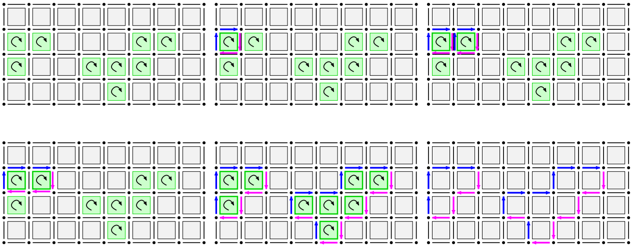


- oriented k-cells form k-dimensional bases
- k-chains are formal sums of k-cells (coefficient $\mathbb{Z}$)

  $\sum_i +o_i^n$ is a digital shape

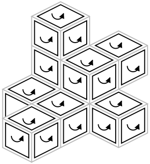
  $\sum +s_j^{n-1} + \sum -s_{j'}^{n-1}$ is a digital surface
- boundary operator Δ, with $\Delta\Delta = 0$, based on cell parities

Application to digital boundaries I

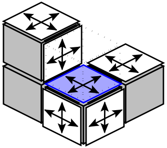


- digital shape is a subset X of $\mathbb{Z}^n$ (odd parities)
- its boundary = $n - 1$-chain $\Delta \sum_{x \in X} +x$
- it is a cycle since $\Delta\Delta = 0$

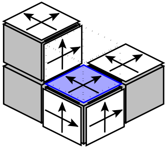
Oriented boundary tracking I



ΔX

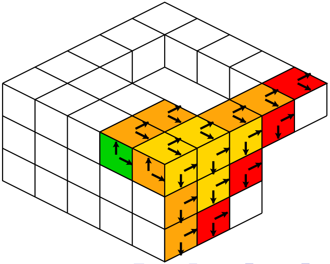


neighbors

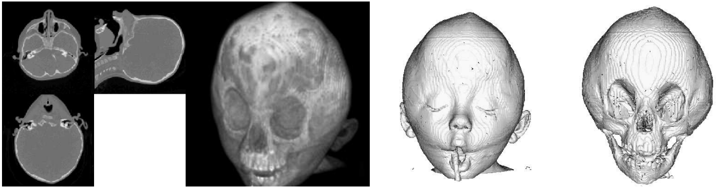


“positive” neighbors

- since $\Delta\Delta X = 0$
- breadth-first traversal of directed bel-adjacency graph
- each bel is visited $\frac{2n-2}{2}$ times
- time complexity $\approx (n-1)V$



Isosurfaces I

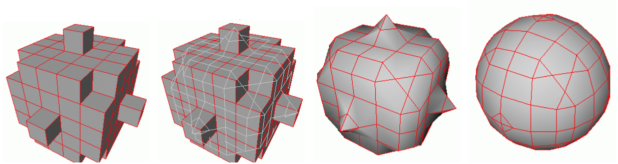


Definition (Isosurface)

Let $I : \mathbb{R}^3 \rightarrow \mathbb{R}$.
Isosurface of value s in I = $\{(x, y, z) \in \mathbb{R}^3, I(x, y, z) = s\}$.

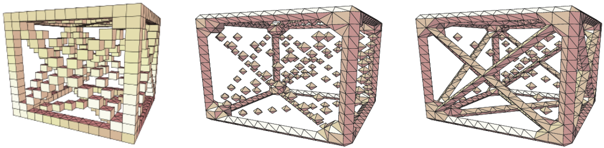
- *marching-cubes* [Lorensen,Cline87], by scanning

Duality isosurfaces / digital surface



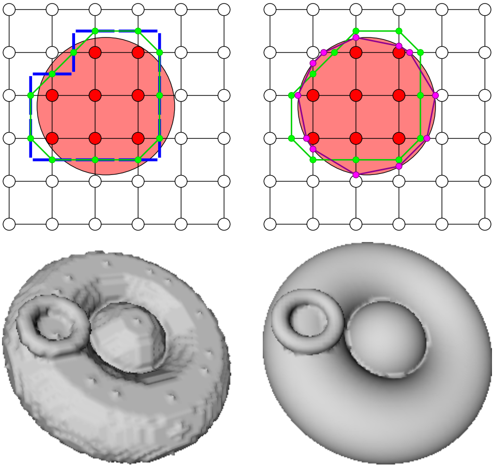
$$X = \{\vec{x} \in \mathbb{Z}^3, I(\vec{x}) \geq s\}$$

bel-adjacency graph with loops defines a comb. 2D surface. In nD, a comb. $n-1$-pseudomanifold without boundary [Lachaud00]



shape interior exterior

Making isosurfaces nice



1. $X = \{\vec{x} \in \mathbb{Z}^3, I(\vec{x}) \geq s\}$
2. track $\Delta \sum_{x \in X} +x$
3. local triangulation
4. move vertices

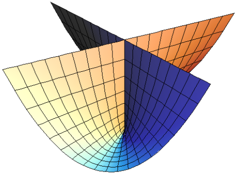
More general isosurfaces

Work in progress

How to visualize $\{(x, y, z) \in \mathbb{R}^3, f(x, y, z) = 0\}$?

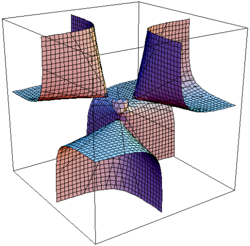
Whitney's umbrella

$x^2 - zy^2 = 0$



Cayley cubic

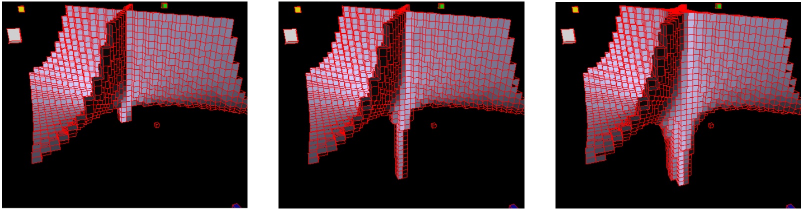
$4(x^2 + y^2 + z^2) + 16xyz - 1 = 0$



Digital surface in Z^4 I

- since $\{f^2 = 0\} = \{f = 0\}$, we cannot rely on a change of sign around the 0-surface
- we introduce $F(x, y, z, t) = f(x, y, z) - t$
- The set $F = 0$ is homeomorphic to a 3-plane
- we sample F at points $(ih, jh, kh, lh' - \frac{1}{2})$, for integers i, j, k, l
- we extract the digital surface $F = 0$ (with $l = 0$ or 1)
  - it is a set S of 3-cells
  - we keep in $\mathrm{Cl}(S)$ the cells included in $t = 0$
  - the obtained complex S' is closed with cells of dim k, $0 \leq k \leq 3$.

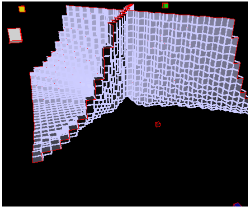
Digital surface in Z^4 II



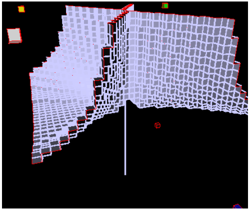
$h' = 0.1$ $h' = 0.5$ $h' = 2.5$

3-complex S' for Whitney's umbrella in $[-5, 5]^3$, $h = \frac{10}{64}$

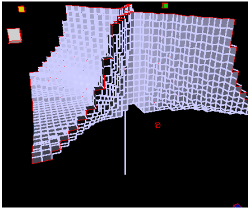
Collapse



$h' = 0.1$



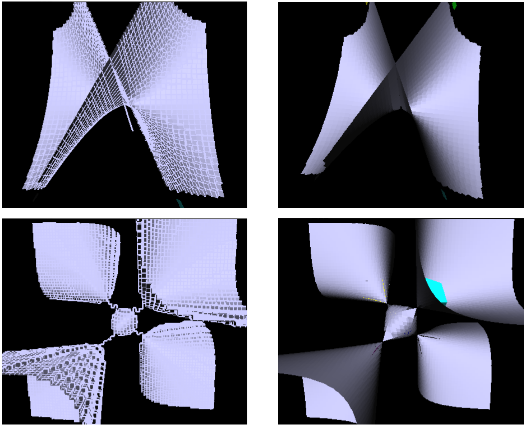
$h' = 0.5$



$h' = 2.5$

- To get a thin complex, we collapse S'
- Collapse : $K \leftarrow S' \setminus T$, T fixed cells
  1. while $\exists(\sigma, \sigma') \in K$, σ maximal cell, σ' free face of σ
     1. $K \leftarrow K \setminus \{\sigma, \sigma'\}$
- the new complex K is homotopic to S'

Projection onto $\{f = 0\}$



- Projected with Newton-Raphson