

# Natural Deduction

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## Define "logic" [fuc, ]

it is possible that  $A$  belongs also to every  $B$ , then if the propositions are stated contrariwise and it is assumed that  $A$  belongs to every  $B$  and to no  $C$ , though the nonpositions are wholly false they will yield a true conclusion. Similarly if  $A$  belongs

logic(n.) A highly contagious disease believed to have originated in ancient Greece, the only known cures for which are poststructuralism and religion.

it is possible that  $A$  should belong to some  $B$  and to some  $C$ , and  $B$  to no  $C$ , e.g. animal to some white things and to some black things, etc.

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# Natural Deduction

- ① Logical Formalisms
- ② Natural Deduction
- ③ Additional Material

## Preamble

The following slides are implicitly dedicated to **classical** logic.

# Logical Formalisms

## 1 Logical Formalisms

- Syntax
- Proof Types
- Proof Systems

## 2 Natural Deduction

## 3 Additional Material

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# Syntax

## 1 Logical Formalisms

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# Terminal Symbols

## Propositional Calculus

Constants  $a, b, c, \dots$

Propositional Variables  $A, B, C, \dots$

Unary Connective  $\neg$

Binary Connectives  $\wedge, \vee, \Rightarrow$

Punctuation  $(,), [, ]$ .

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# Terminal Symbols

## Predicate calculus

Individual Variables  $x, y, z, \dots$

Functions  $f, g, h, \dots$ , with a fixed arity

Predicates  $P, Q, R, \dots$ , with a fixed arity

Quantifiers  $\forall, \exists$

Punctuation  $\cdot$ .

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## Propositional Formulas

$$\langle \text{formula} \rangle ::= \langle \text{propositional variable} \rangle$$
$$| \quad \neg \langle \text{formula} \rangle$$
$$| \quad \langle \text{formula} \rangle \wedge \langle \text{formula} \rangle$$
$$| \quad \langle \text{formula} \rangle \vee \langle \text{formula} \rangle$$
$$| \quad \langle \text{formula} \rangle \Rightarrow \langle \text{formula} \rangle$$

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## Terms

$$\langle \text{term} \rangle ::= \langle \text{constant} \rangle$$
$$| \quad \langle \text{function} \rangle (\langle \text{term} \rangle, \dots)$$

With the proper arity.

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## First Order Formulas

$$\langle \text{formula} \rangle ::= \langle \text{propositional variable} \rangle$$
$$| \quad \neg \langle \text{formula} \rangle$$
$$| \quad \langle \text{formula} \rangle \wedge \langle \text{formula} \rangle$$
$$| \quad \langle \text{formula} \rangle \vee \langle \text{formula} \rangle$$
$$| \quad \langle \text{formula} \rangle \Rightarrow \langle \text{formula} \rangle$$
$$| \quad \langle \text{predicate} \rangle (\langle \text{term} \rangle, \dots)$$
$$| \quad \forall \langle \text{individual variable} \rangle \cdot \langle \text{formula} \rangle$$
$$| \quad \exists \langle \text{individual variable} \rangle \cdot \langle \text{formula} \rangle$$

With the proper arity.

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## Syntactic Conventions

Associativity

- $\wedge, \vee$  are left-associative (unimportant)
- $\Rightarrow$  is right-associative (very important)

Precedence (increasing)

- ①  $\forall, \exists$
- ②  $\Rightarrow$
- ③  $\vee$
- ④  $\wedge$
- ⑤  $\neg$

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## Free Variables

$$\begin{aligned} \text{FV}(X) &= \emptyset \\ \text{FV}(P(x_1, x_2, \dots, x_n)) &= \{x_1, x_2, \dots, x_n\} \\ \text{FV}(\neg A) &= \text{FV}(A) \\ \text{FV}(A \vee B) &= \text{FV}(A) \cup \text{FV}(B) \\ \text{FV}(A \wedge B) &= \text{FV}(A) \cup \text{FV}(B) \\ \text{FV}(A \Rightarrow B) &= \text{FV}(A) \cup \text{FV}(B) \\ \text{FV}(\forall x \cdot A) &= \text{FV}(A) - \{x\} \\ \text{FV}(\exists x \cdot A) &= \text{FV}(A) - \{x\} \end{aligned}$$

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## Proof Types

### 1 Logical Formalisms

- Syntax
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### 2 Natural Deduction

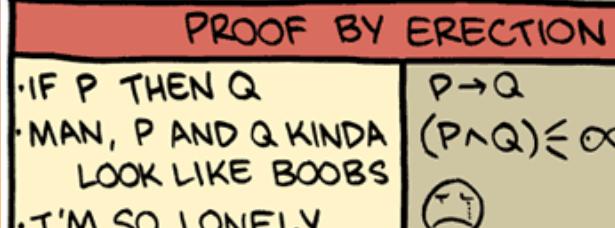
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## Different Proof Types



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## Different Proof Types



## 1 Logical Formalisms

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- Hilbertian Systems
- Natural Deduction
- Sequent Calculus
- Natural Deduction in Sequent Calculus

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# Axioms

- **Axioms** are formulas that are considered true a priori

$$\forall x \cdot x + 0 = x$$

- **Axiom schemes** use meta-variables  
(that range over a specific domain)

$$X + Y = Y + X$$

- Axiom schemes are used when quantifiers are not welcome

$$SXYZ \rightarrow XZ(YZ)$$

$$KXY \rightarrow X$$

- Axiom schemes are used when quantifiers do not apply

$$A \vee \neg A$$

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# Inference Rules

$$\frac{H_1 \quad H_2 \quad \dots \quad H_n}{C} \text{ Rule name}$$

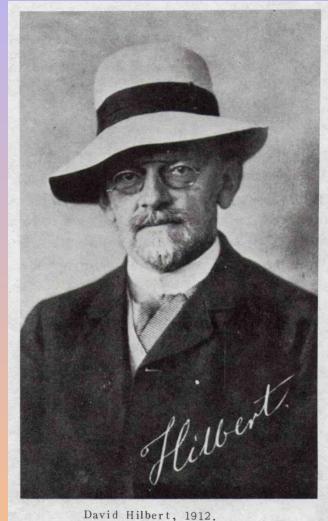
$$\frac{}{A} \text{ Axiom name}$$

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## Logical Formalisms



David Hilbert (1862–1943)

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## Hilbertian System

- A single inference rule: the **modus ponens**

$$\frac{A \quad A \Rightarrow B}{B} \text{ modus ponens}$$

- Many axioms to define the connectives

$$A \Rightarrow B \Rightarrow A \wedge B \quad A \wedge B \Rightarrow A \quad A \wedge B \Rightarrow B$$

$$A \Rightarrow A \vee B \quad B \Rightarrow A \vee B$$

$$A \vee B \Rightarrow (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C$$

$$A \Rightarrow B \Rightarrow A \quad (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C$$

$$A \vee \neg A \quad A \Rightarrow \neg A \Rightarrow B$$

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## Hilbertian System: Prove $A \Rightarrow A$

$$\frac{\overline{(A \Rightarrow ((A \Rightarrow A) \Rightarrow A)) \Rightarrow (A \Rightarrow A \Rightarrow A) \Rightarrow A \Rightarrow A} \quad \overline{A \Rightarrow (A \Rightarrow A) \Rightarrow A}}{\overline{(A \Rightarrow A \Rightarrow A) \Rightarrow A \Rightarrow A} \quad \overline{A \Rightarrow A \Rightarrow A}} \frac{}{A \Rightarrow A}$$

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## Natural Deduction

### 1 Logical Formalisms

### 2 Natural Deduction

- Syntax
- Normalization

### 3 Additional Material

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## Syntax

1 Logical Formalisms

2 Natural Deduction

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## Deductions

What's this?

$A$

A deduction of  $A$  under the hypothesis  $A$ .

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## Deduction

### Deduction

A **deduction** is a tree whose root ( $A$ ) is the **conclusion** and whose **active leafs** ( $\Gamma$ ) is the set of **hypotheses**.

$$\begin{array}{c} \Gamma \\ \vdots \\ A \end{array}$$

Any formula  $A$  is a valid hypothesis.

### Proof (Demonstration)

A **proof** is a deduction without hypotheses.

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## Implication

$$\frac{\begin{array}{c} [A] \\ \vdots \\ B \end{array}}{A \Rightarrow B} \Rightarrow \mathcal{I} \quad \frac{\begin{array}{c} A \\ A \Rightarrow B \end{array}}{B} \frac{\vdots}{\vdots} \Rightarrow \mathcal{E}$$

Deduction theorem, and Modus Ponens.

Note the connection with (left) contraction: any number of  $A$  (including 0) is discharged.

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## Proving $A \Rightarrow A$ in Natural Deduction

$$\frac{[A]}{A \Rightarrow A} \Rightarrow I$$

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## Conjunction

$$\frac{\begin{array}{c} \vdots \\ A \quad B \end{array}}{A \wedge B} \wedge I \quad \frac{\vdots}{A \wedge B} \wedge I E \quad \frac{\vdots}{A \wedge B} \wedge r E$$

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## Universal Quantification

$$\frac{\vdots}{\forall x \cdot A} \forall I \quad y \notin FV(hyp(A)) \quad \frac{\vdots}{A[t/x]} \forall E$$

$$\frac{\frac{[A]}{\forall x \cdot A} \forall I}{A \Rightarrow \forall x \cdot A} \Rightarrow I \quad \frac{\frac{[A]}{A \Rightarrow A} \Rightarrow I}{\forall x \cdot (A \Rightarrow A)} \forall I$$

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## Absurd

$$\frac{\vdots}{\perp} \perp E$$

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## Disjunction

$$\frac{\vdots}{A} \vee I\mathcal{I} \quad \frac{\vdots}{B} \vee r\mathcal{I} \quad \frac{\vdots}{A \vee B} \quad \frac{[A] \quad [B]}{\vdots \quad \vdots} \quad \frac{C \quad C}{\vdots} \quad \frac{[A] \quad [B]}{C} \vee \mathcal{E}$$

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## Existential Quantification

$$\frac{\vdots}{A[t/x]} \exists \mathcal{I} \quad \frac{\vdots \quad A \quad B}{B} \exists \mathcal{E} \quad y \notin FV(B, hyp(B))$$

For elimination,  $y \notin hyp(B)$ , i.e., not in the hypotheses other than the discharged  $A$ .

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## Negation

$$\frac{[A]}{\vdots} \perp \quad \frac{\vdots \quad A \quad \neg A}{\vdots \quad \vdots} \quad \frac{\vdots}{\perp} \neg \mathcal{E}$$

Plus one of these equivalent formulations of the fact that **classical** negation is involutive.

$$\frac{\vdots}{A \vee \neg A} XM \quad \frac{\vdots}{\neg \neg A} \neg \neg \quad \frac{[\neg A] \quad [\neg A]}{\vdots \quad \vdots} \quad \frac{B \quad \neg B}{\vdots} \quad \frac{A}{A} \quad \text{Contradiction}$$

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## Normalization

① Logical Formalisms

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## Cut

Cut: Introduction of a connective followed by its elimination.

$$\frac{\begin{array}{c} \vdots \\ A \\ \vdots \\ B \\ \vdots \end{array}}{\frac{A \wedge B}{\frac{A \wedge B}{A}}} \wedge I\mathcal{E}$$

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## Normalization

The normalization process eliminates the cuts.

$$\frac{\begin{array}{c} \vdots \\ A \\ \vdots \\ B \\ \vdots \end{array}}{\frac{A \wedge B}{\frac{A \wedge B}{A}}} \wedge I\mathcal{E} \rightsquigarrow \frac{\vdots}{A}$$

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## Normalizing Conjunctions

$$\frac{\begin{array}{c} \vdots \\ A \\ \vdots \\ B \\ \vdots \end{array}}{\frac{A \wedge B}{\frac{A \wedge B}{A}}} \wedge I\mathcal{E} \rightsquigarrow \frac{\vdots}{A}$$

$$\frac{\begin{array}{c} \vdots \\ A \\ \vdots \\ B \\ \vdots \end{array}}{\frac{A \wedge B}{\frac{A \wedge B}{B}}} \wedge r\mathcal{E} \rightsquigarrow \frac{\vdots}{B}$$

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## Normalizing Implications

$$\frac{\begin{array}{c} [A] \\ \vdots \\ B \\ \vdots \end{array}}{\frac{A \Rightarrow B}{\frac{A \Rightarrow B}{B}}} \Rightarrow I\mathcal{E} \rightsquigarrow \frac{\vdots}{B}$$

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## Normalizing Universal Quantifiers

$$\frac{\vdots \quad A}{\forall x \cdot A} \forall I \quad \sim \quad \vdots \quad A[t/x]$$

$x$  must not be free in the hypotheses, otherwise the reduction would change them.

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## Normalizing Disjunction

$$\frac{\vdots \quad A \quad [A] \quad [B]}{\frac{A \vee B \quad C \quad C}{C} \vee E} \rightsquigarrow \begin{array}{c} \vdots \\ A \\ \vdots \\ C \\ \vdots \end{array}$$

$$\frac{\vdots \quad B \quad [A] \quad [B]}{\frac{B \vee A \quad C \quad C}{C} \vee E} \rightsquigarrow \begin{array}{c} \vdots \\ B \\ \vdots \\ C \\ \vdots \end{array}$$

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## Additional Material

① Logical Formalisms

② Natural Deduction

③ Additional Material

## Logicians in a Bar

Three logicians walk into a bar,  
and the bartender asks “Would you all like a drink?”

The first one says, “Maybe.”

The second one says, “Maybe.”

The third one says, “Yes.”

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## Connecteurs

Didacticiel Exercices

Les connecteurs logiques sont des éléments fondamentaux pour former des propositions mathématiques à partir de deux propositions quelconques A et B :

- l'implication, A implique B, notée " $A \Rightarrow B$ "
- la conjonction, A et B, notée " $A \wedge B$ "
- la disjonction, A ou B, notée " $A \vee B$ "
- la négation, non A, notée " $\neg A$ "

Chacun de ces connecteurs est associé à deux règles :

- une règle permettant de justifier (ou démontrer) la proposition : comment justifier " $A \wedge B$ " ?
- une règle permettant de déduire une nouvelle proposition : que peut-on déduire de " $A \wedge B$ " ?

### 1. PROUVER UNE CONJONCTION



La conjonction de deux propositions A B, notée " $A \wedge B$ " et lue "A et B", est vraie si A est vraie et B est vraie.

Prouver " $A \wedge B$ " se réduit donc à fournir deux preuves : une de A et une de B.

Quelles que soient les propositions A B,  
 $A \Rightarrow B \Rightarrow (A \wedge B)$



Commencer

### 2. DÉDUIRE D'UNE CONJONCTION



Que peut-on déduire de la conjonction " $A \wedge B$ " ?

Puisque A et B sont vraies, on peut déduire indépendamment A ou B.

On conserve toujours au moins une conjonction, soit A ou B.

Quelles que soient les propositions A B,  
 $(A \wedge B) \Rightarrow (B \wedge A)$



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<http://edukera.appspot.com>  
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## EXERCICE 8



Logique constructiviste

Démontrer :

Quelles que soient les propositions A B C,  
 $(A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C$

Soit la proposition A

Soit la proposition B

Soit la proposition C

Conclusion

$(A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C \quad (1)$

Justifier

Reste à justifier (1)

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## Bibliography Notes

[Girard et al., 1989]

A short (160p.) book addressing all the concerns of this course, and more (especially Linear Logic). Easy and pleasant to read. Now available for free.

[Girard, 2004]

A much more comprehensive book focusing on logic and its connections with computer science. In French.

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