Natural Deduction

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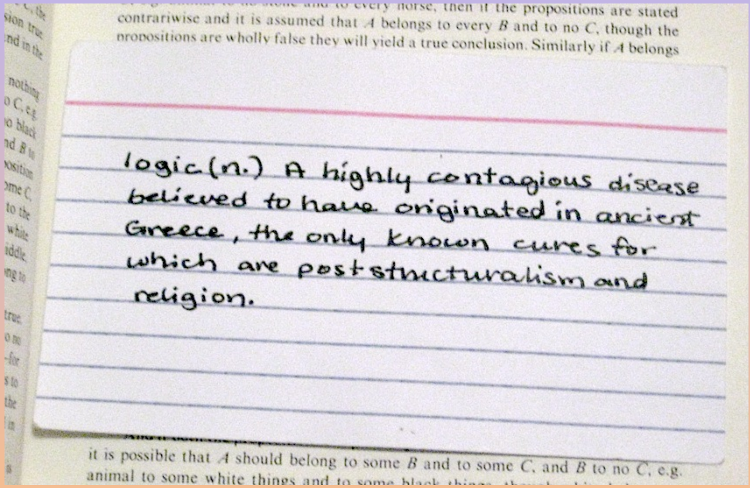
EPITA — École Pour l'Informatique et les Techniques Avancées

June 14, 2016

Define "logic" [fuc,]



Natural Deduction

Preamble

The following slides are implicitly dedicated to classical logic.

Logical Formalisms

1. Logical Formalisms
   - Syntax
   - Proof Types
   - Proof Systems
2. Natural Deduction
3. Additional Material

Syntax

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Terminal Symbols

Propositional Calculus

Constants	$a, b, c, \ldots$
Propositional Variables	$A, B, C, \ldots$
Unary Connective	$\neg$
Binary Connectives	$\wedge, \vee, \Rightarrow$
Punctuation	$(,), [,]$.

Terminal Symbols

Predicate calculus

Individual Variables	$x, y, z, \ldots$
Functions	$f, g, h, \ldots$, with a fixed arity
Predicates	$P, Q, R, \ldots$, with a fixed arity
Quantifiers	$\forall, \exists$
Punctuation	$\cdot$.

Propositional Formulas

$$\begin{array}{rcl}\langle\text{formula}\rangle & ::= & \langle\text{propositional variable}\rangle \\ & | & \neg\langle\text{formula}\rangle \\ & | & \langle\text{formula}\rangle \wedge \langle\text{formula}\rangle \\ & | & \langle\text{formula}\rangle \vee \langle\text{formula}\rangle \\ & | & \langle\text{formula}\rangle \Rightarrow \langle\text{formula}\rangle\end{array}$$

Terms

$$\begin{array}{rcl}\langle\text{term}\rangle & ::= & \langle\text{constant}\rangle \\ & | & \langle\text{function}\rangle(\langle\text{term}\rangle, \ldots)\end{array}$$

With the proper arity.

First Order Formulas

$$\begin{array}{rcl}\langle\text{formula}\rangle & ::= & \langle\text{propositional variable}\rangle \\ & | & \neg\langle\text{formula}\rangle \\ & | & \langle\text{formula}\rangle \wedge \langle\text{formula}\rangle \\ & | & \langle\text{formula}\rangle \vee \langle\text{formula}\rangle \\ & | & \langle\text{formula}\rangle \Rightarrow \langle\text{formula}\rangle \\ & | & \langle\text{predicate}\rangle(\langle\text{term}\rangle, \ldots) \\ & | & \forall\langle\text{individual variable}\rangle \cdot \langle\text{formula}\rangle \\ & | & \exists\langle\text{individual variable}\rangle \cdot \langle\text{formula}\rangle\end{array}$$

With the proper arity.

Syntactic Conventions

Associativity
- $\wedge, \vee$ are left-associative (unimportant)
- $\Rightarrow$ is right-associative (very important)

Precedence (increasing)
1. $\forall, \exists$
2. $\Rightarrow$
3. $\vee$
4. $\wedge$
5. $\neg$

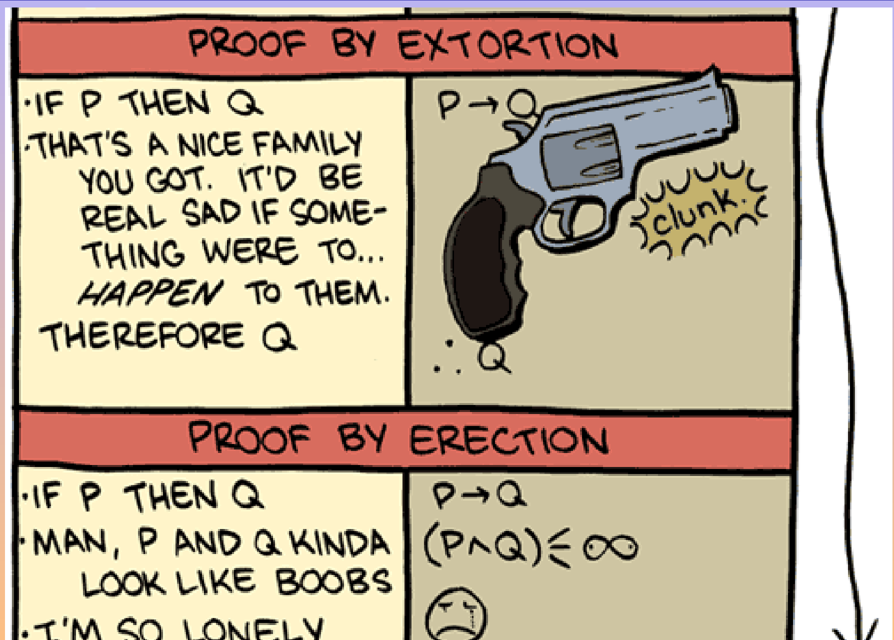
Free Variables

$$
\begin{aligned}
\mathrm{FV}(X) &= \emptyset \\
\mathrm{FV}(P(x_1, x_2, \cdots, x_n)) &= \{x_1, x_2, \cdots, x_n\} \\
\mathrm{FV}(\neg A) &= \mathrm{FV}(A) \\
\mathrm{FV}(A \vee B) &= \mathrm{FV}(A) \cup \mathrm{FV}(B) \\
\mathrm{FV}(A \wedge B) &= \mathrm{FV}(A) \cup \mathrm{FV}(B) \\
\mathrm{FV}(A \Rightarrow B) &= \mathrm{FV}(A) \cup \mathrm{FV}(B) \\
\mathrm{FV}(\forall x \cdot A) &= \mathrm{FV}(A) - \{x\} \\
\mathrm{FV}(\exists x \cdot A) &= \mathrm{FV}(A) - \{x\}
\end{aligned}
$$

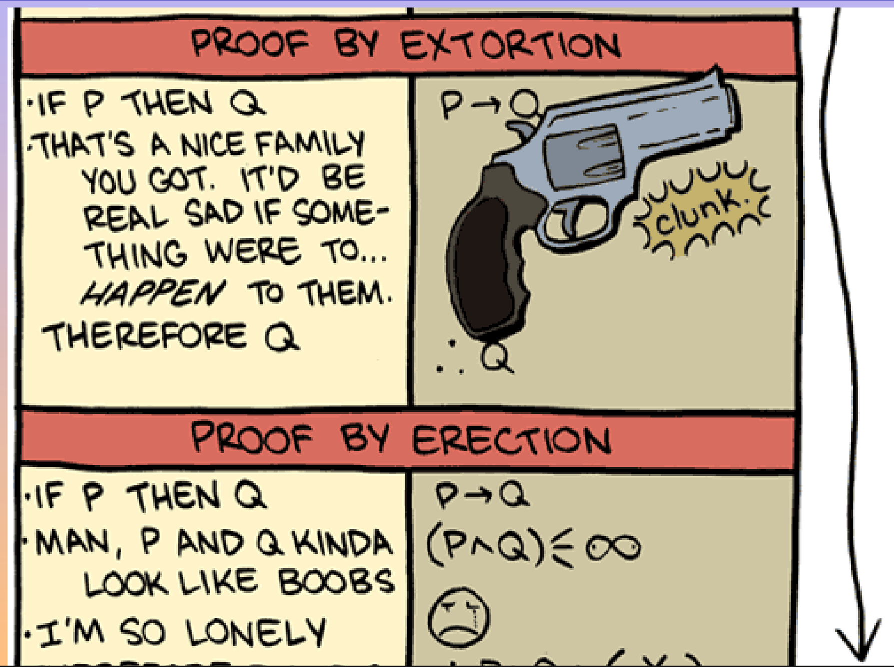
Proof Types

1. Logical Formalisms
   - Syntax
   - Proof Types
   - Proof Systems
2. Natural Deduction
3. Additional Material

Different Proof Types



Different Proof Types



Proof Systems

Proof Systems

- Hilbertian Systems
- Natural Deduction
- Sequent Calculus
- Natural Deduction in Sequent Calculus

Axioms

- Axioms are formulas that are considered true a priori

$$\forall x \cdot x + 0 = x$$

- Axiom schemes use meta-variables
  (that range over a specific domain)

$$X + Y = Y + X$$

- Axiom schemes are used when quantifiers are not welcome

$$\begin{aligned} SXYZ &\rightarrow XZ(YZ) \\ KXY &\rightarrow X \end{aligned}$$

- Axiom schemes are used when quantifiers do not apply

$$A \vee \neg A$$

Inference Rules

$$\frac{H_1 \quad H_2 \quad \cdots \quad H_n}{C}\text{Rule name}$$

$$\frac{}{A}\text{Axiom name}$$

Logical Formalisms

David Hilbert (1862–1943)

Hilbertian System

- A single inference rule: the modus ponens

$$\frac{A \qquad A \Rightarrow B}{B}\ \textit{modus ponens}$$

- Many axioms to define the connectives

$$A \Rightarrow B \Rightarrow A \wedge B \qquad A \wedge B \Rightarrow A \qquad A \wedge B \Rightarrow B$$

$$A \Rightarrow A \vee B \qquad B \Rightarrow A \vee B$$
$$A \vee B \Rightarrow \ (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C$$

$$A \Rightarrow B \Rightarrow A \qquad (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C$$

$$A \vee \neg A \qquad A \Rightarrow \neg A \Rightarrow B$$

Hilbertian System: Prove $A \Rightarrow A$

$$\dfrac{\dfrac{\dfrac{}{(A \Rightarrow ((A \Rightarrow A) \Rightarrow A)) \Rightarrow (A \Rightarrow A \Rightarrow A) \Rightarrow A \Rightarrow A} \qquad \dfrac{}{A \Rightarrow (A \Rightarrow A) \Rightarrow A}}{(A \Rightarrow A \Rightarrow A) \Rightarrow A \Rightarrow A} \qquad \dfrac{}{A \Rightarrow A \Rightarrow A}}{A \Rightarrow A}$$

Natural Deduction

Syntax

Deduction

Deduction

A deduction is a tree whose root (A) is the conclusion and whose active leafs (Γ) is the set of hypotheses.

$$\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}$$

Any formula A is a valid hypothesis.

Proof (Demonstration)

A proof is a deduction without hypotheses.

Deductions

What's this?

$$A$$

A deduction of A under the hypothesis A.

Implication

$$\dfrac{\begin{array}{c}[A]\\ \vdots\\ B\end{array}}{A \Rightarrow B}\Rightarrow\mathcal{I} \qquad \dfrac{\begin{array}{c}\vdots\\ A\end{array} \quad \begin{array}{c}\vdots\\ A \Rightarrow B\end{array}}{B}\Rightarrow\mathcal{E}$$

Deduction theorem, and Modus Ponens.
Note the connection with (left) contraction: any number of A (including 0) is discharged.

Proving $A \Rightarrow A$ in Natural Deduction

$$\frac{[A]}{A \Rightarrow A} \Rightarrow\mathcal{I}$$

Conjunction

$$\frac{\begin{matrix}\vdots \\ A\end{matrix} \quad \begin{matrix}\vdots \\ B\end{matrix}}{A \wedge B} \wedge\mathcal{I} \qquad \frac{\begin{matrix}\vdots \\ A \wedge B\end{matrix}}{A} \wedge l\mathcal{E} \qquad \frac{\begin{matrix}\vdots \\ A \wedge B\end{matrix}}{B} \wedge r\mathcal{E}$$

Universal Quantification

$$\frac{\begin{matrix}\vdots \\ A[y/x]\end{matrix}}{\forall x \cdot A} \forall\mathcal{I} \quad y \notin \mathrm{FV}(hyp(A)) \qquad \frac{\begin{matrix}\vdots \\ \forall x \cdot A\end{matrix}}{A[t/x]} \forall\mathcal{E}$$

$$\frac{\dfrac{[A]}{\forall x \cdot A} \forall\mathcal{I}}{A \Rightarrow \forall x \cdot A} \Rightarrow\mathcal{I} \qquad \frac{\dfrac{[A]}{A \Rightarrow A} \Rightarrow\mathcal{I}}{\forall x \cdot (A \Rightarrow A)} \forall\mathcal{I}$$

Absurd

$$\frac{\begin{matrix}\vdots \\ \bot\end{matrix}}{A} \bot\mathcal{E}$$

Disjunction

$$\dfrac{\begin{matrix}\vdots\\ A\end{matrix}}{A \vee B}\,\vee l\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots\\ B\end{matrix}}{A \vee B}\,\vee r\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots\\ A \vee B\end{matrix} \quad \begin{matrix}[A]\\ \vdots\\ C\end{matrix} \quad \begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{C}\,\vee\mathcal{E}$$

Existential Quantification

$$\dfrac{\begin{matrix}\vdots\\ A[t/x]\end{matrix}}{\exists x \cdot A}\,\exists\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots\\ \exists x \cdot A\end{matrix} \quad \begin{matrix}[A[y/x]]\\ \vdots\\ B\end{matrix}}{B}\,\exists\mathcal{E} \quad y \notin \mathrm{FV}(B, hyp(B))$$

For elimination, $y \notin hyp(B)$, i.e., not in the hypotheses other than the discharged A.

Negation

$$\dfrac{\begin{matrix}[A]\\ \vdots\\ \bot\end{matrix}}{\neg A}\,\neg\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots\\ A\end{matrix} \quad \begin{matrix}\vdots\\ \neg A\end{matrix}}{\bot}\,\neg\mathcal{E}$$

Plus one of these equivalent formulations of the fact that classical negation is involutive.

$$\dfrac{}{A \vee \neg A}\,\mathrm{XM} \qquad \dfrac{\begin{matrix}\vdots\\ \neg\neg A\end{matrix}}{A}\,\neg\neg \qquad \dfrac{\begin{matrix}[\neg A]\\ \vdots\\ B\end{matrix} \quad \begin{matrix}[\neg A]\\ \vdots\\ \neg B\end{matrix}}{A}\,\mathrm{Contradiction}$$

Normalization

Cut

Cut: Introduction of a connective followed by its elimination.

$$\begin{array}{c}\dfrac{\dfrac{\begin{matrix}\vdots & & \vdots\\ A & & B\end{matrix}}{A \wedge B}\,\wedge\mathcal{I}}{A}\,\wedge l\mathcal{E}\\ \vdots\end{array}$$

Normalization

The normalization process eliminates the cuts.

$$\dfrac{\dfrac{\begin{matrix}\vdots & & \vdots\\ A & & B\end{matrix}}{A \wedge B}\,\wedge\mathcal{I}}{A}\,\wedge l\mathcal{E} \quad \leadsto \quad \begin{array}{c}\vdots\\ A\\ \vdots\end{array}$$

Normalizing Conjunctions

$$\begin{array}{c}\dfrac{\dfrac{\begin{matrix}\vdots & & \vdots\\ A & & B\end{matrix}}{A \wedge B}\,\wedge\mathcal{I}}{A}\,\wedge l\mathcal{E}\\ \vdots\end{array} \quad \leadsto \quad \begin{array}{c}\vdots\\ A\\ \vdots\end{array}$$

$$\begin{array}{c}\dfrac{\dfrac{\begin{matrix}\vdots & & \vdots\\ A & & B\end{matrix}}{A \wedge B}\,\wedge\mathcal{I}}{B}\,\wedge r\mathcal{E}\\ \vdots\end{array} \quad \leadsto \quad \begin{array}{c}\vdots\\ B\\ \vdots\end{array}$$

Normalizing Implications

$$\begin{array}{c}\dfrac{\begin{matrix}\vdots\\ A\end{matrix} \quad \dfrac{\begin{matrix}[A]\\ \vdots\\ B\end{matrix}}{A \Rightarrow B}\,\Rightarrow\mathcal{I}}{B}\,\Rightarrow\mathcal{E}\\ \vdots\end{array} \quad \leadsto \quad \begin{array}{c}\vdots\\ A\\ \vdots\\ B\\ \vdots\end{array}$$

Normalizing Universal Quantifiers

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ A\end{array}}{\forall x \cdot A}\,\forall\mathcal{I}}{\begin{array}{c}A[t/x]\\ \vdots\end{array}}\,\forall\mathcal{E} \quad\rightsquigarrow\quad \begin{array}{c}\vdots\\ A[t/x]\\ \vdots\end{array}$$

x must not be free in the hypotheses, otherwise the reduction would change them.

Normalizing Disjunction

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ A\end{array}}{A \vee B}\,\vee l\mathcal{I} \quad \begin{array}{c}[A]\\ \vdots\\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots\\ C\end{array}}{\begin{array}{c}C\\ \vdots\end{array}}\,\vee\mathcal{E} \quad\rightsquigarrow\quad \begin{array}{c}\vdots\\ A\\ \vdots\\ C\\ \vdots\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ B\end{array}}{A \vee B}\,\vee r\mathcal{I} \quad \begin{array}{c}[A]\\ \vdots\\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots\\ C\end{array}}{\begin{array}{c}C\\ \vdots\end{array}}\,\vee\mathcal{E} \quad\rightsquigarrow\quad \begin{array}{c}\vdots\\ B\\ \vdots\\ C\\ \vdots\end{array}$$

Additional Material

Logicians in a Bar

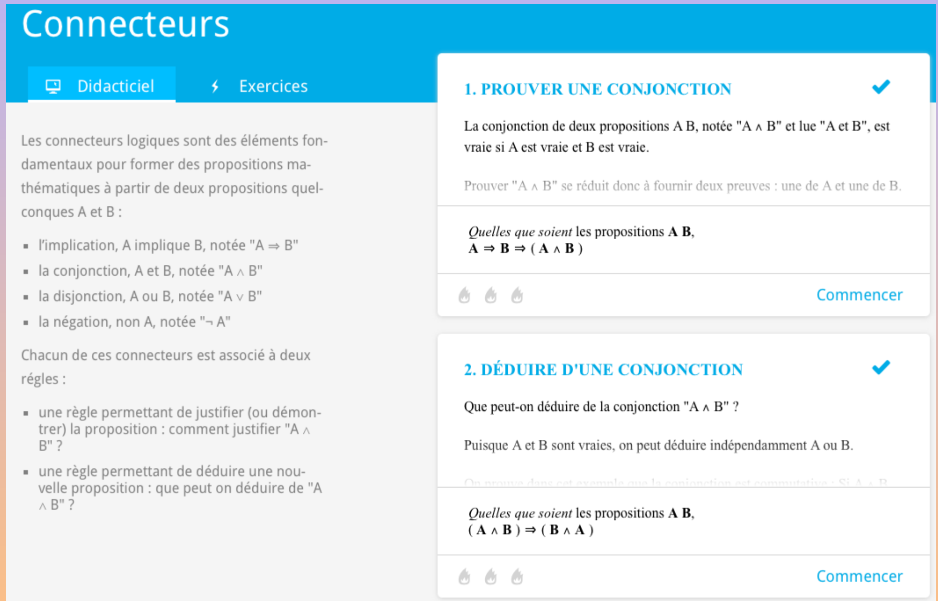
Three logicians walk into a bar,
and the bartender asks “Would you all like a drink?”

The first one says, “Maybe.”

The second one says, “Maybe.”

The third one says, “Yes.”

Edukera



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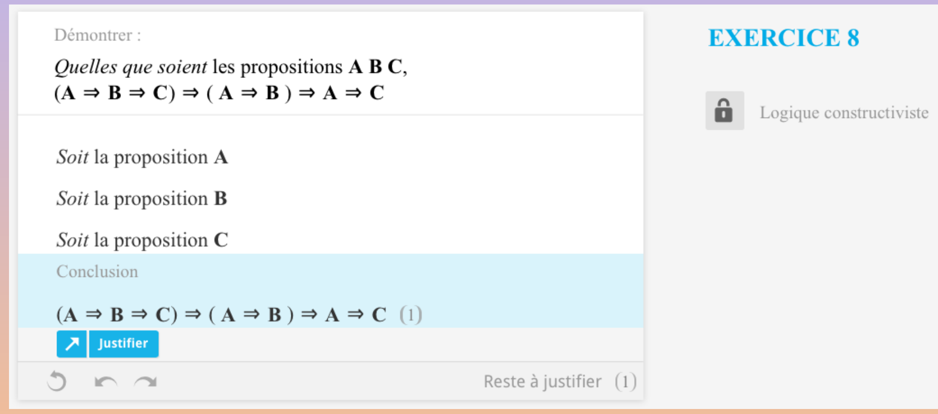

Edukera

`http://edukera.appspot.com`

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Bibliography Notes

[Girard et al., 1989]
A short (160p.) book addressing all the concerns of this course, and more (especially Linear Logic). Easy and pleasant to read. Now available for free.

[Girard, 2004]
A much more comprehensive book focusing on logic and its connections with computer science. In French.

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