Sequent Calculus
Cut Elimination

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EPITA — École Pour l'Informatique et les Techniques Avancées

June 14, 2016

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Preamble

The following slides are implicitly dedicated to classical logic.

Problems of Natural Deduction

Normalization

Proofs hard to find

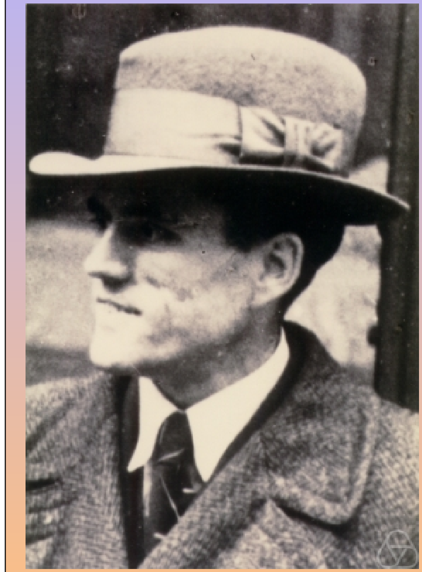
Some elimination rules used formulas coming out of the blue.

$$\dfrac{\begin{matrix}\vdots \\ A \vee B\end{matrix} \quad \begin{matrix}[A] \\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}}{C} \vee\mathcal{E}$$

Negation is awkward

Sequent Calculus

Gerhard Karl Erich Gentzen (1909–1945)



German logician and mathematician.
SA (1933)
An assistant of David Hilbert in Göttingen (1935–1939).
Joined the Nazi Party (1937).
Worked on the V2.
Died of malnutrition in a prison camp (August 4, 1945).

Syntax

Sequents

Sequent

A sequent is an expression $\Gamma \vdash \Delta$, where Γ, Δ are (finite) sequences of formulas.

Variants:

Sets Γ, Δ can be taken as (finite) sets.
- Simplifies the structural group
- Prevents close examination of... the structural rules

Single Sided Sequents can be forced to be $\vdash \Gamma$
- Half the number of rules is needed
- Not for intuitionistic systems

Reading a Sequent

$\Gamma \vdash \Delta$ If all the formulas of Γ are true, then one of the formulas of Δ is true.

- Commas on the left hand side stand for "and"
- Turnstile, $\vdash$, stands for "implies"
- Commas on the right hand side stand for "or"

Some Special Sequents

$\Gamma \vdash A$	A is true under the hypotheses Γ
$\vdash A$	A is true
$\Gamma \vdash$	Γ is in contradiction
$A \vdash$	$\neg A$
$\vdash$	contradiction

LK — Classical Sequent Calculus

LK — Classical Sequent Calculus

LK — Gentzen 1934

logistischer klassischer Kalkül.

- There are several possible exposures
- Different sets of inference rules
- We follow [Girard, 2011]

Identity Group

$$\frac{}{A \vdash A}\,\mathrm{Id} \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

Gentzen's Hauptsatz

The cut rule is redundant, i.e., any sequent provable with cuts is provable without.

Structural Group

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \tau(\Delta)}\vdash\mathrm{X} \qquad \frac{\Gamma \vdash \Delta}{\sigma(\Gamma) \vdash \Delta}\mathrm{X}\vdash$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\vdash\mathrm{W} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\mathrm{W}\vdash$$

$$\frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\vdash\mathrm{C} \qquad \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\mathrm{C}\vdash$$

Logical Group: Negation

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\vdash\neg \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\neg\vdash$$

Logical Group: Conjunction

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\vdash\wedge \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}l\wedge\vdash \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}r\wedge\vdash$$

Additive $\&$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'}\vdash\wedge \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\wedge\vdash$$

Multiplicative $\otimes$

Logical Group: Disjunction

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\vdash l\vee \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\vdash r\vee \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\vee\vdash$$

Additive $\oplus$

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\vdash\vee \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'}\vee\vdash$$

Multiplicative $\parr$

Logical Group: Implication

$$\frac{\Gamma \vdash \Delta, A \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \Rightarrow B \vdash \Delta, \Delta'} \Rightarrow\vdash \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta} \vdash\Rightarrow$$

Prove $A \wedge B \vdash A \wedge B$

$$\dfrac{\dfrac{\overline{A \vdash A}}{A \wedge B \vdash A}\, l\wedge\vdash \qquad \dfrac{\overline{B \vdash B}}{A \wedge B \vdash B}\, r\wedge\vdash}{A \wedge B \vdash A \wedge B} \vdash\wedge$$

Prove $A \wedge B \vdash A \vee B$

$$\dfrac{\dfrac{\overline{A \vdash A}}{A \wedge B \vdash A}\, l\wedge\vdash}{A \wedge B \vdash A \vee B} \vdash r\vee$$

Prove $A \vee B \vdash A \vee B$

$$\dfrac{\dfrac{\overline{A \vdash A}}{A \vdash A \vee B} \vdash l\vee \qquad \dfrac{\overline{B \vdash B}}{B \vdash A \vee B} \vdash r\vee}{A \vee B \vdash A \vee B} \vee\vdash$$

Prove the equivalence of the two ∧ rules

Additive — Multiplicative

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \vdash \wedge + \quad \equiv \quad \frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'} \vdash \wedge \times$$

$$\frac{\dfrac{\dfrac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma, \Gamma \vdash A \wedge B, \Delta, \Delta} \vdash \wedge \times}{\Gamma, \Gamma \vdash A \wedge B, \Delta} \vdash \mathrm{C}}{\Gamma \vdash A \wedge B, \Delta} \mathrm{C} \vdash$$

$$\frac{\dfrac{\dfrac{\Gamma \vdash A, \Delta}{\Gamma, \Gamma' \vdash A, \Delta} \mathrm{W} \vdash}{\Gamma, \Gamma' \vdash A, \Delta, \Delta'} \vdash \mathrm{W} \quad \dfrac{\dfrac{\Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash B, \Delta'} \mathrm{W} \vdash}{\Gamma, \Gamma' \vdash B, \Delta, \Delta'} \vdash \mathrm{W}}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'} \vdash \wedge +$$

Logical Group: Quantifiers

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash \forall x \cdot A, \Delta} \vdash \forall \qquad \frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x \cdot A \vdash \Delta} \forall \vdash$$

$$\frac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x \cdot A, \Delta} \vdash \exists \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, \exists x \cdot A \vdash \Delta} \exists \vdash$$

In $\vdash \forall$ and $\exists \vdash$, $x \notin \mathrm{FV}(\Gamma, \Delta)$.

Single Sided

Defining the Negation

- Alternatively, one can define the negation as a notation instead of defining it by inference rules.

$$\begin{aligned} \neg(\neg p) &:= p \\ \neg(A \wedge B) &:= \neg A \vee \neg B \\ \neg(A \vee B) &:= \neg A \wedge \neg B \\ \neg(\forall x \cdot A) &:= \exists x \cdot \neg A \\ \neg(\exists x \cdot A) &:= \forall x \cdot \neg A \end{aligned}$$

- Then define the sequents as $\vdash \Gamma$
- I.e., $\Gamma \vdash \Delta \rightsquigarrow \vdash \neg\Gamma, \Delta$

Single Sided

The Full Sequent Calculus

$$\frac{}{\vdash \neg A, A} \mathrm{Id} \qquad \frac{\vdash \Gamma, A \quad \vdash \neg A, \Delta}{\vdash \Gamma, \Delta} \mathrm{Cut}$$

$$\frac{\vdash \Gamma}{\vdash \sigma(\Gamma)} \mathrm{X} \qquad \frac{\vdash \Gamma}{\vdash A, \Gamma} \mathrm{W} \qquad \frac{\vdash A, A, \Gamma}{\vdash A, \Gamma} \mathrm{C}$$

$$\frac{\vdash A, \Delta}{\vdash A \vee B, \Delta} l\vee \qquad \frac{\vdash B, \Delta}{\vdash A \vee B, \Delta} r\vee \qquad \frac{\vdash A, \Delta \quad \vdash B, \Delta}{\vdash A \wedge B, \Delta} \wedge$$

$$\frac{\vdash A, \Delta}{\vdash \forall x \cdot A, \Delta} \forall \qquad \frac{\vdash A[t/x], \Delta}{\vdash \exists x \cdot A, \Delta} \exists$$

Cut Elimination

Subformula Property

What can be told from the last rule of a proof?

- If the last rule is a cut,

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

  nothing can be said!
- Otherwise. . .

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \vdash \wedge \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\, l\wedge \vdash \qquad \cdots$$

  premisses can only use subformulas of the conclusion!

Cut Elimination

- replace "complex" cuts by simpler cuts (smaller formulas)
- until the cut is on the simplest form, the identity
- where it is not longer needed!

Cut Elimination

Logical Rules

$$\frac{\dfrac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \vdash \wedge \qquad \dfrac{\Gamma', A \vdash \Delta'}{\Gamma', A \wedge B \vdash \Delta'}\, l\wedge \vdash}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

$$\leadsto$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

For all the connectives.

Cut Elimination
Removal of a Cut

$$\dfrac{\dfrac{}{A \vdash A}\text{Identity} \qquad \dfrac{\vdots}{\Gamma, A \vdash \Delta}}{\Gamma, A \vdash \Delta}\text{Cut}$$

$$\rightsquigarrow$$

$$\dfrac{\vdots}{\Gamma, A \vdash \Delta}$$

Cut Elimination
Commutations

$$\dfrac{\dfrac{\Gamma, B \vdash A, \Delta \quad \Gamma, C \vdash A, \Delta}{\Gamma, B \vee C \vdash A, \Delta}\vee\vdash \qquad \Gamma', A \vdash \Delta'}{\Gamma, B \vee C, \Gamma' \vdash \Delta, \Delta'}\text{Cut}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\Gamma, B \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, B, \Gamma' \vdash \Delta, \Delta'}\text{Cut} \qquad \dfrac{\Gamma, C \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, C, \Gamma' \vdash \Delta, \Delta'}\text{Cut}}{\Gamma, B \vee C, \Gamma' \vdash \Delta, \Delta'}\vee\vdash$$

Beware of the duplication!

Cut Elimination
Structural Rules: Weakening

$$\dfrac{\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\vdash\text{W} \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\text{Cut}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\Gamma \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta}\text{W}\vdash}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\vdash\text{W}$$

Cut Elimination
Structural Rules: Contraction

$$\dfrac{\dfrac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\vdash\text{C} \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\text{Cut}$$

Nice!

$$\rightsquigarrow$$

but wrong

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash A, A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash A, \Delta, \Delta'}\text{Cut} \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma', \Gamma' \vdash \Delta, \Delta', \Delta'}\text{Cut}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\text{C}$$

might loop for ever

Cut Elimination

Structural Rules: Complications with Contractions

$$\dfrac{\dfrac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\,{\vdash}\mathrm{C} \qquad \dfrac{\Gamma', A, A \vdash \Delta'}{\Gamma', A \vdash \Delta'}\,\mathrm{C}{\vdash}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash A, A, \Delta \qquad \dfrac{\Gamma', A, A \vdash \Delta'}{\Gamma', A \vdash \Delta'}\,\mathrm{C}{\vdash}}{\Gamma, \Gamma' \vdash A, \Delta, \Delta'}\,\mathrm{Cut} \qquad \dfrac{\Gamma', A, A \vdash \Delta'}{\Gamma', A \vdash \Delta'}\,\mathrm{C}{\vdash}}{\Gamma, \Gamma', \Gamma' \vdash \Delta, \Delta', \Delta'}\,\mathrm{Cut}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{C}$$

Natural Deduction in Sequent Calculus

Recommended readings

[Girard et al., 1989], Chapters 5 & 13
: A short (160p.) book addressing all the concerns of this course, and more (especially Linear Logic).
Easy and pleasant to read. Available for free.

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