Exercises on λ-calculus and Deduction Systems

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Exercises on λ-calculus and Deduction Systems

1. λ-calculus

2. Deduction Systems

λ-calculus

1. **λ-calculus**
   - Untyped λ-calculus
   - Simply Typed λ-calculus

2. Deduction Systems

Untyped λ-calculus

1. **λ-calculus**
   - Untyped λ-calculus
   - Simply Typed λ-calculus

2. Deduction Systems

Substitutions

$$[\lambda z \cdot zz/x]\lambda y \cdot xy \equiv \lambda y \cdot (\lambda z \cdot zz)y$$
$$[yy/z](\lambda xy \cdot zy) \equiv \lambda xu \cdot yyu$$

β-Reductions

$$
\begin{aligned}
(\lambda x \cdot xyx)\lambda z \cdot z &\rightarrow (\lambda z \cdot z)y(\lambda z \cdot z) \\
(\lambda x \cdot x)((\lambda y \cdot y)x) &\rightarrow (\lambda x \cdot x)(x) \\
(\lambda x \cdot x)((\lambda y \cdot y)x) &\rightarrow ((\lambda y \cdot y)x) \\
(\lambda x \cdot x)((\lambda y \cdot y)x) &\overset{*}{\rightarrow} x \\
(\lambda x \cdot xx)((\lambda x \cdot xx)y) &\overset{*}{\rightarrow} yy(yy) \\
(\lambda x \cdot xx)((\lambda x \cdot x)y) &\overset{*}{\rightarrow} yy \\
(\lambda x \cdot x)((\lambda x \cdot xx)y) &\overset{*}{\rightarrow} yy
\end{aligned}
$$

Simply Typed λ-calculus

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2. Deduction Systems

Simply Typed λ-calculus

Type derivations are trees built from the following nodes.

$$
\frac{M : \sigma \to \tau \quad N : \sigma}{MN : \tau}
\qquad
\frac{\begin{array}{c}[x : \sigma]\\ \vdots \\ M : \tau\end{array}}{\lambda x \cdot M : \sigma \to \tau}
$$

Type Statements
Type $\lambda fx \cdot f(fx)$

$$\vdash \lambda fx \cdot f(fx) : (\sigma \to \sigma) \to \sigma \to \sigma$$

$$\cfrac{[f : \sigma \to \sigma]^{(2)} \quad \cfrac{\cfrac{[f : \sigma \to \sigma]^{(2)} \quad [x : \sigma]^{(1)}}{fx : \sigma}}{f(fx) : \sigma}}{\cfrac{\lambda x \cdot f(fx) : \sigma \to \sigma}{\lambda fx \cdot f(fx) : (\sigma \to \sigma) \to \sigma \to \sigma} \, (2)} \, (1)$$

Type Statements
Type $\lambda xy \cdot x$

$$\vdash \lambda xy \cdot x : \sigma \to \tau \to \sigma$$

$$\cfrac{\cfrac{\cfrac{[x : \sigma]^{(1)}}{\lambda y \cdot x : \tau \to \sigma}}{\lambda xy \cdot x : \sigma \to \tau \to \sigma}}{} \, (1)$$

Deduction Systems

1 λ-calculus

2 Deduction Systems
  - Natural Deduction
  - Sequent Calculus

Natural Deduction

1 λ-calculus

2 Deduction Systems
  - Natural Deduction
  - Sequent Calculus

Intuitionistic Natural Deduction

$$\frac{\begin{array}{c}[A]\\ \vdots\\ B\end{array}}{A \Rightarrow B}\Rightarrow\mathcal{I} \qquad \frac{A \quad A \Rightarrow B}{B}\Rightarrow\mathcal{E} \qquad \frac{\bot}{A}\bot\mathcal{E} \qquad \neg A := A \Rightarrow \bot$$

$$\frac{A \quad B}{A \wedge B}\wedge\mathcal{I} \qquad \frac{A \wedge B}{A}\wedge l\mathcal{E} \qquad \frac{A \wedge B}{B}\wedge r\mathcal{E}$$

$$\frac{A}{A \vee B}\vee l\mathcal{I} \qquad \frac{B}{A \vee B}\vee r\mathcal{I} \qquad \frac{A \vee B \quad \begin{array}{c}[A]\\ \vdots\\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots\\ C\end{array}}{C}\vee\mathcal{E}$$

Prove $A \wedge B \Rightarrow B \wedge A$

$$\cfrac{\cfrac{\cfrac{[A \wedge B]^1}{B}\wedge r\mathcal{E} \quad \cfrac{[A \wedge B]^1}{A}\wedge l\mathcal{E}}{B \wedge A}\wedge\mathcal{I}}{A \wedge B \Rightarrow B \wedge A}\Rightarrow\mathcal{I}_1$$

Prove $A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$

$$\cfrac{\cfrac{A \wedge (B \vee C)}{B \vee C}\wedge r\mathcal{E} \quad \cfrac{\cfrac{\cfrac{A \wedge (B \vee C)}{A}\wedge l\mathcal{E} \quad [B]^1}{A \wedge B}\wedge\mathcal{I}}{(A \wedge B) \vee (A \wedge C)}\vee l\mathcal{I} \quad \cfrac{\cfrac{\cfrac{A \wedge (B \vee C)}{A}\wedge l\mathcal{E} \quad [C]^1}{A \wedge C}\wedge\mathcal{I}}{(A \wedge B) \vee (A \wedge C)}\vee r\mathcal{I}}{(A \wedge B) \vee (A \wedge C)}\vee\mathcal{E}_1$$

Prove $(A \Rightarrow A) \Rightarrow A \Rightarrow A$ (LOFO-2005)

Remember, $\Rightarrow$ is right-associative.

$$\cfrac{\cfrac{\cfrac{\cfrac{[A]^2 \quad [A \Rightarrow A]^1}{A}\Rightarrow\mathcal{E}}{A \Rightarrow A}\Rightarrow\mathcal{I}_2}{(A \Rightarrow A) \Rightarrow A \Rightarrow A}\Rightarrow\mathcal{I}_1}{} \qquad \cfrac{\cfrac{\cfrac{[A]^1}{A \Rightarrow A}\Rightarrow\mathcal{I}_1}{(A \Rightarrow A) \Rightarrow A \Rightarrow A}\Rightarrow\mathcal{I}}{}$$

$$\cfrac{[A \Rightarrow A]^1}{(A \Rightarrow A) \Rightarrow A \Rightarrow A}\Rightarrow\mathcal{I}_1$$

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Prove $A \Rightarrow B, B \Rightarrow C \vdash A \Rightarrow (B \wedge C)$ (LOFO-2005)

$$\cfrac{\cfrac{\cfrac{[A]^1 \quad A \Rightarrow B}{B} \Rightarrow \mathcal{E} \quad \cfrac{\cfrac{[A]^1 \quad A \Rightarrow B}{B} \Rightarrow \mathcal{E} \quad B \Rightarrow C}{C} \Rightarrow \mathcal{E}}{B \wedge C} \wedge \mathcal{I}}{A \Rightarrow (B \wedge C)} \Rightarrow \mathcal{I}_1$$

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Prove $A \vee B, \neg B \vdash A$ (Intuitionistic) (LOFO-2005)

Recall that $\neg B := B \Rightarrow \bot$.

$$\cfrac{A \vee B \quad [A]^1 \quad \cfrac{\cfrac{[B]^1 \quad B \Rightarrow \bot}{\bot} \Rightarrow \mathcal{E}}{A} \bot \mathcal{E}}{A} \vee \mathcal{E}_1$$

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Sequent Calculus

1. λ-calculus

2. Deduction Systems
   - Natural Deduction
   - Sequent Calculus

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Classical Sequent Calculus

$$\cfrac{\Gamma \vdash \Delta}{\Gamma \vdash \tau(\Delta)} \vdash \mathrm{X} \qquad \cfrac{\Gamma \vdash \Delta}{\sigma(\Gamma) \vdash \Delta} \mathrm{X} \vdash \qquad \cfrac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \vdash \mathrm{W} \qquad \cfrac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \mathrm{W} \vdash \qquad \cfrac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \vdash \mathrm{C} \qquad \cfrac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \mathrm{C} \vdash$$

$$\cfrac{}{F \vdash F} \mathrm{Id} \qquad \cfrac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \mathrm{Cut}$$

$$\cfrac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \vdash \neg \qquad \cfrac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \neg \vdash$$

$$\cfrac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \vdash \wedge \qquad \cfrac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} l \wedge \vdash \qquad \cfrac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} r \wedge \vdash$$

$$\cfrac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta} \vdash l \vee \qquad \cfrac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vdash r \vee \qquad \cfrac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \vee \vdash$$

$$\cfrac{\Gamma \vdash \Delta, A \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \Rightarrow B \vdash \Delta, \Delta'} \Rightarrow \vdash \qquad \cfrac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta} l \Rightarrow$$

Prove $A \wedge B \vdash A \wedge B$

$$\cfrac{\cfrac{\overline{A \vdash A}}{A \wedge B \vdash A}\, l\wedge\vdash \qquad \cfrac{\overline{B \vdash B}}{A \wedge B \vdash B}\, r\wedge\vdash}{A \wedge B \vdash A \wedge B}\, \vdash\wedge$$

Prove $A \wedge B \vdash A \vee B$

$$\cfrac{\cfrac{\overline{A \vdash A}}{A \wedge B \vdash A}\, l\wedge\vdash}{A \wedge B \vdash A \vee B}\, \vdash r\vee$$

Prove $A \vee B \vdash A \vee B$

$$\cfrac{\cfrac{\overline{A \vdash A}}{A \vdash A \vee B}\, \vdash l\vee \qquad \cfrac{\overline{B \vdash B}}{B \vdash A \vee B}\, \vdash r\vee}{A \vee B \vdash A \vee B}\, \vee\vdash$$

Prove $(F \wedge G) \Rightarrow H \vdash (F \Rightarrow H) \vee (G \Rightarrow H)$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\overline{F \vdash F}}{F, G \vdash F}\, W\vdash \quad \cfrac{\overline{G \vdash G}}{F, G \vdash G}\, W\vdash}{F, G \vdash F \wedge G}\, \vdash\wedge \quad \overline{H \vdash H}}{F, G, (F \wedge G) \Rightarrow H \vdash H}\, \Rightarrow\vdash}{F, G, (F \wedge G) \Rightarrow H \vdash H, H}\, \vdash W}{F, (F \wedge G) \Rightarrow H \vdash H, G \Rightarrow H}\, \vdash\Rightarrow}{(F \wedge G) \Rightarrow H \vdash F \Rightarrow H, G \Rightarrow H}\, \vdash\Rightarrow}{(F \wedge G) \Rightarrow H \vdash (F \Rightarrow H) \vee (G \Rightarrow H), G \Rightarrow H}\, \vdash r\vee}{(F \wedge G) \Rightarrow H \vdash (F \Rightarrow H) \vee (G \Rightarrow H), (F \Rightarrow H) \vee (G \Rightarrow H)}\, \vdash l\vee}{(F \wedge G) \Rightarrow H \vdash (F \Rightarrow H) \vee (G \Rightarrow H)}\, \vdash C$$

Prove $(A \Rightarrow A) \Rightarrow A \Rightarrow A$ (LOFO-2005)

$$
\cfrac{\cfrac{\cfrac{\overline{A \vdash A} \qquad \overline{A \vdash A}}{A, A \Rightarrow A \vdash A} \Rightarrow\vdash}{A \Rightarrow A \vdash A \Rightarrow A} \vdash\Rightarrow}{\vdash (A \Rightarrow A) \Rightarrow A \Rightarrow A} \vdash\Rightarrow
$$

Prove $A \Rightarrow B, B \Rightarrow C \vdash A \Rightarrow (B \wedge C)$ (LOFO-2005)

$$
\cfrac{\cfrac{\cfrac{\cfrac{\overline{A \vdash A} \quad \overline{B \vdash B}}{A, A \Rightarrow B \vdash B} \Rightarrow\vdash}{A, A \Rightarrow B, B \Rightarrow C \vdash B} \mathrm{W}\vdash \qquad \cfrac{\cfrac{\overline{A \vdash A} \quad \overline{C \vdash C}}{A, B \Rightarrow C \vdash C} \Rightarrow\vdash}{A, A \Rightarrow B, B \Rightarrow C \vdash C} \mathrm{W}\vdash}{A, A \Rightarrow B, A, B \Rightarrow C \vdash B \wedge C} \wedge\vdash}{A, A \Rightarrow B, B \Rightarrow C \vdash B \wedge C} \mathrm{C}\vdash
$$

Prove $A \vee B, \neg B \vdash A$ (Classical) (LOFO-2005)

$$
\cfrac{\cfrac{\overline{A \vdash A}}{A, \neg B \vdash A} \mathrm{W}\vdash \qquad \cfrac{\cfrac{\cfrac{\overline{B \vdash B}}{B \vdash B, A} \vdash\mathrm{W}}{B, \neg B \vdash A} \vdash\neg}{B, \neg B \vdash A}}{A \vee B, \neg B \vdash A} \vee\vdash
$$