Intuitionistic Logic

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Intuitionistic Logic

Constructivity

Constructivity

- Classical logic does not build truth
- it discovers a preexisting truth
- Classical logic assumes facts are either true or false
- $\vdash A \vee \neg A$ Excluded middle, *tertium non datur*

Excluded Middle

$$\frac{}{A \vee \neg A}\,\mathrm{XM}$$

$$\frac{\begin{array}{c}\vdots \\ \neg\neg A\end{array}}{A}\,\neg\neg$$

$$\frac{\begin{array}{c}[\neg A] \\ \vdots \\ B\end{array} \qquad \begin{array}{c}[\neg A] \\ \vdots \\ \neg B\end{array}}{A}\,\mathrm{Contradiction}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{\vdash \neg A, A}\vdash\neg}{\vdash A \vee \neg A, A}\vdash r\vee}{\dfrac{\vdash A \vee \neg A, A \vee \neg A}{\vdash A \vee \neg A}\vdash \mathrm{C}}\vdash l\vee$$

Reductio ad Absurdum

In Real Life

A You should respect C's belief, for all beliefs are of equal validity and cannot be denied.

B What about D's belief? (Where D believes something that is considered to be wrong by most people, such as nazism or the world being flat)

A I agree it is right to deny D's belief.

B If it is right to deny D's belief, it is not true that no belief can be denied. Therefore, I can deny C's belief if I can give reasons that suggest it too is incorrect.

Reductio ad Absurdum

In Real Life

A You should respect C's belief, for all beliefs are of equal validity and cannot be denied.

B
1. I deny that belief of yours and believe it to be invalid.
2. According to your statement, this belief of mine (1) is valid, like all other beliefs.
3. However, your statement also contradicts and invalidates mine, being the exact opposite of it.
4. The conclusions of 2 and 3 are incompatible and contradictory, so your statement is logically absurd.

Reductio ad Absurdum

Mathematics: The Smallest Positive Rational

There is no smallest positive rational.

1. Suppose there exists one such rational r
2. $r/2$ is rational and positive
3. $r/2 < r$
4. Contradiction □

Reductio ad Absurdum

Mathematics: $\sqrt{2}$ is irrational

$\sqrt{2}$ is irrational.

1. Assume $\sqrt{2}$ is rational: $\exists a, b$ integers st. $a/b = \sqrt{2}$
2. a, b can be taken coprime
3. $\therefore a^2/b^2 = 2$ and $a^2 = 2b^2$
4. $\therefore a^2$ is even ($a^2 = 2b^2$)
5. $\therefore a$ is even
6. Because a is even, $\exists k$ st. $a = 2k$.
7. We insert the last equation of (3) in (6): $2b^2 = (2k)^2$ is equivalent to $2b^2 = 4k^2$ is equivalent to $b^2 = 2k^2$.
8. Since $2k^2$ is even, b^2 is even, hence, b is even
9. By (5) and (8) a, b are even
10. Contradicts 2 □

Constructivity

Mathematics: Rationality and Power

There are irrational positive numbers a, b such that a^b is rational.

1. $\sqrt{2}$ is known to be irrational
2. Consider $\sqrt{2}^{\sqrt{2}}$:
   1. If it is rational, take $a = b = \sqrt{2}$
   2. Otherwise, take $a = \sqrt{2}^{\sqrt{2}}$, $b = \sqrt{2}$, $a^b = 2$ □

But it is not known which numbers.
We proved $A \vee B$, but neither A nor B.

Constructivity

Mathematics: Unknown Numbers

Let σ be the number defined below. Its value is unknown, but it is rational.

For each decimal digit of π, write 3. Stop if the sequence 0123456789 is found.

1. If 0123456789 occurs in π, then $\sigma = 0,3\ldots3 = \frac{10^k - 1}{3.10^k}$
2. If it does not, $\sigma = 0,3\ldots = 1/3$

We proved $\exists x.P(x)$, but know no $t : P(t)$.

Constructivity

Disjunction Property

If $A \vee B$ is provable, then either A or B is provable, and reading the proof tells which one.

Existence Property

If $\exists x \cdot A(x)$ is provable, then reading the proof allows to exhibit a witness t (i.e., such that $A(t)$).

Intuitionistic Logic

1 Constructivity

2 Intuitionistic Logic
- NJ: Intuitionistic Natural Deduction
- LJ: Intuitionistic Sequent Calculus

Intuitionistic Logic

Luitzen Egbertus Jan Brouwer (1881–1966)

Intuitionistic Logic

- Classical logic focuses on truth (hence truth values)
- Intuitionistic logic focuses on provability (hence proofs)
- A is true if it is provable
- The excluded middle is. . . excluded

$$\cfrac{\cfrac{\cfrac{\cfrac{\overline{A \vdash A}}{\vdash \neg A, A}\vdash\neg}{\vdash A \vee \neg A, A}\vdash r\vee}{\vdash A \vee \neg A, A \vee \neg A}\vdash l\vee}{\vdash A \vee \neg A}\vdash \mathrm{C}$$

NJ: Intuitionistic Natural Deduction

1 Constructivity

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Intuitionistic Natural Deduction

- Natural deduction supports very well intuitionistic logic.
- In fact, classical logic does not fit well in natural deduction.

$$\dfrac{\begin{array}{c}[A]\\ \vdots \\ \bot\end{array}}{\neg A}\,\neg\mathcal{I} \qquad \dfrac{\begin{array}{c}\vdots \\ A\end{array} \quad \begin{array}{c}\vdots \\ \neg A\end{array}}{\bot}\,\neg\mathcal{E}$$

$$\dfrac{}{A \vee \neg A}\,\mathrm{XM} \qquad \dfrac{\begin{array}{c}\vdots \\ \neg\neg A\end{array}}{A}\,\neg\neg \qquad \dfrac{\begin{array}{c}[\neg A]\\ \vdots \\ B\end{array} \quad \begin{array}{c}[\neg A]\\ \vdots \\ \neg B\end{array}}{A}\,\mathrm{Contradiction}$$

Intuitionistic Negation

$$\dfrac{\begin{array}{c}[A]\\ \vdots \\ \bot\end{array}}{\neg A}\,\neg\mathcal{I} \qquad \dfrac{\begin{array}{c}\vdots \\ A\end{array} \quad \begin{array}{c}\vdots \\ \neg A\end{array}}{\bot}\,\neg\mathcal{E}$$

$$\dfrac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \Rightarrow B}\,\Rightarrow\mathcal{I} \qquad \dfrac{\begin{array}{c}\vdots \\ A\end{array} \quad \begin{array}{c}\vdots \\ A \Rightarrow B\end{array}}{B}\,\Rightarrow\mathcal{E}$$

So define $\neg A := A \Rightarrow \bot$.

Prove $A \vdash \neg\neg A$

$$\dfrac{\dfrac{A \quad [A \Rightarrow \bot]^1}{\bot}\,\Rightarrow\mathcal{E}}{(A \Rightarrow \bot) \Rightarrow \bot}\,\Rightarrow\mathcal{I}_1$$

Prove $\neg\neg\neg A \vdash \neg A$

$$\dfrac{\dfrac{\dfrac{\dfrac{[A]^2 \quad [A \Rightarrow \bot]^1}{\bot}\,\Rightarrow\mathcal{E}}{(A \Rightarrow \bot) \Rightarrow \bot}\,\Rightarrow\mathcal{I}_1 \qquad ((A \Rightarrow \bot) \Rightarrow \bot) \Rightarrow \bot}{\bot}\,\Rightarrow\mathcal{E}}{A \Rightarrow \bot}\,\Rightarrow\mathcal{I}_2$$

Intuitionistic Natural Deduction

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \Rightarrow B} \Rightarrow\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots \\ A\end{matrix} \quad \begin{matrix}\vdots \\ A \Rightarrow B\end{matrix}}{B} \Rightarrow\mathcal{E} \qquad \dfrac{\begin{matrix}\vdots \\ \bot\end{matrix}}{A} \bot\mathcal{E}$$

$$\dfrac{A \quad B}{A \wedge B} \wedge\mathcal{I} \qquad \dfrac{A \wedge B}{A} \wedge l\mathcal{E} \qquad \dfrac{A \wedge B}{B} \wedge r\mathcal{E}$$

$$\dfrac{\begin{matrix}\vdots \\ A\end{matrix}}{A \vee B} \vee l\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots \\ B\end{matrix}}{A \vee B} \vee r\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots \\ A \vee B\end{matrix} \quad \begin{matrix}[A]\\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C} \vee\mathcal{E}$$

LJ: Intuitionistic Sequent Calculus

LJ — Intuitionistic Sequent Calculus

LJ — Gentzen 1934

Logistischer intuitionistischer Kalkül

LK: Identity Group

$$\dfrac{}{A \vdash A}\,\mathrm{Id} \qquad \dfrac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

LK: Structural Group

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \tau(\Delta)} \vdash \mathrm{X} \qquad \frac{\Gamma \vdash \Delta}{\sigma(\Gamma) \vdash \Delta} \mathrm{X}\vdash$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \vdash \mathrm{W} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \mathrm{W}\vdash$$

$$\frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \vdash \mathrm{C} \qquad \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \mathrm{C}\vdash$$

LK: Logical Group: Negation

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \vdash\neg \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \neg\vdash$$

LK: Logical Group: Conjunction

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \vdash\wedge \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} l\wedge\vdash \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} r\wedge\vdash$$

LK: Logical Group: Disjunction

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta} \vdash l\vee \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vdash r\vee \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \vee\vdash$$

LK: Logical Group: Implication

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \Rightarrow B \vdash \Delta, \Delta'} \Rightarrow\vdash \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta} \vdash\Rightarrow$$

LJ

$$\frac{}{A \vdash A}\,\mathrm{Id} \qquad \frac{\Gamma \vdash A \quad \Gamma', A \vdash B}{\Gamma, \Gamma' \vdash B}\,\mathrm{Cut}$$

$$\frac{\Gamma \vdash B}{\sigma(\Gamma) \vdash B}\,\mathrm{X}\vdash \qquad \frac{\Gamma \vdash B}{\Gamma, A \vdash B}\,\mathrm{W}\vdash \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,\mathrm{C}\vdash$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \vdash\wedge \qquad \frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C}\, l\wedge\vdash \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \wedge B \vdash C}\, r\wedge\vdash$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \vdash l\vee \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \vdash r\vee \qquad \frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C} \vee\vdash$$

$$\frac{\Gamma \vdash A \quad \Gamma', B \vdash C}{\Gamma, \Gamma', A \Rightarrow B \vdash C} \Rightarrow\vdash \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \vdash\Rightarrow$$

Prove $A \vdash \neg\neg A$

$$\frac{\dfrac{\dfrac{}{A \vdash A} \qquad \dfrac{}{\bot \vdash \bot}}{A, A \Rightarrow \bot \vdash \bot} \Rightarrow\vdash}{A \vdash (A \Rightarrow \bot) \Rightarrow \bot} \vdash\Rightarrow$$

Prove $\neg\neg\neg A \vdash \neg A$

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A} \qquad \dfrac{}{\bot \vdash \bot}}{A, A \Rightarrow \bot \vdash \bot} \Rightarrow\vdash}{A \vdash (A \Rightarrow \bot) \Rightarrow \bot} \vdash\Rightarrow \qquad \dfrac{}{\bot' \vdash \bot'}}{A, ((A \Rightarrow \bot) \Rightarrow \bot) \Rightarrow \bot' \vdash \bot'} \Rightarrow\vdash}{((A \Rightarrow \bot) \Rightarrow \bot) \Rightarrow \bot' \vdash A \Rightarrow \bot'} \vdash\Rightarrow$$

Therefore, in intuistionistic logic $\neg\neg\neg A \equiv \neg A$, but $\neg\neg A \not\equiv A$.

Recommended Readings

[van Atten, 2014]
The history of intuitionistic logic.

Bibliography I

van Atten, M. (2014).
The development of intuitionistic logic.
In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*.
Spring 2014 edition.