Simply Typed λ-Calculus

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About these lecture notes

Many of these slides are largely inspired from Andrew D. Ker's lecture notes [Ker, 2005a, Ker, 2005b]. Some slides are even straightforward copies.

Simply Typed λ-Calculus

Types

1. Types
   - Untyped λ-calculus
   - Paradoxes
   - Church vs. Curry
2. $\lambda^{\rightarrow}$: Type Assignments

Types

Alonzo Church (1903–1995)

Haskell Curry (1900–1982)

Types

Types first appeared with

- Curry (1934) for Combinatory Logic
- Church (1940)

Types are syntactic objects assigned to terms:

$$M : A \qquad M \text{ has type } A$$

For instance:

$$I : A \rightarrow A$$

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λ-terms

Λ, set of λ-terms

$$\frac{}{x \in \Lambda}\, x \in \mathcal{V} \qquad \frac{M \in \Lambda \quad N \in \Lambda}{(MN) \in \Lambda} \qquad \frac{M \in \Lambda}{(\lambda x \cdot M) \in \Lambda}\, x \in \mathcal{V}$$

The $\lambda\beta$ Formal System

$$\frac{}{M = M} \qquad \frac{M = N}{N = M} \qquad \frac{M = N \quad N = L}{M = L}$$

$$\frac{M = M' \quad N = N'}{MN = M'N'} \qquad \frac{M = N}{\lambda x \cdot M = \lambda x \cdot N}$$

$$\frac{}{(\lambda x \cdot M)N = [N/x]M}$$

Properties of $\lambda\beta$

β-reduction is Church-Rosser.

Any term has (at most) a unique NF.

β-reduction is not normalizing.

Some terms have no NF (Ω).

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Paradoxes

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Self application

What is the computational meaning of $\lambda x \cdot xx$?

- Stop considering anything can be applied to anything
- A function and its argument have different behaviors

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Church vs. Curry

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Simple Types

- A set of type variables
  $\alpha, \beta, \ldots$
- A symbol $\rightarrow$ for functions
  $\alpha \rightarrow \alpha$, $\alpha \rightarrow (\beta \rightarrow \gamma)$, $(\alpha \rightarrow \beta) \rightarrow \gamma$, ...
- Possibly constants for "primitive" types
  ι for integers, etc.

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Simple Types

By convention $\to$ is right-associative:

$$\alpha \to \beta \to \gamma = \alpha \to (\beta \to \gamma)$$

This matches the right-associativity of λ:

$$\lambda x \cdot \lambda y \cdot M = \lambda x \cdot (\lambda y \cdot M)$$

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Alonzo Style, or Haskell Way?

Church: Typed λ-calculus

$$\frac{x : \alpha}{\lambda x^{\alpha} \cdot x : \alpha \to \alpha}$$

Curry: λ-calculus with Types

$$\frac{x : \alpha}{\lambda x \cdot x : \alpha \to \alpha}$$

$\mathcal{TV}$ a set of type variables $\alpha, \beta, \ldots$

Simple Types

The set $\mathcal{T}$ of types $\sigma, \tau, \ldots$:

$$\frac{}{\alpha \in \mathcal{T}}\ \alpha \in \mathcal{TV} \qquad \frac{\sigma \in \mathcal{T} \quad \tau \in \mathcal{T}}{(\sigma \to \tau) \in \mathcal{T}}$$

Type Contexts

Statement

A statement $M : \sigma$ is a pair with $M \in \Lambda, \sigma \in \mathcal{T}$.
M is the subject, σ the predicate.

Type Context, Basis

A type context Γ is a finite set of statements over distinct variables $\{x_1 : \sigma_1, \ldots\}$.

Assignment

The variable x is assigned the type σ in Γ iff $x : \sigma \in \Gamma$.

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Type Contexts

Type Context Restrictions

$\Gamma - x$ is the Γ with all assignment $x : \sigma$ removed.
$\Gamma \upharpoonright M$ is $\Gamma - \mathrm{FV}(M)$.

Type Deductions

A Natural Presentation

Type derivations are trees built from the following nodes.

$$\frac{M : \sigma \to \tau \quad N : \sigma}{}$$

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$$\frac{\begin{array}{c} x : \sigma \\ \vdots \\ M : \tau \end{array}}{}$$

A Natural Presentation

Type derivations are trees built from the following nodes.

$$\frac{M : \sigma \to \tau \quad N : \sigma}{MN : \tau} \qquad \frac{\begin{array}{c}[x : \sigma] \\ \vdots \\ M : \tau\end{array}}{\lambda x \cdot M : \sigma \to \tau}$$

Type Statement

Type Statement

A statement $M : \sigma$ is derivable from the type context Γ,

$$\Gamma \vdash M : \sigma$$

if there is a derivation of $M : \sigma$ whose non-canceled assumptions are in Γ.

Type Statements

Prove

$$\vdash \lambda fx \cdot f(fx) : (\sigma \to \sigma) \to \sigma \to \sigma$$

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$$\dfrac{\dfrac{[f : \sigma \to \sigma]^{(2)} \quad \dfrac{[f : \sigma \to \sigma]^{(2)} \quad [x : \sigma]^{(1)}}{fx : \sigma}}{\dfrac{f(fx) : \sigma}{\lambda x \cdot f(fx) : \sigma \to \sigma}\,(1)}}{\lambda fx \cdot f(fx) : (\sigma \to \sigma) \to \sigma \to \sigma}\,(2)$$

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$$\cfrac{\cfrac{[x : \sigma]^{(1)}}{\lambda y \cdot x : \tau \to \sigma}}{\lambda xy \cdot x : \sigma \to \tau \to \sigma}\,(1)$$

Alternative Presentation of Type Derivations

Type Derivations

$$\frac{}{\{x:\sigma\} \mapsto x:\sigma}$$

$$\frac{\Gamma \mapsto M:\sigma\to\tau \quad \Delta \mapsto N:\sigma}{\Gamma\cup\Delta \mapsto MN:\tau} \quad \Gamma,\Delta \text{ consistent}$$

$$\frac{\Gamma \mapsto M:\tau}{\Gamma\setminus\{x:\sigma\} \mapsto \lambda x\cdot M:\sigma\to\tau} \quad \Gamma,\{x:\sigma\} \text{ consistent}$$

$$\frac{\Gamma \mapsto M:\tau}{\Gamma,\Delta \vdash M:\tau}$$

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$$\dfrac{\dfrac{[x : \sigma]^{(1)}}{\lambda y \cdot x : \tau \to \sigma}}{\lambda xy \cdot x : \sigma \to \tau \to \sigma}\,(1)$$

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$$\dfrac{\dfrac{\vdots}{\{x : \sigma_1\} \mapsto x : \tau \to \sigma_2} \qquad \dfrac{\vdots}{\{x : \sigma_1\} \mapsto x : \tau}}{\dfrac{\{x : \sigma_1\} \mapsto xx : \sigma_2}{\mapsto \lambda x \cdot xx : \sigma} \; \sigma = \sigma_1 \to \sigma_2}$$

Typability

Typability

A term M is typable if there exists a type σ such that $\vdash M : \sigma$.

- ω, Ω are not typable.
- S, K, I are typable.
- Y is not typable!

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Subject Construction Lemma I

Consider the derivation π for $M : \sigma$.

- If $M = x$, then $\Gamma = \{x : \sigma\}$ and

$$\pi = \frac{}{\{x : \sigma\} \mapsto x : \sigma}$$

- If $M = NL$, then

$$\pi = \frac{\Gamma \upharpoonright N \mapsto N : \tau \to \sigma \quad \Gamma \upharpoonright L \mapsto L : \tau}{\Gamma \mapsto NL : \sigma}$$

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Subject Construction Lemma I

Consider the derivation π for $M : \sigma$.

- If $M = \lambda x \cdot N$, then $\sigma = \sigma_1 \to \sigma_2$ and
  - If $x \in \mathrm{FV}(N)$

$$\pi = \frac{\Gamma \cup \{x : \sigma_1\} \mapsto N : \sigma_2}{\Gamma \mapsto \lambda x \cdot N : \sigma_1 \to \sigma_2}$$

  - If $x \notin \mathrm{FV}(N)$

$$\pi = \frac{\Gamma \mapsto N : \sigma_2}{\Gamma \mapsto \lambda x \cdot N : \sigma_1 \to \sigma_2}$$

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Derivations are not Unique

They are for β-normal forms.

Subject Reduction Theorem

Conversions and Types

α-Invariance

If $\Gamma \mapsto M : \sigma$ and $M \equiv_\alpha N$ then $\Gamma \mapsto N : \sigma$.

Substitution

If

- $\Gamma \cup \{x : \tau\} \vdash M : \sigma$,
- $\Delta \vdash N : \tau$,
- Γ, Δ consistent,
- $x \notin \mathrm{FV}(N)$

then

$$\Gamma \cup \Delta \vdash [N/x]M : \sigma$$

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$$\Gamma \cup \Delta \vdash [N/x]M : \sigma$$

Subject Reduction Theorem

Subject Reduction Theorem

If $\Gamma \vdash M : \sigma$ and $M \to_\beta N$ then $\Gamma \vdash N : \sigma$.

What about the converse?

Subject Expansion

If $\Gamma \vdash N : \sigma$ and $M \to_\beta N$ then $\Gamma \vdash M : \sigma$.

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Ω is not typable, but $KI\Omega \rightarrow I$.

Reducibility

Types and Normalization

$\lambda^{\to}$ is Strongly Normalizing

All typable terms are β-strongly normalizing.

Typability

1. Types
2. $\lambda^{\rightarrow}$: Type Assignments
   - Types
   - Type Deductions
   - Subject Reduction Theorem
   - Reducibility
   - Typability

$\vdash M : \sigma$ suffices

Note that with $\Gamma = \{x_1 : \sigma_1, \ldots, x_n : \sigma_n\}$

$$\Gamma \vdash M : \tau$$

is equivalent to

$$\vdash \lambda x_1 \ldots x_n \cdot M : \sigma_1 \to \ldots \to \sigma_n \to \tau$$

Decidability of Type Assignment

Type checking Given M, σ, does $\vdash M : \sigma$?

$$\vdash M : \sigma?$$

Typability Given M, does there exist σ such that $\vdash M : \sigma$?

$$\vdash M :?$$

Inhabitation Given σ, does there exist M such that $\vdash M : \sigma$?

$$\vdash ? : \sigma$$

Decidability of Type Assignment

Type Checking is Decidable

It is decidable whether a statement of $\lambda^{\rightarrow}$ is provable.

Typability is Decidable

It is decidable whether a term of $\lambda^{\rightarrow}$ has a type.

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Type Checking is Decidable

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Typability is Decidable

It is decidable whether a term of $\lambda^{\rightarrow}$ has a type.

Types are not Unique...

Typable terms do not have unique types.

$$\dfrac{\dfrac{\overline{\{x:\sigma\} \mapsto x:\sigma}}{\{x:\sigma\} \mapsto \lambda y \cdot x : \tau \to \sigma}}{\mapsto K : \sigma \to \tau \to \sigma}$$

is valid for *any* σ, τ, including $\sigma = \sigma' \to \sigma'', \tau = (\tau' \to \tau'') \to \tau'$ etc.

Types are not Unique...

but some are more Unique than others ...

All the types of K are instances of $\sigma \to \tau \to \sigma$.

Bibliography Notes

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