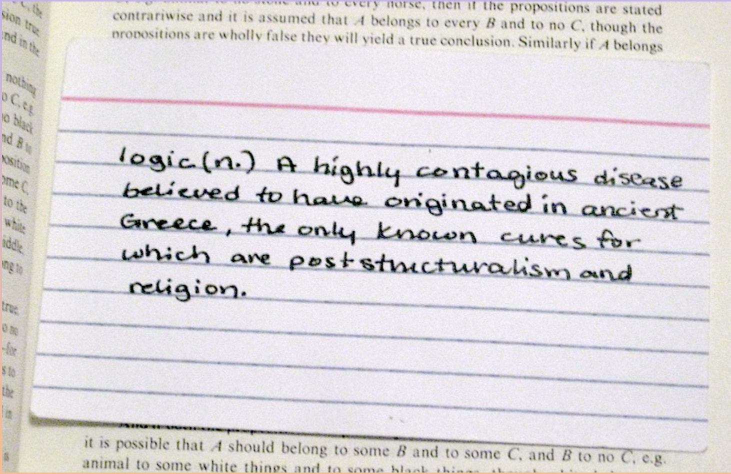
Natural Deduction

Akim Demaille akim@lrde.epita.fr

EPITA — École Pour l'Informatique et les Techniques Avancées

June 10, 2016

Define “logic” [fuc,]



Natural Deduction

Preamble

The following slides are implicitly dedicated to classical logic.

Terminal Symbols

Propositional Calculus

Constants	$a, b, c, \ldots$
Propositional Variables	$A, B, C, \ldots$
Unary Connective	$\neg$
Binary Connectives	$\wedge, \vee, \Rightarrow$
Punctuation	(,), [,].

Terminal Symbols

Propositional Calculus

Constants	$a, b, c, \ldots$
Propositional Variables	$A, B, C, \ldots$
Unary Connective	$\neg$
Binary Connectives	$\wedge, \vee, \Rightarrow$
Punctuation	(,), [,].

Terminal Symbols

Propositional Calculus

Constants	$a, b, c, \ldots$
Propositional Variables	$A, B, C, \ldots$
Unary Connective	$\neg$
Binary Connectives	$\wedge, \vee, \Rightarrow$
Punctuation	(,), [,].

Terminal Symbols

Propositional Calculus

Constants	$a, b, c, \ldots$
Propositional Variables	$A, B, C, \ldots$
Unary Connective	$\neg$
Binary Connectives	$\wedge, \vee, \Rightarrow$
Punctuation	(,), [,].

Terminal Symbols

Propositional Calculus

Constants	$a, b, c, \ldots$
Propositional Variables	$A, B, C, \ldots$
Unary Connective	$\neg$
Binary Connectives	$\wedge, \vee, \Rightarrow$
Punctuation	(,), [,].

Terminal Symbols

Predicate calculus

Individual Variables $x, y, z, \ldots$

Functions $f, g, h, \ldots$, with a fixed arity

Predicates $P, Q, R, \ldots$, with a fixed arity

Quantifiers $\forall, \exists$

Punctuation $\cdot$.

Terminal Symbols

Predicate calculus

Individual Variables $x, y, z, \ldots$

Functions $f, g, h, \ldots$, with a fixed arity

Predicates $P, Q, R, \ldots$, with a fixed arity

Quantifiers $\forall, \exists$

Punctuation $\cdot$.

Terminal Symbols

Predicate calculus

Individual Variables	$x, y, z, \ldots$
Functions	$f, g, h, \ldots$, with a fixed arity
Predicates	$P, Q, R, \ldots$, with a fixed arity
Quantifiers	$\forall, \exists$
Punctuation	$\cdot$.

Terminal Symbols

Predicate calculus

Individual Variables	$x, y, z, \ldots$
Functions	$f, g, h, \ldots$, with a fixed arity
Predicates	$P, Q, R, \ldots$, with a fixed arity
Quantifiers	$\forall, \exists$
Punctuation	$\cdot.$

Terminal Symbols

Predicate calculus

Individual Variables	$x, y, z, \ldots$
Functions	$f, g, h, \ldots$, with a fixed arity
Predicates	$P, Q, R, \ldots$, with a fixed arity
Quantifiers	$\forall, \exists$
Punctuation	$\cdot$.

Propositional Formulas

$$
\begin{array}{rcl}
\langle\text{formula}\rangle & ::= & \langle\text{propositional variable}\rangle \\
& | & \neg\langle\text{formula}\rangle \\
& | & \langle\text{formula}\rangle \wedge \langle\text{formula}\rangle \\
& | & \langle\text{formula}\rangle \vee \langle\text{formula}\rangle \\
& | & \langle\text{formula}\rangle \Rightarrow \langle\text{formula}\rangle
\end{array}
$$

Terms

$$\begin{array}{rcl} \langle \text{term} \rangle & ::= & \langle \text{constant} \rangle \\ & | & \langle \text{function} \rangle(\langle \text{term} \rangle, \ldots) \end{array}$$

With the proper arity.

First Order Formulas

$$
\begin{array}{rcl}
\langle \text{formula} \rangle & ::= & \langle \text{propositional variable} \rangle \\
& | & \neg \langle \text{formula} \rangle \\
& | & \langle \text{formula} \rangle \wedge \langle \text{formula} \rangle \\
& | & \langle \text{formula} \rangle \vee \langle \text{formula} \rangle \\
& | & \langle \text{formula} \rangle \Rightarrow \langle \text{formula} \rangle \\
& | & \langle \text{predicate} \rangle (\langle \text{term} \rangle, \ldots) \\
& | & \forall \langle \text{individual variable} \rangle \cdot \langle \text{formula} \rangle \\
& | & \exists \langle \text{individual variable} \rangle \cdot \langle \text{formula} \rangle
\end{array}
$$

With the proper arity.

Syntactic Conventions

Associativity

- $\wedge, \vee$ are left-associative (unimportant)
- $\Rightarrow$ is right-associative (very important)

Precedence (increasing)

1. $\forall, \exists$
2. $\Rightarrow$
3. $\vee$
4. $\wedge$
5. $\neg$

Syntactic Conventions

Associativity
- $\wedge, \vee$ are left-associative (unimportant)
- $\Rightarrow$ is right-associative (very important)

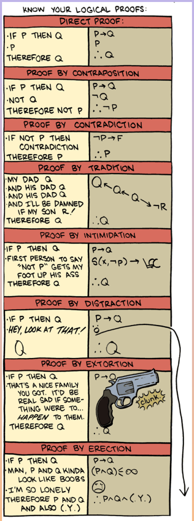
Precedence (increasing)

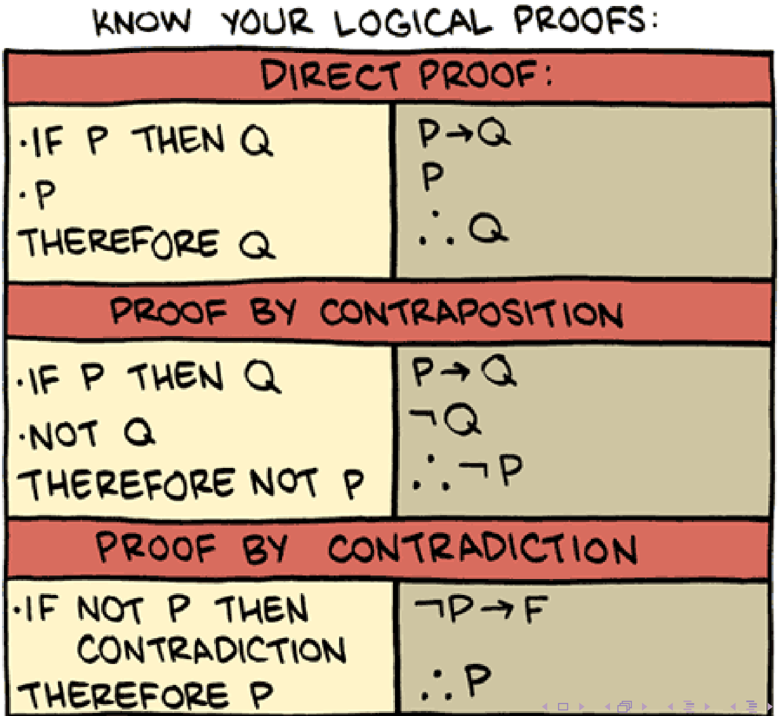
1. $\forall, \exists$
2. $\Rightarrow$
3. $\vee$
4. $\wedge$
5. $\neg$

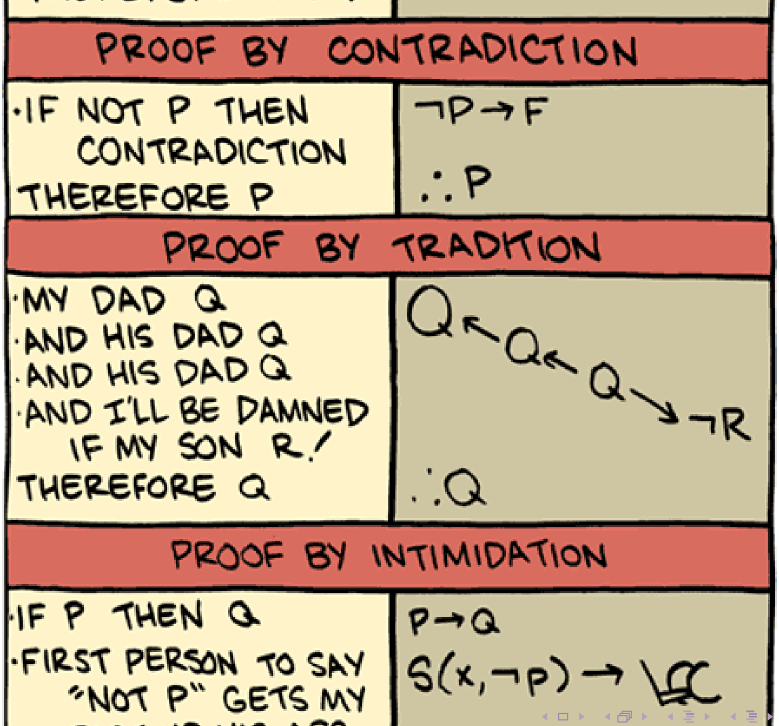
Free Variables

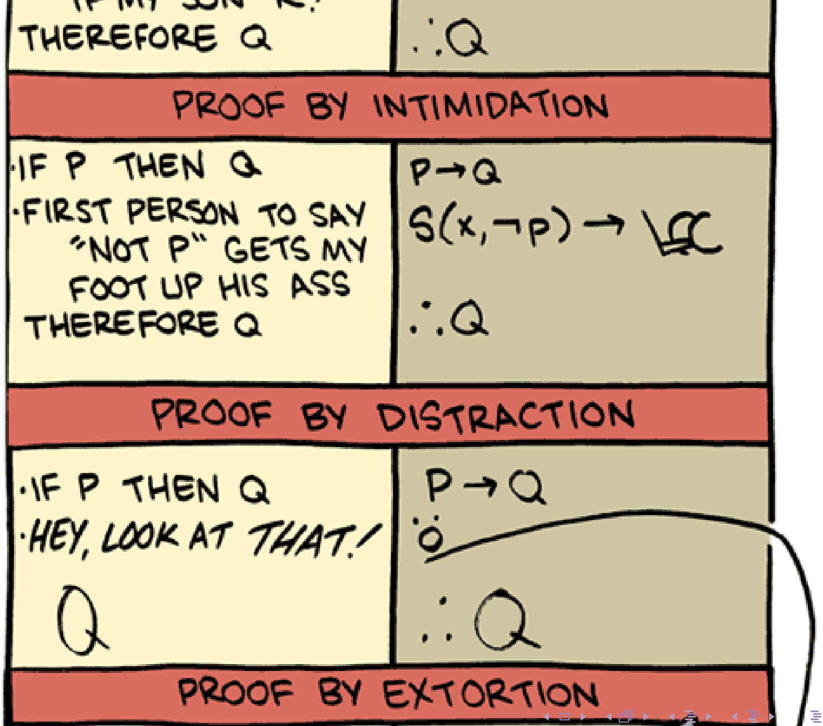
$$
\begin{aligned}
\mathrm{FV}(X) &= \emptyset \\
\mathrm{FV}(P(x_1, x_2, \cdots, x_n)) &= \{x_1, x_2, \cdots, x_n\} \\
\mathrm{FV}(\neg A) &= \mathrm{FV}(A) \\
\mathrm{FV}(A \vee B) &= \mathrm{FV}(A) \cup \mathrm{FV}(B) \\
\mathrm{FV}(A \wedge B) &= \mathrm{FV}(A) \cup \mathrm{FV}(B) \\
\mathrm{FV}(A \Rightarrow B) &= \mathrm{FV}(A) \cup \mathrm{FV}(B) \\
\mathrm{FV}(\forall x \cdot A) &= \mathrm{FV}(A) - \{x\} \\
\mathrm{FV}(\exists x \cdot A) &= \mathrm{FV}(A) - \{x\}
\end{aligned}
$$

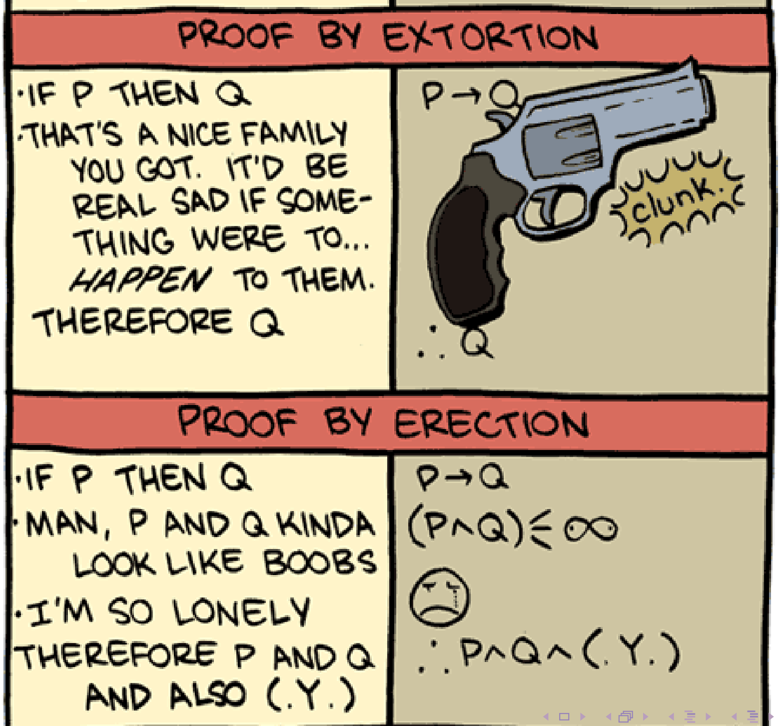
Different Proof Types




KNOW YOUR LOGICAL PROOFS:
DIRECT PROOF:
·IF P THEN Q
·P
THEREFORE Q
P→Q
P
∴ Q
PROOF BY CONTRAPOSITION
·IF P THEN Q
·NOT Q
THEREFORE NOT P
P→Q
¬Q
∴ ¬P
PROOF BY CONTRADICTION
·IF NOT P THEN CONTRADICTION
THEREFORE P
¬P→F
∴ P


PROOF BY CONTRADICTION
•IF NOT P THEN CONTRADICTION
THEREFORE P
¬P → F
∴ P
PROOF BY TRADITION
•MY DAD Q
•AND HIS DAD Q
•AND HIS DAD Q
•AND I'LL BE DAMNED IF MY SON R!
THEREFORE Q
Q ← Q ← Q → ¬R
∴ Q
PROOF BY INTIMIDATION
•IF P THEN Q
•FIRST PERSON TO SAY "NOT P" GETS MY
P → Q
S(x, ¬p) →


THEREFORE Q
∴Q
PROOF BY INTIMIDATION
•IF P THEN Q
•FIRST PERSON TO SAY "NOT P" GETS MY FOOT UP HIS ASS
THEREFORE Q
P→Q
S(x,¬p) →
∴Q
PROOF BY DISTRACTION
•IF P THEN Q
•HEY, LOOK AT THAT!
Q
P→Q
∴Q
PROOF BY EXTORTION


PROOF BY EXTORTION
•IF P THEN Q
•THAT'S A NICE FAMILY YOU GOT. IT'D BE REAL SAD IF SOMETHING WERE TO... HAPPEN TO THEM.
THEREFORE Q
P→Q
clunk.
∴ Q
PROOF BY ERECTION
•IF P THEN Q
•MAN, P AND Q KINDA LOOK LIKE BOOBS
•I'M SO LONELY
THEREFORE P AND Q AND ALSO (.Y.)
P→Q
(P∧Q)⇐∞
∴ P∧Q∧(.Y.)

1. Logical Formalisms
   - Syntax
   - Proof Types
   - Proof Systems
2. Natural Deduction
3. Additional Material

Proof Systems

- Hilbertian Systems
- Natural Deduction
- Sequent Calculus
- Natural Deduction in Sequent Calculus

- Hilbertian Systems
- Natural Deduction
- Sequent Calculus
- Natural Deduction in Sequent Calculus

Proof Systems

- Hilbertian Systems
- Natural Deduction
- Sequent Calculus
- Natural Deduction in Sequent Calculus

Proof Systems

- Hilbertian Systems
- Natural Deduction
- Sequent Calculus
- Natural Deduction in Sequent Calculus

Axioms

- Axioms are formulas that are considered true a priori

$$\forall x \cdot x + 0 = x$$

- Axiom schemes use meta-variables (that range over a specific domain)

$$X + Y = Y + X$$

- Axiom schemes are used when quantifiers are not welcome

$$\begin{aligned} SXYZ &\rightarrow XZ(YZ) \\ KXY &\rightarrow X \end{aligned}$$

- Axiom schemes are used when quantifiers do not apply

$$A \vee \neg A$$

Axioms

- Axioms are formulas that are considered true a priori

$$\forall x \cdot x + 0 = x$$

- Axiom schemes use meta-variables
  (that range over a specific domain)

$$X + Y = Y + X$$

- Axiom schemes are used when quantifiers are not welcome

$$\begin{aligned} SXYZ &\rightarrow XZ(YZ) \\ KXY &\rightarrow X \end{aligned}$$

- Axiom schemes are used when quantifiers do not apply

$$A \vee \neg A$$

Axioms

- Axioms are formulas that are considered true a priori

$$\forall x \cdot x + 0 = x$$

- Axiom schemes use meta-variables
  (that range over a specific domain)

$$X + Y = Y + X$$

- Axiom schemes are used when quantifiers are not welcome

$$\begin{aligned} SXYZ &\rightarrow XZ(YZ) \\ KXY &\rightarrow X \end{aligned}$$

- Axiom schemes are used when quantifiers do not apply

$$A \vee \neg A$$

Axioms

- Axioms are formulas that are considered true a priori

$$\forall x \cdot x + 0 = x$$

- Axiom schemes use meta-variables
  (that range over a specific domain)

$$X + Y = Y + X$$

- Axiom schemes are used when quantifiers are not welcome

$$\begin{aligned} SXYZ &\rightarrow XZ(YZ) \\ KXY &\rightarrow X \end{aligned}$$

- Axiom schemes are used when quantifiers do not apply

$$A \vee \neg A$$

Inference Rules

$$\frac{H_1 \quad H_2 \quad \cdots \quad H_n}{C} \text{Rule name}$$

$$\frac{}{A} \text{Axiom name}$$

Logical Formalisms

David Hilbert (1862–1943)

Hilbertian System

- A single inference rule: the modus ponens

$$\frac{A \qquad A \Rightarrow B}{B}\ \textit{modus ponens}$$

- Many axioms to define the connectives

$$A \Rightarrow B \Rightarrow A \wedge B \qquad A \wedge B \Rightarrow A \qquad A \wedge B \Rightarrow B$$

$$A \Rightarrow A \vee B \qquad B \Rightarrow A \vee B$$
$$A \vee B \Rightarrow \ (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C$$

$$A \Rightarrow B \Rightarrow A \qquad (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C$$

$$A \vee \neg A \qquad A \Rightarrow \neg A \Rightarrow B$$

Hilbertian System

- A single inference rule: the modus ponens

$$\frac{A \qquad A \Rightarrow B}{B}\ \textit{modus ponens}$$

- Many axioms to define the connectives

$$A \Rightarrow B \Rightarrow A \wedge B \qquad A \wedge B \Rightarrow A \qquad A \wedge B \Rightarrow B$$

$$A \Rightarrow A \vee B \qquad B \Rightarrow A \vee B$$
$$A \vee B \Rightarrow \ (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C$$

$$A \Rightarrow B \Rightarrow A \qquad (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C$$

$$A \vee \neg A \qquad A \Rightarrow \neg A \Rightarrow B$$

Hilbertian System

- A single inference rule: the modus ponens

$$\frac{A \qquad A \Rightarrow B}{B}\ \textit{modus ponens}$$

- Many axioms to define the connectives

$$A \Rightarrow B \Rightarrow A \wedge B \qquad A \wedge B \Rightarrow A \qquad A \wedge B \Rightarrow B$$

$$A \Rightarrow A \vee B \qquad B \Rightarrow A \vee B$$

$$A \vee B \Rightarrow (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C$$

$$A \Rightarrow B \Rightarrow A \qquad (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C$$

$$\Rightarrow A \vee \neg A \qquad A \Rightarrow \neg A \Rightarrow B$$

Hilbertian System: Prove $A \Rightarrow A$

Hilbertian System: Prove $A \Rightarrow A$

$$\dfrac{\dfrac{\dfrac{}{(A \Rightarrow ((A \Rightarrow A) \Rightarrow A)) \Rightarrow (A \Rightarrow A \Rightarrow A) \Rightarrow A \Rightarrow A} \quad \dfrac{}{A \Rightarrow (A \Rightarrow A) \Rightarrow A}}{(A \Rightarrow A \Rightarrow A) \Rightarrow A \Rightarrow A} \quad \dfrac{}{A \Rightarrow A \Rightarrow A}}{A \Rightarrow A}$$

Natural Deduction

Syntax

Deduction

A deduction is a tree whose root (A) is the conclusion and whose leafs (Γ) is the set of hypotheses.

$$\begin{array}{c}\Gamma \\ \vdots \\ A\end{array}$$

Any formula A is a valid hypothesis.

Proof (Demonstration)

A proof is a deduction without hypotheses.

Deduction

Deduction

A deduction is a tree whose root (A) is the conclusion and whose active leafs (Γ) is the set of hypotheses.

$$\begin{array}{c}\Gamma \\ \vdots \\ A\end{array}$$

Any formula A is a valid hypothesis.

Proof (Demonstration)

A proof is a deduction without hypotheses.

What's this?

$$A$$

Deductions

What's this?

$$A$$

A deduction of A under the hypothesis A.

Implication

$$\dfrac{\begin{array}{c}[A] \\ \vdots \\ B\end{array}}{A \Rightarrow B} \Rightarrow\mathcal{I} \qquad \dfrac{\begin{array}{c}\vdots \\ A\end{array} \quad \begin{array}{c}\vdots \\ A \Rightarrow B\end{array}}{B} \Rightarrow\mathcal{E}$$

Deduction theorem, and Modus Ponens.

Note the connection with (left) contraction: any number of A (including 0) is discharged.

Implication

$$\dfrac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \Rightarrow B} \Rightarrow\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots \\ A\end{matrix} \qquad \begin{matrix}\vdots \\ A \Rightarrow B\end{matrix}}{B} \Rightarrow\mathcal{E}$$

Deduction theorem, and Modus Ponens.
Note the connection with (left) contraction: any number of A (including 0) is discharged.

Implication

$$\dfrac{\begin{array}{c}[A]\\ \vdots\\ B\end{array}}{A \Rightarrow B}\Rightarrow\mathcal{I} \qquad \dfrac{\begin{array}{c}\vdots\\ A\end{array}\quad\begin{array}{c}\vdots\\ A \Rightarrow B\end{array}}{B}\Rightarrow\mathcal{E}$$

Deduction theorem, and Modus Ponens.

Note the connection with (left) contraction: any number of A (including 0) is discharged.

Implication

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \Rightarrow B} \Rightarrow\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots \\ A\end{matrix} \qquad \begin{matrix}\vdots \\ A \Rightarrow B\end{matrix}}{B} \Rightarrow\mathcal{E}$$

Deduction theorem, and Modus Ponens.
Note the connection with (left) contraction: any number of A (including 0) is discharged.

Proving $A \Rightarrow A$ in Natural Deduction

Proving $A \Rightarrow A$ in Natural Deduction

$$\frac{[A]}{A \Rightarrow A} \Rightarrow\mathcal{I}$$

Conjunction

$$\dfrac{\begin{matrix}\vdots & & \vdots \\ A & & B\end{matrix}}{A \wedge B}\wedge\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots \\ A \wedge B\end{matrix}}{A}\wedge l\mathcal{E} \qquad \dfrac{\begin{matrix}\vdots \\ A \wedge B\end{matrix}}{B}\wedge r\mathcal{E}$$

Conjunction

$$\dfrac{\begin{matrix}\vdots & \vdots \\ A & B\end{matrix}}{A \wedge B}\wedge\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots \\ A \wedge B\end{matrix}}{A}\wedge l\mathcal{E} \qquad \dfrac{\begin{matrix}\vdots \\ A \wedge B\end{matrix}}{B}\wedge r\mathcal{E}$$

Universal Quantification

$$\cfrac{\begin{matrix}\vdots\\ A[y/x]\end{matrix}}{\forall x \cdot A}\,\forall\mathcal{I} \quad y \notin \mathrm{FV}(hyp(A)) \qquad\qquad \cfrac{\begin{matrix}\vdots\\ \forall x \cdot A\end{matrix}}{A[t/x]}\,\forall\mathcal{E}$$

$$\cfrac{\cfrac{[A]}{\forall x \cdot A}\,\forall\mathcal{I}}{A \Rightarrow \forall x \cdot A}\,\Rightarrow\mathcal{I} \qquad\qquad \cfrac{\cfrac{[A]}{A \Rightarrow A}\,\Rightarrow\mathcal{I}}{\forall x \cdot (A \Rightarrow A)}\,\forall\mathcal{I}$$

Universal Quantification

$$\dfrac{\begin{matrix}\vdots \\ A[y/x]\end{matrix}}{\forall x \cdot A}\,\forall\mathcal{I} \quad y \notin \mathrm{FV}(hyp(A)) \qquad \dfrac{\begin{matrix}\vdots \\ \forall x \cdot A\end{matrix}}{A[t/x]}\,\forall\mathcal{E}$$

$$\dfrac{\dfrac{[A]}{\forall x \cdot A}\,\forall\mathcal{I}}{A \Rightarrow \forall x \cdot A}\,{\Rightarrow}\mathcal{I} \qquad \dfrac{\dfrac{[A]}{A \Rightarrow A}\,{\Rightarrow}\mathcal{I}}{\forall x \cdot (A \Rightarrow A)}\,\forall\mathcal{I}$$

Universal Quantification

$$\frac{\begin{array}{c}\vdots\\ A[y/x]\end{array}}{\forall x \cdot A}\,\forall\mathcal{I} \quad y \notin \mathrm{FV}(hyp(A)) \qquad \frac{\begin{array}{c}\vdots\\ \forall x \cdot A\end{array}}{A[t/x]}\,\forall\mathcal{E}$$

$$\dfrac{\dfrac{[A]}{\forall x \cdot A}\,\forall\mathcal{I}}{A \Rightarrow \forall x \cdot A}\,\Rightarrow\mathcal{I} \qquad \dfrac{\dfrac{[A]}{A \Rightarrow A}\,\Rightarrow\mathcal{I}}{\forall x \cdot (A \Rightarrow A)}\,\forall\mathcal{I}$$

Absurd

$$\frac{\begin{array}{c}\vdots \\ \bot\end{array}}{A}\ \bot\mathcal{E}$$

Disjunction

$$\dfrac{\begin{matrix}\vdots \\ A\end{matrix}}{A \vee B}\ \vee l\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots \\ B\end{matrix}}{A \vee B}\ \vee r\mathcal{I} \qquad \dfrac{\begin{matrix} \\ \vdots \\ A \vee B\end{matrix} \quad \begin{matrix}[A] \\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}}{C}\ \vee\mathcal{E}$$

Existential Quantification

$$\dfrac{\begin{matrix}\vdots \\ A[t/x]\end{matrix}}{\exists x \cdot A}\,\exists\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots \\ \exists x \cdot A\end{matrix} \qquad \begin{matrix}[A[y/x]] \\ \vdots \\ B\end{matrix}}{B}\,\exists\mathcal{E} \quad y \notin \mathrm{FV}(B, hyp(B))$$

For elimination, $y \notin hyp(B)$, i.e., not in the hypotheses other than the discharged A.

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ \bot\end{matrix}}{\neg A}\,\neg\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots \\ A\end{matrix} \qquad \begin{matrix}\vdots \\ \neg A\end{matrix}}{\bot}\,\neg\mathcal{E}$$

Negation

$$\dfrac{\begin{matrix}[A]\\ \vdots\\ \bot\end{matrix}}{\neg A}\ \neg\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots\\ A\end{matrix} \qquad \begin{matrix}\vdots\\ \neg A\end{matrix}}{\bot}\ \neg\mathcal{E}$$

Plus one of these equivalent formulations of the fact that classical negation is involutive.

$$\dfrac{}{A \vee \neg A}\ \mathrm{XM} \qquad \dfrac{\begin{matrix}\vdots\\ \neg\neg A\end{matrix}}{A}\ \neg\neg \qquad \dfrac{\begin{matrix}[\neg A]\\ \vdots\\ B\end{matrix} \qquad \begin{matrix}[\neg A]\\ \vdots\\ \neg B\end{matrix}}{A}\ \mathrm{Contradiction}$$

Cut

Cut: Introduction of a connective followed by its elimination.

$$\dfrac{\dfrac{\begin{matrix}\vdots & & \vdots \\ A & & B\end{matrix}}{A \wedge B}\wedge\mathcal{I}}{\begin{matrix}A \\ \vdots\end{matrix}}\wedge l\mathcal{E}$$

Normalization

The normalization process eliminates the cuts.

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ A\end{matrix} \qquad \begin{matrix}\vdots \\ B\end{matrix}}{A \wedge B}\,\wedge\mathcal{I}}{A}\,\wedge l\mathcal{E} \quad \rightsquigarrow \quad \begin{matrix}\vdots \\ A \\ \vdots\end{matrix}$$

Normalizing Conjunctions

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ A\end{matrix} \qquad \begin{matrix}\vdots \\ B\end{matrix}}{A \wedge B} \wedge\mathcal{I}}{\begin{matrix}A \\ \vdots\end{matrix}} \wedge l\mathcal{E} \quad \leadsto \quad \begin{matrix}\vdots \\ A \\ \vdots\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ A\end{matrix} \qquad \begin{matrix}\vdots \\ B\end{matrix}}{A \wedge B} \wedge\mathcal{I}}{\begin{matrix}B \\ \vdots\end{matrix}} \wedge r\mathcal{E} \quad \leadsto \quad \begin{matrix}\vdots \\ B \\ \vdots\end{matrix}$$

Normalizing Implications

$$\dfrac{\begin{array}{c}\vdots\\ A\end{array} \qquad \dfrac{\begin{array}{c}[A]\\ \vdots\\ B\end{array}}{A \Rightarrow B}\Rightarrow\mathcal{I}}{\begin{array}{c}B\\ \vdots\end{array}}\Rightarrow\mathcal{E} \qquad \rightsquigarrow \qquad \begin{array}{c}\vdots\\ A\\ \vdots\\ B\\ \vdots\end{array}$$

Normalizing Universal Quantifiers

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ A\end{array}}{\forall x \cdot A}\,\forall\mathcal{I}}{\begin{array}{c}A[t/x]\\ \vdots\end{array}}\,\forall\mathcal{E} \quad\leadsto\quad \begin{array}{c}\vdots\\ A[t/x]\\ \vdots\end{array}$$

x must not be free in the hypotheses, otherwise the reduction would change them.

Normalizing Disjunction

$$\begin{matrix}\dfrac{\dfrac{\begin{matrix}\vdots\\ A\end{matrix}}{A \vee B}\,\vee l\mathcal{I} \qquad \begin{matrix}[A]\\ \vdots\\ C\end{matrix} \qquad \begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{C}\,\vee\mathcal{E}\\ \vdots\end{matrix} \qquad \rightsquigarrow \qquad \begin{matrix}\vdots\\ A\\ \vdots\\ C\\ \vdots\end{matrix}$$

$$\begin{matrix}\dfrac{\dfrac{\begin{matrix}\vdots\\ B\end{matrix}}{A \vee B}\,\vee r\mathcal{I} \qquad \begin{matrix}[A]\\ \vdots\\ C\end{matrix} \qquad \begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{C}\,\vee\mathcal{E}\\ \vdots\end{matrix} \qquad \rightsquigarrow \qquad \begin{matrix}\vdots\\ B\\ \vdots\\ C\\ \vdots\end{matrix}$$

Additional Material

Logicians in a Bar

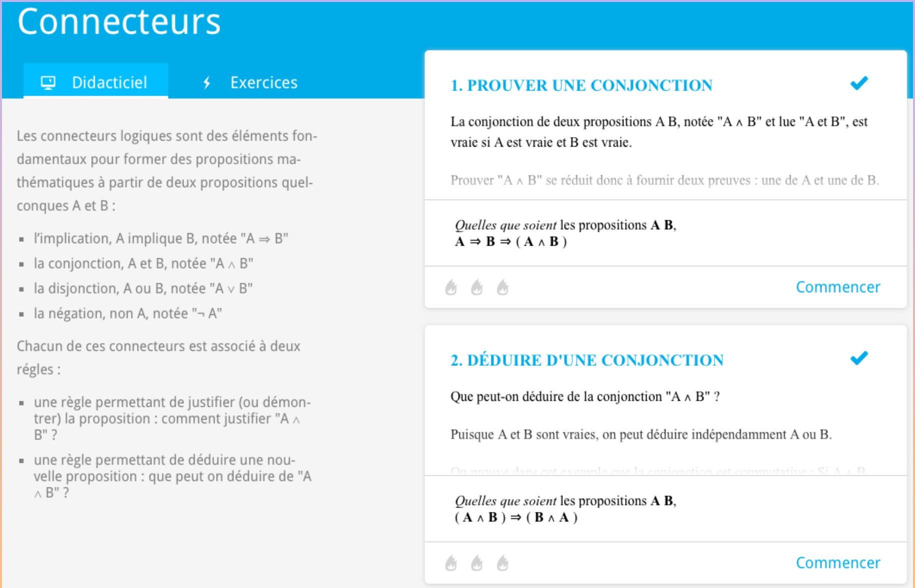
Three logicians walk into a bar,
and the bartender asks “Would you all like a drink?”

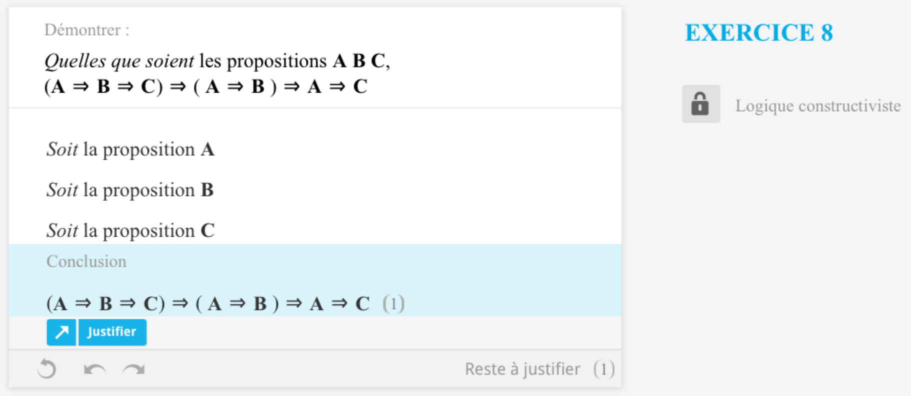
The first one says, “Maybe.”

The second one says, “Maybe.”

The third one says, “Yes.”

Edukera


Connecteurs
Didacticiel
Exercices
Les connecteurs logiques sont des éléments fondamentaux pour former des propositions mathématiques à partir de deux propositions quelconques A et B :
l'implication, A implique B, notée "A ⇒ B"
la conjonction, A et B, notée "A ∧ B"
la disjonction, A ou B, notée "A ∨ B"
la négation, non A, notée "¬ A"
Chacun de ces connecteurs est associé à deux régles :
une règle permettant de justifier (ou démontrer) la proposition : comment justifier "A ∧ B" ?
une règle permettant de déduire une nouvelle proposition : que peut on déduire de "A ∧ B" ?
1. PROUVER UNE CONJONCTION
La conjonction de deux propositions A B, notée "A ∧ B" et lue "A et B", est vraie si A est vraie et B est vraie.
Prouver "A ∧ B" se réduit donc à fournir deux preuves : une de A et une de B.
Quelles que soient les propositions A B,
A ⇒ B ⇒ (A ∧ B)
Commencer
2. DÉDUIRE D'UNE CONJONCTION
Que peut-on déduire de la conjonction "A ∧ B" ?
Puisque A et B sont vraies, on peut déduire indépendamment A ou B.
Quelles que soient les propositions A B,
(A ∧ B) ⇒ (B ∧ A)
Commencer


Démontrer :
Quelles que soient les propositions A B C,
(A ⇒ B ⇒ C) ⇒ (A ⇒ B) ⇒ A ⇒ C
Soit la proposition A
Soit la proposition B
Soit la proposition C
Conclusion
(A ⇒ B ⇒ C) ⇒ (A ⇒ B) ⇒ A ⇒ C (1)
Justifier
Reste à justifier (1)
EXERCICE 8
Logique constructiviste

Edukera

`http://edukera.appspot.com`

8245EEC

Bibliography Notes

[Girard et al., 1989]
A short (160p.) book addressing all the concerns of this course, and more (especially Linear Logic). Easy and pleasant to read. Now available for free.

[Girard, 2004]
A much more comprehensive book focusing on logic and its connections with computer science. In French.

Bibliography I

Fuck theory — Experiments in visceral philosophy.
http://fucktheory.tumblr.com/post/12492878134/dictionary-of-philosophy-logic.

Girard, J.-Y. (2004).
Cours de Logique, Rome, Automne 2004.
http://logica.uniroma3.it/uif/corso/.

Girard, J.-Y., Lafont, Y., and Taylor, P. (1989).
Proofs and Types.
Cambridge University Press.
http://www.cs.man.ac.uk/~pt/stable/Proofs+Types.html.