Sequent Calculus
Cut Elimination

Akim Demaille akim@lrde.epita.fr

EPITA — École Pour l'Informatique et les Techniques Avancées

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1. Problems of Natural Deduction
2. Sequent Calculus
3. Natural Deduction in Sequent Calculus

Preamble

The following slides are implicitly dedicated to classical logic.

Problems of Natural Deduction

1. Problems of Natural Deduction
2. Sequent Calculus
3. Natural Deduction in Sequent Calculus

Normalization

Proofs hard to find

Some elimination rules used formulas coming out of the blue.

$$\dfrac{\begin{matrix} \vdots \\ A \vee B \end{matrix} \qquad \begin{matrix} [A] \\ \vdots \\ C \end{matrix} \qquad \begin{matrix} [B] \\ \vdots \\ C \end{matrix}}{C} \vee\mathcal{E}$$

Negation is awkward

Sequent Calculus

Gerhard Karl Erich Gentzen (1909–1945)

German logician and mathematician.
SA (1933)
An assistant of David Hilbert in Göttingen (1935–1939).
Joined the Nazi Party (1937).
Worked on the V2.
Died of malnutrition in a prison camp (August 4, 1945).

Syntax

Sequents

Sequent

A sequent is an expression $\Gamma \vdash \Delta$, where Γ, Δ are (finite) sequences of formulas.

Variants:

Sets Γ, Δ can be taken as (finite) sets.

- Simplifies the structural group
- Prevents close examination of ... the structural rules

Single Sided Sequents can be forced to be $\vdash \Gamma$

- Half the number of rules is needed
- Not for intuitionistic systems

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Reading a Sequent

$\Gamma \vdash \Delta$

If all the formulas of Γ are true,
then one of the formulas of Δ is true.

- Commas on the left hand side stand for "and"
- Turnstile, $\vdash$, stands for "implies"
- Commas on the right hand side stand for "or"

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Some Special Sequents

$\Gamma \vdash A$ A is true under the hypotheses Γ

Some Special Sequents

$\Gamma \vdash A$	A is true under the hypotheses Γ
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Some Special Sequents

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$\vdash A$	A is true
$\Gamma \vdash$	Γ is in contradiction
$A \vdash$	$\neg A$
$\vdash$	contradiction

LK — Gentzen 1934

logistischer klassischer Kalkül.

- There are several possible exposures
- Different sets of inference rules
- We follow [Girard, 2011]

Identity Group

$$\frac{}{A \vdash A}\,\text{Id} \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\text{Cut}$$

Gentzen's Hauptsatz

The cut rule is redundant, i.e., any sequent provable with cuts is provable without.

Identity Group

$$\frac{}{A \vdash A}\text{Id} \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\text{Cut}$$

Gentzen's Hauptsatz

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Structural Group

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \tau(\Delta)}\ \vdash\mathrm{X} \qquad \frac{\Gamma \vdash \Delta}{\sigma(\Gamma) \vdash \Delta}\ \mathrm{X}\vdash$$

Structural Group

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \tau(\Delta)} \vdash\mathrm{X} \qquad \frac{\Gamma \vdash \Delta}{\sigma(\Gamma) \vdash \Delta} \mathrm{X}\vdash$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \vdash\mathrm{W} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \mathrm{W}\vdash$$

Structural Group

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \tau(\Delta)} \vdash \mathrm{X} \qquad \frac{\Gamma \vdash \Delta}{\sigma(\Gamma) \vdash \Delta} \mathrm{X}\vdash$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \vdash \mathrm{W} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \mathrm{W}\vdash$$

$$\frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \vdash \mathrm{C} \qquad \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \mathrm{C}\vdash$$

Logical Group: Negation

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\vdash\neg \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\neg\vdash$$

Logical Group: Conjunction

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \vdash \wedge$$

Logical Group: Conjunction

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \vdash\wedge$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} l\wedge\vdash$$

Logical Group: Conjunction

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \vdash\wedge$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \; l\wedge\vdash$$

$$\frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \; r\wedge\vdash$$

Logical Group: Conjunction

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \vdash\wedge$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} l\wedge\vdash$$

$$\frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} r\wedge\vdash$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'} \vdash\wedge$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge\vdash$$

Logical Group: Conjunction

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \vdash\wedge$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} l\wedge\vdash$$

$$\frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} r\wedge\vdash$$

Additive

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'} \vdash\wedge$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge\vdash$$

Multiplicative

Logical Group: Conjunction

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \vdash\wedge$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} l\wedge\vdash$$

$$\frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} r\wedge\vdash$$

Additive

$\&$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'} \vdash\wedge$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge\vdash$$

Multiplicative

$\otimes$

Logical Group: Disjunction

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta} \vdash I\vee$$

Logical Group: Disjunction

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta} \vdash l\vee$$

$$\frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vdash r\vee$$

Logical Group: Disjunction

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta} \vdash l\vee$$

$$\frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vdash r\vee$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \vee\vdash$$

Logical Group: Disjunction

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta} \vdash l\vee$$

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$$\frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'} \vee\vdash$$

Logical Group: Disjunction

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta} \vdash l\vee \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \vee\vdash$$

$$\frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vdash r\vee$$

Additive

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vdash\vee \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'} \vee\vdash$$

Multiplicative

Logical Group: Disjunction

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \lor B, \Delta} \vdash l\lor$$

$$\frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \lor B, \Delta} \vdash r\lor$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \lor B \vdash \Delta} \lor\vdash$$

Additive

$\oplus$

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \lor B, \Delta} \vdash\lor$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \lor B \vdash \Delta, \Delta'} \lor\vdash$$

Multiplicative

⅋

Logical Group: Implication

$$\frac{\Gamma \vdash \Delta, A \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \Rightarrow B \vdash \Delta, \Delta'} \Rightarrow\!\vdash$$

Logical Group: Implication

$$\frac{\Gamma \vdash \Delta, A \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \Rightarrow B \vdash \Delta, \Delta'} \Rightarrow\vdash \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta} \vdash\Rightarrow$$

Prove $A \wedge B \vdash A \wedge B$

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A \wedge B \vdash A}\, l\wedge\vdash \qquad \dfrac{\dfrac{}{B \vdash B}}{A \wedge B \vdash B}\, r\wedge\vdash}{A \wedge B \vdash A \wedge B}\, \vdash\wedge$$

Prove $A \wedge B \vdash A \wedge B$

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A \wedge B \vdash A}\ l\wedge\vdash \qquad \dfrac{\dfrac{}{B \vdash B}}{A \wedge B \vdash B}\ r\wedge\vdash}{A \wedge B \vdash A \wedge B}\ \vdash\wedge$$

Prove $A \wedge B \vdash A \vee B$

$$\dfrac{\dfrac{\overline{A \vdash A}}{A \wedge B \vdash A}\, l\wedge\vdash}{A \wedge B \vdash A \vee B}\, \vdash r\vee$$

Prove $A \wedge B \vdash A \vee B$

$$\dfrac{\dfrac{\overline{A \vdash A}}{A \wedge B \vdash A}\, l\wedge\vdash}{A \wedge B \vdash A \vee B}\, \vdash r\vee$$

Prove $A \vee B \vdash A \vee B$

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A \vdash A \vee B}\vdash l\vee \qquad \dfrac{\dfrac{}{B \vdash B}}{B \vdash A \vee B}\vdash r\vee}{A \vee B \vdash A \vee B}\vee\vdash$$

Prove $A \vee B \vdash A \vee B$

$$
\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A \vdash A \vee B}\vdash l\vee \qquad \dfrac{\dfrac{}{B \vdash B}}{B \vdash A \vee B}\vdash r\vee}{A \vee B \vdash A \vee B}\vee\vdash
$$

Prove the equivalence of the two $\wedge$ rules

Additive	Multiplicative

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \vdash \wedge + \quad \equiv \quad \frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'} \vdash \wedge \times$$

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Additive	Multiplicative

$$\dfrac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \vdash \wedge + \quad \equiv \quad \dfrac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'} \vdash \wedge \times$$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma, \Gamma \vdash A \wedge B, \Delta, \Delta} \vdash \wedge \times}{\Gamma, \Gamma \vdash A \wedge B, \Delta} \vdash \mathrm{C}}{\Gamma \vdash A \wedge B, \Delta} \mathrm{C} \vdash$$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash A, \Delta}{\Gamma, \Gamma' \vdash A, \Delta} \mathrm{W} \vdash}{\Gamma, \Gamma' \vdash A, \Delta, \Delta'} \vdash \mathrm{W} \quad \dfrac{\dfrac{\Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash B, \Delta'} \mathrm{W} \vdash}{\Gamma, \Gamma' \vdash B, \Delta, \Delta'} \vdash \mathrm{W}}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'} \vdash \wedge +$$

Logical Group: Quantifiers

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash \forall x \cdot A, \Delta} \vdash \forall \qquad \frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x \cdot A \vdash \Delta} \forall \vdash$$

Logical Group: Quantifiers

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash \forall x \cdot A, \Delta} \vdash \forall \qquad \frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x \cdot A \vdash \Delta} \forall \vdash$$

$$\frac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x \cdot A, \Delta} \vdash \exists \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, \exists x \cdot A \vdash \Delta} \exists \vdash$$

In $\vdash \forall$ and $\exists \vdash$, $x \notin \mathrm{FV}(\Gamma, \Delta)$.

- Alternatively, one can define the negation as a notation instead of defining it by inference rules.

$$
\begin{aligned}
\neg(\neg p) &:= p \\
\neg(A \wedge B) &:= \neg A \vee \neg B \\
\neg(A \vee B) &:= \neg A \wedge \neg B \\
\neg(\forall x \cdot A) &:= \exists x \cdot \neg A \\
\neg(\exists x \cdot A) &:= \forall x \cdot \neg A
\end{aligned}
$$

- Then define the sequents as $\vdash \Gamma$
- I.e., $\Gamma \vdash \Delta \leadsto \vdash \neg\Gamma, \Delta$

Single Sided

Defining the Negation

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$$\frac{}{\vdash \neg A, A}\ \mathrm{Id} \qquad \frac{\vdash \Gamma, A \quad \vdash \neg A, \Delta}{\vdash \Gamma, \Delta}\ \mathrm{Cut}$$

$$\frac{\vdash \Gamma}{\vdash \sigma(\Gamma)}\ \mathrm{X} \qquad \frac{\vdash \Gamma}{\vdash A, \Gamma}\ \mathrm{W} \qquad \frac{\vdash A, A, \Gamma}{\vdash A, \Gamma}\ \mathrm{C}$$

$$\frac{\vdash A, \Delta}{\vdash A \vee B, \Delta}\ l\vee \qquad \frac{\vdash B, \Delta}{\vdash A \vee B, \Delta}\ r\vee \qquad \frac{\vdash A, \Delta \quad \vdash B, \Delta}{\vdash A \wedge B, \Delta}\ \wedge$$

$$\frac{\vdash A, \Delta}{\vdash \forall x \cdot A, \Delta}\ \forall \qquad \frac{\vdash A[t/x], \Delta}{\vdash \exists x \cdot A, \Delta}\ \exists$$

Subformula Property

What can be told from the last rule of a proof?

- If the last rule is a cut,

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

- Otherwise. . .

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \land B, \Delta} \vdash \land \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \land B \vdash \Delta}\, l\land\vdash \qquad \cdots$$

Subformula Property

What can be told from the last rule of a proof?

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  nothing can be said!

- Otherwise. . .

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \vdash \wedge \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\, l\wedge\vdash \qquad \cdots$$

Subformula Property

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Subformula Property

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- Otherwise...

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  premisses can only use subformulas of the conclusion!

Cut Elimination

- replace "complex" cuts by simpler cuts (smaller formulas)
- until the cut is on the simplest form, the identity
- where it is not longer needed!

Cut Elimination

Logical Rules

$$\dfrac{\dfrac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \vdash \wedge \qquad \dfrac{\Gamma', A \vdash \Delta'}{\Gamma', A \wedge B \vdash \Delta'} \, l\wedge \vdash}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \, \mathrm{Cut}$$

$$\leadsto$$

$$\dfrac{\dfrac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \vdash \wedge \qquad \dfrac{\Gamma', A \vdash \Delta'}{\Gamma', A \wedge B \vdash \Delta'} l\wedge \vdash}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \mathrm{Cut}$$

$$\leadsto$$

$$\dfrac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \mathrm{Cut}$$

$$\dfrac{\dfrac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \vdash \wedge \qquad \dfrac{\Gamma', A \vdash \Delta'}{\Gamma', A \wedge B \vdash \Delta'} l\wedge \vdash}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{Cut}$$

$$\rightsquigarrow$$

$$\dfrac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{Cut}$$

For all the connectives.

$$\dfrac{\dfrac{}{A \vdash A}\,\text{Identity} \qquad \dfrac{\vdots}{\Gamma, A \vdash \Delta}}{\Gamma, A \vdash \Delta}\,\text{Cut}$$

$$\rightsquigarrow$$

Cut Elimination

Removal of a Cut

$$\dfrac{\dfrac{}{A \vdash A}\text{Identity} \qquad \dfrac{\vdots}{\Gamma, A \vdash \Delta}}{\Gamma, A \vdash \Delta}\text{Cut}$$

$$\rightsquigarrow$$

$$\dfrac{\vdots}{\Gamma, A \vdash \Delta}$$

Cut Elimination

Commutations

$$\dfrac{\dfrac{\Gamma, B \vdash A, \Delta \quad \Gamma, C \vdash A, \Delta}{\Gamma, B \vee C \vdash A, \Delta} \vee\vdash \qquad \Gamma', A \vdash \Delta'}{\Gamma, B \vee C, \Gamma' \vdash \Delta, \Delta'} \text{Cut}$$

$$\rightsquigarrow$$

Cut Elimination

Commutations

$$\dfrac{\dfrac{\Gamma, B \vdash A, \Delta \quad \Gamma, C \vdash A, \Delta}{\Gamma, B \vee C \vdash A, \Delta}\vee\vdash \qquad \Gamma', A \vdash \Delta'}{\Gamma, B \vee C, \Gamma' \vdash \Delta, \Delta'}\text{Cut}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\Gamma, B \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, B, \Gamma' \vdash \Delta, \Delta'}\text{Cut} \qquad \dfrac{\Gamma, C \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, C, \Gamma' \vdash \Delta, \Delta'}\text{Cut}}{\Gamma, B \vee C, \Gamma' \vdash \Delta, \Delta'}\vee\vdash$$

$$\dfrac{\dfrac{\Gamma, B \vdash A, \Delta \quad \Gamma, C \vdash A, \Delta}{\Gamma, B \vee C \vdash A, \Delta} \vee\vdash \qquad \Gamma', A \vdash \Delta'}{\Gamma, B \vee C, \Gamma' \vdash \Delta, \Delta'} \mathrm{Cut}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\Gamma, B \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, B, \Gamma' \vdash \Delta, \Delta'} \mathrm{Cut} \qquad \dfrac{\Gamma, C \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, C, \Gamma' \vdash \Delta, \Delta'} \mathrm{Cut}}{\Gamma, B \vee C, \Gamma' \vdash \Delta, \Delta'} \vee\vdash$$

Beware of the duplication!

Cut Elimination

Structural Rules: Weakening

$$\dfrac{\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\vdash\mathrm{W} \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\mathrm{Cut}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\,{\vdash}\mathrm{W} \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\Gamma \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta}\,\mathrm{W}{\vdash}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,{\vdash}\mathrm{W}$$

Cut Elimination

Structural Rules: Contraction

$$\dfrac{\dfrac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\vdash\mathrm{C} \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\mathrm{Cut}$$

$$\rightsquigarrow$$

Cut Elimination

Structural Rules: Contraction

$$
\cfrac{\cfrac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\,{\vdash}\mathrm{C} \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}
$$

$$
\rightsquigarrow
$$

$$
\cfrac{\cfrac{\cfrac{\Gamma \vdash A, A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash A, \Delta, \Delta'}\,\mathrm{Cut} \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma', \Gamma' \vdash \Delta, \Delta', \Delta'}\,\mathrm{Cut}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{C}
$$

$$\dfrac{\dfrac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\,\vdash\mathrm{C} \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

Nice!

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash A, A, \Delta \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash A, \Delta, \Delta'}\,\mathrm{Cut} \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma', \Gamma' \vdash \Delta, \Delta', \Delta'}\,\mathrm{Cut}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{C}$$

$$\dfrac{\dfrac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\vdash\mathrm{C} \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

Nice!

$$\leadsto$$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash A, A, \Delta \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash A, \Delta, \Delta'}\,\mathrm{Cut} \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma', \Gamma' \vdash \Delta, \Delta', \Delta'}\,\mathrm{Cut}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{C}$$

but
wrong

$$\dfrac{\dfrac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\vdash\mathrm{C} \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\mathrm{Cut}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash A, A, \Delta \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash A, \Delta, \Delta'}\mathrm{Cut} \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma', \Gamma' \vdash \Delta, \Delta', \Delta'}\mathrm{Cut}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\mathrm{C}$$

Nice!

but wrong

might loop for ever

Cut Elimination

Structural Rules: Complications with Contractions

$$\dfrac{\dfrac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\vdash\mathrm{C} \qquad \dfrac{\Gamma', A, A \vdash \Delta'}{\Gamma', A \vdash \Delta'}\mathrm{C}\vdash}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\mathrm{Cut}$$

$$\rightsquigarrow$$

$$
\dfrac{\dfrac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\,\vdash\mathrm{C} \qquad \dfrac{\Gamma', A, A \vdash \Delta'}{\Gamma', A \vdash \Delta'}\,\mathrm{C}\vdash}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}
$$

$$
\rightsquigarrow
$$

$$
\dfrac{\dfrac{\dfrac{\Gamma \vdash A, A, \Delta \qquad \dfrac{\Gamma', A, A \vdash \Delta'}{\Gamma', A \vdash \Delta'}\,\mathrm{C}\vdash}{\Gamma, \Gamma' \vdash A, \Delta, \Delta'}\,\mathrm{Cut} \qquad \dfrac{\Gamma', A, A \vdash \Delta'}{\Gamma', A \vdash \Delta'}\,\mathrm{C}\vdash}{\Gamma, \Gamma', \Gamma' \vdash \Delta, \Delta', \Delta'}\,\mathrm{Cut}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{C}
$$

Recommended readings

[Girard et al., 1989], Chapters 5 & 13

A short (160p.) book addressing all the concerns of this course, and more (especially Linear Logic).
Easy and pleasant to read. Available for free.

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