Exercises on λ-calculus and Deduction Systems

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λ-calculus

1. λ-calculus
   - Untyped λ-calculus
   - Simply Typed λ-calculus
2. Deduction Systems

Substitutions

$$[\lambda z \cdot zz/x]\lambda y \cdot xy \quad \equiv$$

Substitutions

$$
\begin{aligned}
[\lambda z \cdot zz/x]\lambda y \cdot xy &\equiv \lambda y \cdot (\lambda z \cdot zz)y \\
[yy/z](\lambda xy \cdot zy) &\equiv
\end{aligned}
$$

Substitutions

$$
\begin{aligned}
[\lambda z \cdot zz/x]\lambda y \cdot xy &\equiv \lambda y \cdot (\lambda z \cdot zz)y \\
[yy/z](\lambda xy \cdot zy) &\equiv \lambda xu \cdot yyu
\end{aligned}
$$

β-Reductions

$$(\lambda x \cdot xyx)\lambda z \cdot z \quad \rightarrow$$

β-Reductions

$$\begin{aligned} (\lambda x \cdot xyx)\lambda z \cdot z &\to (\lambda z \cdot z)y(\lambda z \cdot z) \\ (\lambda x \cdot x)((\lambda y \cdot y)x) &\to \end{aligned}$$

β-Reductions

$$
\begin{aligned}
(\lambda x \cdot xyx)\lambda z \cdot z &\rightarrow (\lambda z \cdot z)y(\lambda z \cdot z) \\
(\lambda x \cdot x)((\lambda y \cdot y)x) &\rightarrow (\lambda x \cdot x)(x) \\
(\lambda x \cdot x)((\lambda y \cdot y)x) &\rightarrow
\end{aligned}
$$

β-Reductions

$$
\begin{aligned}
(\lambda x \cdot xyx)\lambda z \cdot z &\rightarrow (\lambda z \cdot z)y(\lambda z \cdot z) \\
(\lambda x \cdot x)((\lambda y \cdot y)x) &\rightarrow (\lambda x \cdot x)(x) \\
(\lambda x \cdot x)((\lambda y \cdot y)x) &\rightarrow ((\lambda y \cdot y)x) \\
(\lambda x \cdot x)((\lambda y \cdot y)x) &\stackrel{*}{\rightarrow}
\end{aligned}
$$

β-Reductions

$$
\begin{aligned}
(\lambda x \cdot xyx)\lambda z \cdot z &\rightarrow (\lambda z \cdot z)y(\lambda z \cdot z) \\
(\lambda x \cdot x)((\lambda y \cdot y)x) &\rightarrow (\lambda x \cdot x)(x) \\
(\lambda x \cdot x)((\lambda y \cdot y)x) &\rightarrow ((\lambda y \cdot y)x) \\
(\lambda x \cdot x)((\lambda y \cdot y)x) &\xrightarrow{*} x \\
(\lambda x \cdot xx)((\lambda x \cdot xx)y) &\xrightarrow{*}
\end{aligned}
$$

β-Reductions

$$
\begin{array}{rcl}
(\lambda x \cdot xyx)\lambda z \cdot z & \rightarrow & (\lambda z \cdot z)y(\lambda z \cdot z) \\
(\lambda x \cdot x)((\lambda y \cdot y)x) & \rightarrow & (\lambda x \cdot x)(x) \\
(\lambda x \cdot x)((\lambda y \cdot y)x) & \rightarrow & ((\lambda y \cdot y)x) \\
(\lambda x \cdot x)((\lambda y \cdot y)x) & \stackrel{*}{\rightarrow} & x \\
(\lambda x \cdot xx)((\lambda x \cdot xx)y) & \stackrel{*}{\rightarrow} & yy(yy) \\
(\lambda x \cdot xx)((\lambda x \cdot x)y) & \stackrel{*}{\rightarrow} &
\end{array}
$$

β-Reductions

$$
\begin{array}{rcl}
(\lambda x \cdot xyx)\lambda z \cdot z & \rightarrow & (\lambda z \cdot z)y(\lambda z \cdot z) \\
(\lambda x \cdot x)((\lambda y \cdot y)x) & \rightarrow & (\lambda x \cdot x)(x) \\
(\lambda x \cdot x)((\lambda y \cdot y)x) & \rightarrow & ((\lambda y \cdot y)x) \\
(\lambda x \cdot x)((\lambda y \cdot y)x) & \stackrel{*}{\rightarrow} & x \\
(\lambda x \cdot xx)((\lambda x \cdot xx)y) & \stackrel{*}{\rightarrow} & yy(yy) \\
(\lambda x \cdot xx)((\lambda x \cdot x)y) & \stackrel{*}{\rightarrow} & yy \\
(\lambda x \cdot x)((\lambda x \cdot xx)y) & \stackrel{*}{\rightarrow} &
\end{array}
$$

β-Reductions

$$
\begin{aligned}
(\lambda x \cdot xyx)\lambda z \cdot z &\rightarrow (\lambda z \cdot z)y(\lambda z \cdot z) \\
(\lambda x \cdot x)((\lambda y \cdot y)x) &\rightarrow (\lambda x \cdot x)(x) \\
(\lambda x \cdot x)((\lambda y \cdot y)x) &\rightarrow ((\lambda y \cdot y)x) \\
(\lambda x \cdot x)((\lambda y \cdot y)x) &\overset{*}{\rightarrow} x \\
(\lambda x \cdot xx)((\lambda x \cdot xx)y) &\overset{*}{\rightarrow} yy(yy) \\
(\lambda x \cdot xx)((\lambda x \cdot x)y) &\overset{*}{\rightarrow} yy \\
(\lambda x \cdot x)((\lambda x \cdot xx)y) &\overset{*}{\rightarrow} yy
\end{aligned}
$$

Simply Typed λ-calculus

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Type derivations are trees built from the following nodes.

$$\frac{M : \sigma \to \tau \quad N : \sigma}{MN : \tau}$$

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$$\frac{M : \sigma \to \tau \quad N : \sigma}{MN : \tau} \qquad \frac{\begin{array}{c} x : \sigma \\ \vdots \\ M : \tau \end{array}}{}$$

Simply Typed λ-calculus

Type derivations are trees built from the following nodes.

$$\frac{M : \sigma \to \tau \quad N : \sigma}{MN : \tau} \qquad \frac{\begin{array}{c}[x : \sigma] \\ \vdots \\ M : \tau\end{array}}{\lambda x \cdot M : \sigma \to \tau}$$

Type Statements

Type $\lambda fx \cdot f(fx)$

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$$\vdash \lambda fx \cdot f(fx) : (\sigma \to \sigma) \to \sigma \to \sigma$$

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Type $\lambda fx \cdot f(fx)$

$$\vdash \lambda fx \cdot f(fx) : (\sigma \to \sigma) \to \sigma \to \sigma$$

$$\dfrac{\dfrac{\dfrac{[f : \sigma \to \sigma]^{(2)} \qquad \dfrac{[f : \sigma \to \sigma]^{(2)} \quad [x : \sigma]^{(1)}}{fx : \sigma}}{f(fx) : \sigma}}{\lambda x \cdot f(fx) : \sigma \to \sigma}\,(1)}{\lambda fx \cdot f(fx) : (\sigma \to \sigma) \to \sigma \to \sigma}\,(2)$$

Type Statements

Type $\lambda xy \cdot x$

Type Statements

Type $\lambda xy \cdot x$

$$\vdash \lambda xy \cdot x : \sigma \to \tau \to \sigma$$

Type Statements

Type $\lambda xy \cdot x$

$$\vdash \lambda xy \cdot x : \sigma \to \tau \to \sigma$$

$$\cfrac{\cfrac{[x : \sigma]^{(1)}}{\lambda y \cdot x : \tau \to \sigma}}{\lambda xy \cdot x : \sigma \to \tau \to \sigma}\ (1)$$

Deduction Systems

Natural Deduction

1. λ-calculus
2. Deduction Systems
   - Natural Deduction
   - Sequent Calculus

Intuitionistic Natural Deduction

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \Rightarrow B} \Rightarrow\mathcal{I} \qquad \dfrac{A \quad A \Rightarrow B}{B} \Rightarrow\mathcal{E} \qquad \dfrac{\bot}{A} \bot\mathcal{E} \qquad \neg A := A \Rightarrow \bot$$

$$\dfrac{A \quad B}{A \wedge B} \wedge\mathcal{I} \qquad \dfrac{A \wedge B}{A} \wedge l\mathcal{E} \qquad \dfrac{A \wedge B}{B} \wedge r\mathcal{E}$$

$$\dfrac{A}{A \vee B} \vee l\mathcal{I} \qquad \dfrac{B}{A \vee B} \vee r\mathcal{I} \qquad \dfrac{A \vee B \quad \begin{matrix}[A]\\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C} \vee\mathcal{E}$$

Prove $A \wedge B \Rightarrow B \wedge A$

Prove $A \wedge B \Rightarrow B \wedge A$

$$\dfrac{\dfrac{\dfrac{[A \wedge B]^1}{B}\,\wedge r\mathcal{E} \qquad \dfrac{[A \wedge B]^1}{A}\,\wedge l\mathcal{E}}{B \wedge A}\,\wedge\mathcal{I}}{A \wedge B \Rightarrow B \wedge A}\,\Rightarrow\mathcal{I}_1$$

Prove $A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$

Prove $A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$

$$
\dfrac{
\dfrac{A \wedge (B \vee C)}{B \vee C}\,\wedge r\mathcal{E}
\qquad
\dfrac{\dfrac{\dfrac{A \wedge (B \vee C)}{A}\,\wedge l\mathcal{E} \qquad [B]^1}{A \wedge B}\,\wedge\mathcal{I}}{(A \wedge B) \vee (A \wedge C)}\,\vee l\mathcal{I}
\qquad
\dfrac{\dfrac{\dfrac{A \wedge (B \vee C)}{A}\,\wedge l\mathcal{E} \qquad [C]^1}{A \wedge C}\,\wedge\mathcal{I}}{(A \wedge B) \vee (A \wedge C)}\,\vee r\mathcal{I}
}{(A \wedge B) \vee (A \wedge C)}\,\vee\mathcal{E}_1
$$

Prove $(A \Rightarrow A) \Rightarrow A \Rightarrow A$ (LOFO-2005)

Remember, $\Rightarrow$ is right-associative.

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Remember, $\Rightarrow$ is right-associative.

$$\dfrac{\dfrac{\dfrac{[A]^2 \quad [A \Rightarrow A]^1}{A} \Rightarrow\mathcal{E}}{A \Rightarrow A} \Rightarrow\mathcal{I}_2}{(A \Rightarrow A) \Rightarrow A \Rightarrow A} \Rightarrow\mathcal{I}_1$$

Prove $(A \Rightarrow A) \Rightarrow A \Rightarrow A$ (LOFO-2005)

Remember, $\Rightarrow$ is right-associative.

$$\dfrac{\dfrac{\dfrac{[A]^2 \quad [A \Rightarrow A]^1}{A} \Rightarrow\mathcal{E}}{A \Rightarrow A} \Rightarrow\mathcal{I}_2}{(A \Rightarrow A) \Rightarrow A \Rightarrow A} \Rightarrow\mathcal{I}_1$$

$$\dfrac{\dfrac{[A]^1}{A \Rightarrow A} \Rightarrow\mathcal{I}_1}{(A \Rightarrow A) \Rightarrow A \Rightarrow A} \Rightarrow\mathcal{I}$$

Prove $(A \Rightarrow A) \Rightarrow A \Rightarrow A$ (LOFO-2005)

Remember, $\Rightarrow$ is right-associative.

$$\dfrac{\dfrac{\dfrac{[A]^2 \quad [A \Rightarrow A]^1}{A} \Rightarrow\mathcal{E}}{A \Rightarrow A} \Rightarrow\mathcal{I}_2}{(A \Rightarrow A) \Rightarrow A \Rightarrow A} \Rightarrow\mathcal{I}_1$$

$$\dfrac{\dfrac{[A]^1}{A \Rightarrow A} \Rightarrow\mathcal{I}_1}{(A \Rightarrow A) \Rightarrow A \Rightarrow A} \Rightarrow\mathcal{I}$$

$$\dfrac{[A \Rightarrow A]^1}{(A \Rightarrow A) \Rightarrow A \Rightarrow A} \Rightarrow\mathcal{I}_1$$

Prove $A \Rightarrow B, B \Rightarrow C \vdash A \Rightarrow (B \land C)$ (LOFO-2005)

Prove $A \Rightarrow B, B \Rightarrow C \vdash A \Rightarrow (B \wedge C)$ (LOFO-2005)

$$
\dfrac{
  \dfrac{
    \dfrac{[A]^1 \quad A \Rightarrow B}{B}\,\Rightarrow\mathcal{E}
    \qquad
    \dfrac{
      \dfrac{[A]^1 \quad A \Rightarrow B}{B}\,\Rightarrow\mathcal{E}
      \quad B \Rightarrow C
    }{C}\,\Rightarrow\mathcal{E}
  }{B \wedge C}\,\wedge\mathcal{I}
}{A \Rightarrow (B \wedge C)}\,\Rightarrow\mathcal{I}_1
$$

Prove $A \vee B, \neg B \vdash A$ (Intuitionistic) (LOFO-2005)

Prove $A \vee B, \neg B \vdash A$ (Intuitionistic) (LOFO-2005)

Recall that $\neg B := B \Rightarrow \bot$.

Prove $A \vee B, \neg B \vdash A$ (Intuitionistic) (LOFO-2005)

Recall that $\neg B := B \Rightarrow \bot$.

$$\dfrac{A \vee B \qquad [A]^1 \qquad \dfrac{\dfrac{[B]^1 \quad B \Rightarrow \bot}{\bot}\Rightarrow\mathcal{E}}{A}\bot\mathcal{E}}{A}\vee\mathcal{E}_1$$

Sequent Calculus

Classical Sequent Calculus

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \tau(\Delta)}\ \vdash\mathrm{X} \qquad \frac{\Gamma \vdash \Delta}{\sigma(\Gamma) \vdash \Delta}\ \mathrm{X}\vdash \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ \vdash\mathrm{W} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ \mathrm{W}\vdash \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\ \vdash\mathrm{C} \qquad \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ \mathrm{C}\vdash$$

$$\frac{}{F \vdash F}\ \mathrm{Id} \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \mathrm{Cut}$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \vdash\neg \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ \neg\vdash$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ \vdash\wedge \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ l\wedge\vdash \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ r\wedge\vdash$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vdash l\vee \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vdash r\vee \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ \vee\vdash$$

$$\frac{\Gamma \vdash \Delta, A \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \Rightarrow B \vdash \Delta, \Delta'}\ \Rightarrow\vdash \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta}\ \vdash\Rightarrow$$

Prove $A \wedge B \vdash A \wedge B$

Prove $A \wedge B \vdash A \wedge B$

$$\dfrac{\dfrac{\overline{A \vdash A}}{A \wedge B \vdash A}\, l\wedge\vdash \qquad \dfrac{\overline{B \vdash B}}{A \wedge B \vdash B}\, r\wedge\vdash}{A \wedge B \vdash A \wedge B}\, \vdash\wedge$$

Prove $A \wedge B \vdash A \vee B$

Prove $A \wedge B \vdash A \vee B$

$$\dfrac{\dfrac{\overline{A \vdash A}}{A \wedge B \vdash A}\ l\wedge\vdash}{A \wedge B \vdash A \vee B}\ \vdash r\vee$$

Prove $A \vee B \vdash A \vee B$

Prove $A \vee B \vdash A \vee B$

$$\dfrac{\dfrac{\overline{A \vdash A}}{A \vdash A \vee B}\vdash l\vee \qquad \dfrac{\overline{B \vdash B}}{B \vdash A \vee B}\vdash r\vee}{A \vee B \vdash A \vee B}\vee\vdash$$

Prove $(F \wedge G) \Rightarrow H \vdash (F \Rightarrow H) \vee (G \Rightarrow H)$

Prove $(F \wedge G) \Rightarrow H \vdash (F \Rightarrow H) \vee (G \Rightarrow H)$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{
            \dfrac{
              \dfrac{
                \dfrac{
                  \dfrac{\dfrac{}{F \vdash F}}{F, G \vdash F}\,\mathrm{W}\vdash
                  \qquad
                  \dfrac{\dfrac{}{G \vdash G}}{F, G \vdash G}\,\mathrm{W}\vdash
                }{F, G \vdash F \wedge G}\,\vdash\wedge
                \qquad
                \dfrac{}{H \vdash H}
              }{F, G, (F \wedge G) \Rightarrow H \vdash H}\,\Rightarrow\vdash
            }{F, G, (F \wedge G) \Rightarrow H \vdash H, H}\,\vdash\mathrm{W}
          }{F, (F \wedge G) \Rightarrow H \vdash H, G \Rightarrow H}\,\vdash\Rightarrow
        }{(F \wedge G) \Rightarrow H \vdash F \Rightarrow H, G \Rightarrow H}\,\vdash\Rightarrow
      }{(F \wedge G) \Rightarrow H \vdash (F \Rightarrow H) \vee (G \Rightarrow H), G \Rightarrow H}\,\vdash r\vee
    }{(F \wedge G) \Rightarrow H \vdash (F \Rightarrow H) \vee (G \Rightarrow H), (F \Rightarrow H) \vee (G \Rightarrow H)}\,\vdash l\vee
  }{(F \wedge G) \Rightarrow H \vdash (F \Rightarrow H) \vee (G \Rightarrow H)}\,\vdash\mathrm{C}
}{}
$$

Prove $(A \Rightarrow A) \Rightarrow A \Rightarrow A$ (LOFO-2005)

Prove $(A \Rightarrow A) \Rightarrow A \Rightarrow A$ (LOFO-2005)

$$
\cfrac{\cfrac{\cfrac{\cfrac{}{A \vdash A} \qquad \cfrac{}{A \vdash A}}{A, A \Rightarrow A \vdash A} \Rightarrow\vdash}{A \Rightarrow A \vdash A \Rightarrow A} \vdash\Rightarrow}{\vdash (A \Rightarrow A) \Rightarrow A \Rightarrow A} \vdash\Rightarrow
$$

Prove $A \Rightarrow B, B \Rightarrow C \vdash A \Rightarrow (B \wedge C)$ (LOFO-2005)

Prove $A \Rightarrow B, B \Rightarrow C \vdash A \Rightarrow (B \wedge C)$ (LOFO-2005)

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{}{A \vdash A} \qquad \dfrac{}{B \vdash B}
      }{A, A \Rightarrow B \vdash B} \Rightarrow\vdash
    }{A, A \Rightarrow B, B \Rightarrow C \vdash B} \mathrm{W}\vdash
    \qquad
    \dfrac{
      \dfrac{
        \dfrac{}{A \vdash A} \qquad \dfrac{}{C \vdash C}
      }{A, B \Rightarrow C \vdash C} \Rightarrow\vdash
    }{A, A \Rightarrow B, B \Rightarrow C \vdash C} \mathrm{W}\vdash
  }{A, A \Rightarrow B, A, B \Rightarrow C \vdash B \wedge C} \wedge\vdash
}{A, A \Rightarrow B, B \Rightarrow C \vdash B \wedge C} \mathrm{C}\vdash
$$

Prove $A \vee B, \neg B \vdash A$ (Classical) (LOFO-2005)

Prove $A \vee B, \neg B \vdash A$ (Classical) (LOFO-2005)

$$
\dfrac{
  \dfrac{\dfrac{}{A \vdash A}}{A, \neg B \vdash A}\,\mathrm{W}\vdash
  \qquad
  \dfrac{\dfrac{\dfrac{}{B \vdash B}}{B \vdash B, A}\,\vdash\mathrm{W}}{B, \neg B \vdash A}\,\vdash\neg
}{A \vee B, \neg B \vdash A}\,\vee\vdash
$$