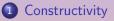
Intuitionistic Logic

Akim Demaille akim@lrde.epita.fr

EPITA — École Pour l'Informatique et les Techniques Avancées

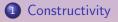
June 10, 2016



2 Intuitionistic Logic

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Constructivity



Intuitionistic Logic

A. Demaille

Intuitionistic Logic

• Classical logic does not build truth

- it discovers a preexisting truth
- Classical logic assumes facts are either true or false
- $\vdash A \lor \neg A$ Excluded middle, *tertium non datur*

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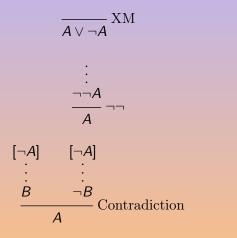
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$$\frac{\overline{A \vdash A}}{\vdash \neg A, A} \vdash \neg$$

$$\frac{\overline{A \vdash A}}{\vdash A \lor \neg A, A} \vdash r \lor$$

$$\frac{\overline{A \vdash A}}{\vdash A \lor \neg A, A \lor \neg A} \vdash I \lor$$

$$\frac{\overline{A \vdash A}}{\vdash A \lor \neg A} \vdash C$$

- A You should respect C's belief, for all beliefs are of equal validity and cannot be denied.
- B What about D's belief? (Where D believes something that is considered to be wrong by most people, such as nazism or the world being flat)
- A I agree it is right to deny D's belief.
- B If it is right to deny D's belief, it is not true that no belief can be denied. Therefore, I can deny C's belief if I can give reasons that suggest it too is incorrect.

- A You should respect C's belief, for all beliefs are of equal validity and cannot be denied.
- B **1** deny that belief of yours and believe it to be invalid.
 - According to your statement, this belief of mine (1) is valid, like all other beliefs.
 - However, your statement also contradicts and invalidates mine, being the exact opposite of it.
 - The conclusions of 2 and 3 are incompatible and contradictory, so your statement is logically absurd.

Reductio ad Absurdum Mathematics: The Smallest Positive Rational

There is no smallest positive rational.

- Suppose there exists one such rational r
- In the second second
- **3** r/2 < r
- Contradiction

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- **3** r/2 < r
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- Suppose there exists one such rational r
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- Assume $\sqrt{2}$ is rational: $\exists a, b$ integers st. $a/b = \sqrt{2}$
- a, b can be taken coprime
- $\bigcirc \therefore a^2/b^2 = 2 \text{ and } a^2 = 2b^2$
- $\therefore a^2$ is even $(a^2 = 2b^2)$
- I: a is even
- **O** Because *a* is even, $\exists k \text{ st. } a = 2k$
- We insert the last equation of (3) in (6): $2b^2 = (2k)^2$ is equivalent to $2b^2 = 4k^2$ is equivalent to $b^2 = 2k^2$.
- Since 2k² is even, b² is even, hence, b is even
- By (5) and (8) a, b are even
- 💿 Contradicts 🛛

- Assume $\sqrt{2}$ is rational: $\exists a, b$ integers st. $a/b = \sqrt{2}$
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 $\sqrt{2}$ is irrational.

• Assume $\sqrt{2}$ is rational: $\exists a, b$ integers st. $a/b = \sqrt{2}$ 2 a, b can be taken coprime **3** ∴ $a^2/b^2 = 2$ and $a^2 = 2b^2$ \bigcirc : a^2 is even $(a^2 = 2b^2)$

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- **6** Because *a* is even, $\exists k \text{ st. } a = 2k$.
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Contradicts

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- **1** √2 is known to be irrational
 2 Consider √2^{√2}:

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- 2 Consider $\sqrt{2}^{\sqrt{2}}$:

1 √2 is known to be irrational
 2 Consider √2^{√2}:

• If it is rational, take $a = b = \sqrt{2}$

2 Otherwise, take $a = \sqrt{2}^{\sqrt{2}}, b = \sqrt{2}, a^b = 2$

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But it is not known which numbers. We proved $A \vee B$, but neither A nor B. Let σ be the number defined below. Its value is unknown, but it is rational.

For each decimal digit of $\pi,$ write 3. Stop if the sequence 0123456789 is found.

• If 0123456789 occurs in π , then $\sigma = 0, 3 \dots 3 = \frac{10^{k} - 1}{3 \dots 10^{k}}$

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We proved $\exists x.P(x)$, but know no t: P(t).

Disjunction Property

If $A \lor B$ is provable, then either A or B is provable, and reading the proof tells which one.

Existence Property

If $\exists x \cdot A(x)$ is provable, then reading the proof allows to exhibit a witness t (i.e., such that A(t)).

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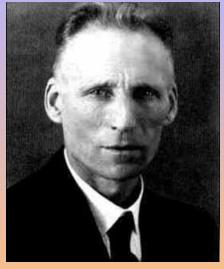
Constructivity

2 Intuitionistic Logic

- NJ: Intuitionistic Natural Deduction
- LJ: Intuitionistic Sequent Calculus

(a)

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Luitzen Egbertus Jan Brouwer (1881–1966)

A. Demaille

Intuitionistic Logic

A (1) > A (2) > A

• Classical logic focuses on truth (hence truth values)

- Intuitionistic logic focuses on provability (hence proofs)
- A is true if it is provable
- The excluded middle is... excluded

$$\begin{array}{c}
A \vdash A \\
\vdash \neg A, A \\
\vdash \neg A, A \\
\vdash A \lor \neg A, A \\
\vdash A \lor \neg A, A \lor \neg A \\
\vdash A \lor \neg A \\
\vdash A \lor \neg A
\end{array}$$

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\vdash A \lor \neg A, A \lor \neg A \\
\vdash A \lor \neg A \\
\vdash A \lor \neg A \\
\vdash C \\
\hline
\end{array}$$

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$$\frac{A \vdash A}{\vdash \neg A, A} \vdash \neg$$

$$\frac{-A \lor \neg A, A}{\vdash A \lor \neg A, A} \vdash r \lor$$

$$\frac{+A \lor \neg A, A \lor \neg A}{\vdash A \lor \neg A} \vdash C$$

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\frac{+ \neg A, A}{\vdash A \lor \neg A, A} \vdash r \lor \\
\frac{- A \lor \neg A, A \lor \neg A}{\vdash A \lor \neg A} \vdash C$$

1 Constructivity

Intuitionistic Logic
 NJ: Intuitionistic Natural Deduction
 LJ: Intuitionistic Sequent Calculus

(a)

Intuitionistic Natural Deduction

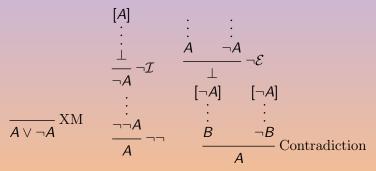
- Natural deduction supports very well intuistionistic logic.
- In fact, classical logic does not fit well in natural deduction.



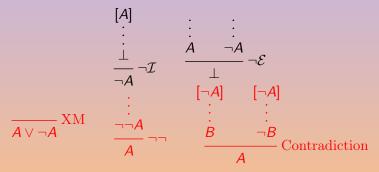
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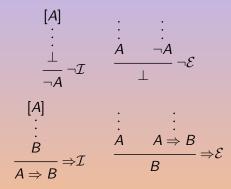
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$$\begin{bmatrix} A \\ \vdots \\ \vdots \\ \neg A \neg \mathcal{I} \end{bmatrix} \xrightarrow{\begin{array}{c} \vdots \\ A \\ \neg A \\ - \end{matrix}} \begin{array}{c} \vdots \\ A \\ \neg A \\ - \end{matrix} \xrightarrow{\begin{array}{c} A \\ \bot \\ - \end{matrix}} \neg \mathcal{E}$$

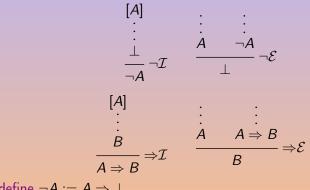
Intuitionistic Negation



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Intuitionistic Negation



So define $\neg A := A \Rightarrow \bot$.

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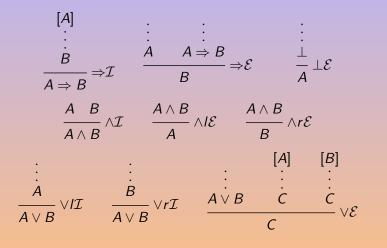
Prove $A \vdash \neg \neg A$

$$\frac{A \quad [A \Rightarrow \bot]^{1}}{\bot} \Rightarrow \mathcal{E}$$
$$\frac{1}{(A \Rightarrow \bot) \Rightarrow \bot} \Rightarrow \mathcal{I}_{1}$$

Prove $\neg\neg\neg A \vdash \neg A$

$$\frac{[A]^2 \quad [A \Rightarrow \bot]^1}{(A \Rightarrow \bot) \Rightarrow \bot} \Rightarrow \mathcal{E} \\
\frac{(A \Rightarrow \bot) \Rightarrow \bot}{(A \Rightarrow \bot) \Rightarrow \bot} \qquad ((A \Rightarrow \bot) \Rightarrow \bot) \Rightarrow \bot \\
\frac{\bot}{A \Rightarrow \bot} \Rightarrow \mathcal{I}_2$$

Intuitionistic Natural Deduction



1 Constructivity

Intuitionistic Logic
 NJ: Intuitionistic Natural Deduction

• LJ: Intuitionistic Sequent Calculus

LJ — Intuitionistic Sequent Calculus

LJ — Gentzen 1934

Logistischer intuitionistischer Kalkül

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Intuitionistic Logic

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LK: Identity Group

$$\frac{1}{A \vdash A} \operatorname{Id} \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \operatorname{Cut}$$

LK: Identity Group

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Intuitionistic Logic

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$$\frac{1}{A \vdash A} \operatorname{Id} \qquad \frac{\Gamma \vdash A \quad \Gamma', A \vdash B}{\Gamma, \Gamma' \vdash B} \operatorname{Cut}$$

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LK: Structural Group

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \tau(\Delta)} \vdash X \qquad \frac{\Gamma \vdash \Delta}{\sigma(\Gamma) \vdash \Delta} X \vdash$$
$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \vdash W \qquad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} W \vdash$$
$$\frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \vdash C \qquad \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} C \vdash$$

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LK: Structural Group



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$$\frac{\Gamma \vdash B}{\sigma(\Gamma) \vdash B} X \vdash$$
$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B} W \vdash$$
$$\frac{\Gamma, A \vdash B}{\Gamma, A \vdash B} C \vdash$$

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$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \vdash \neg \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \neg \vdash$$

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LK: Logical Group: Negation

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \vdash \neg \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \neg \vdash$$

Intuitionistic Logic

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LJ: Logical Group: Negation

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \land B, \Delta} \vdash \land \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \land B \vdash \Delta} I \land \vdash \\ \frac{\Gamma, B \vdash \Delta}{\Gamma, A \land B \vdash \Delta} r \land \vdash$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \land B, \Delta} \vdash \land \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \land B \vdash \Delta} I \land \vdash \Delta$$

$$\frac{\Gamma, B \vdash \Delta}{\Gamma, A \land B \vdash \Delta} r \land \vdash \Delta$$

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Intuitionistic Logic

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$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \land B} \vdash \land \qquad \frac{\Gamma, A \vdash C}{\Gamma, A \land B \vdash C} \land \vdash \\ \frac{\Gamma, B \vdash C}{\Gamma, A \land B \vdash C} \land \vdash$$

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LK: Logical Group: Disjunction

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Intuitionistic Logic

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$$\frac{\Gamma \vdash A}{\Gamma \vdash A \lor B} \vdash I \lor \qquad \qquad \frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \lor B \vdash C} \lor \vdash P \lor$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \Rightarrow B \vdash \Delta, \Delta'} \Rightarrow \vdash \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta} \vdash \Rightarrow$$

E.

LK: Logical Group: Implication

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \Rightarrow B \vdash \Delta, \Delta'} \Rightarrow \vdash \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta} \vdash \Rightarrow$$

Intuitionistic Logic

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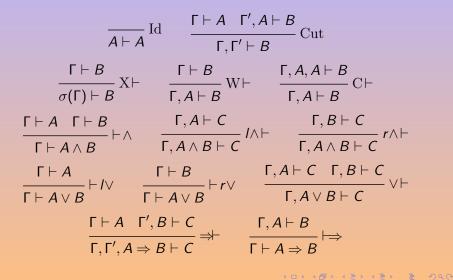
E

$$\frac{\Gamma \vdash A \qquad \Gamma', B \vdash B'}{\Gamma, \Gamma', A \Rightarrow B \vdash B'} \Rightarrow \vdash \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \vdash \Rightarrow$$

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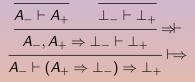
LJ



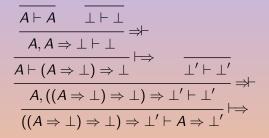
Prove $A \vdash \neg \neg A$

$$\frac{\overrightarrow{A \vdash A} \qquad \overrightarrow{\bot \vdash \bot}}{A, A \Rightarrow \bot \vdash \bot} \Rightarrow \vdash$$

$$\overrightarrow{A \vdash (A \Rightarrow \bot) \Rightarrow \bot} \vdash \Rightarrow$$



Prove $\neg\neg\neg A \vdash \neg A$



Intuitionistic Logic

Therefore, in intuistionistic logic $\neg \neg \neg A \equiv \neg A$, but $\neg \neg A \neq A$.

[van Atten, 2009] The history of intuitionistic logic.

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van Atten, M. (2009).

Stanford Encyclopedia of Philosophy, chapter The Development of Intuitionistic Logic.

The Metaphysics Research Lab.

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