Analog-to-Digital Conversion

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LRDE, EPITA

Being discrete but looking continuous...

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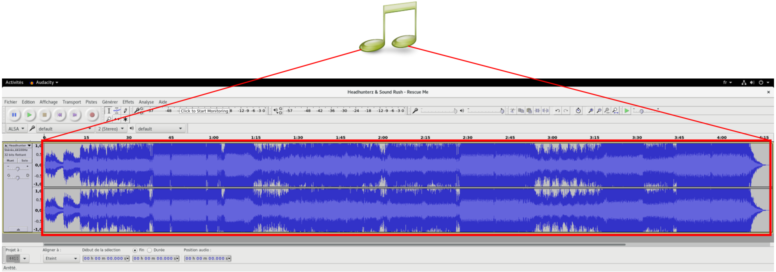

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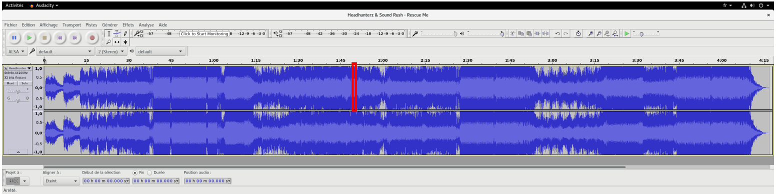

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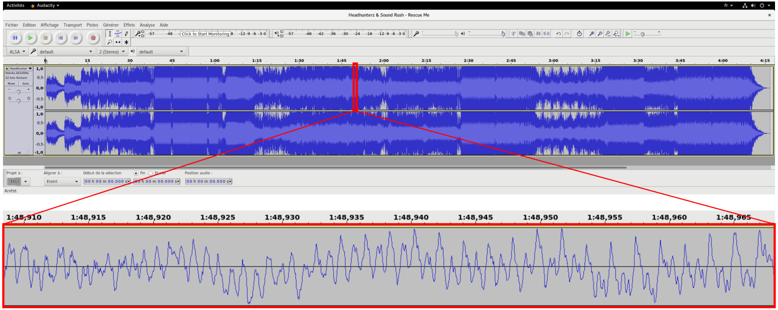

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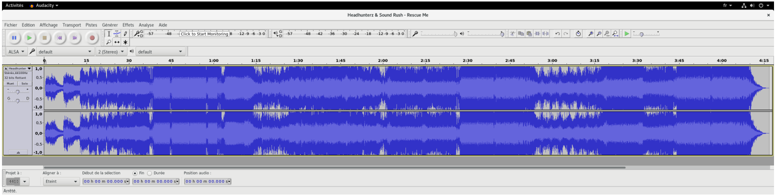

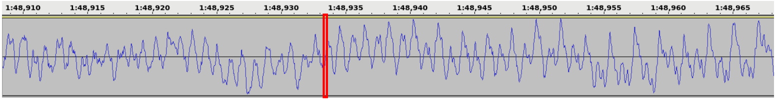

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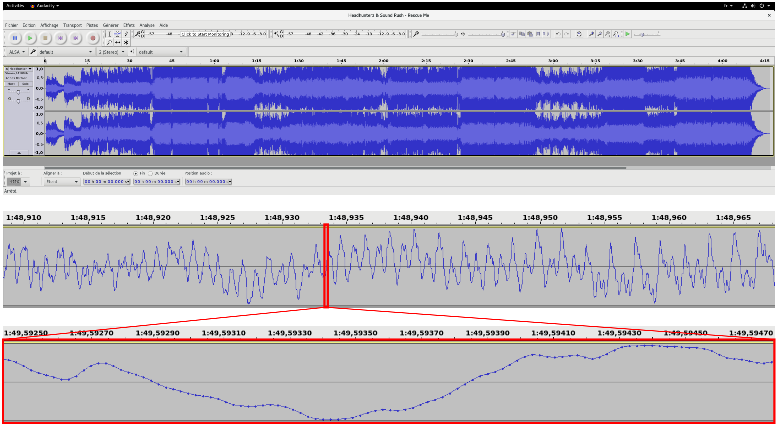
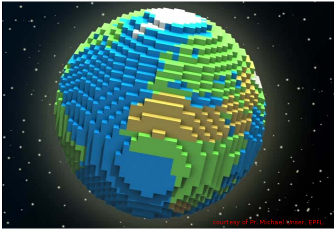


Sampling the real world

Physical phenomena are continous by nature (light, sound, pressure, temperature, current, voltage, etc) and must somehow be discretized in order to be digitally handled and stored on computers.

$\longrightarrow$



From the real world $f_a(x, y, z, t)$ ... to the digital one $f_d(i, j, k, n)$

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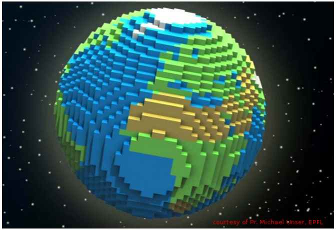

From the real world . . . to the digital one

$f_a(x, y, z, t)$ $f_d(i, j, k, n)$

Can this be done without loosing any information (or as few as possible)?
And if yes, how?

Recorded physical signals are continuous both with respect to their variable(s) (time and/or position) and the values they may take $\Rightarrow f_a : \mathbb{R} \rightarrow \mathbb{R}$.

But they must be converted into discrete-time and discrete-amplitude digital signals in order to be stored and manipulated on computers $\Rightarrow f_d : \mathbb{Z} \rightarrow \mathbb{F}$

continuous world	discrete world
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From the continuous world to the discrete one

The challenge of analog-to-digital conversion

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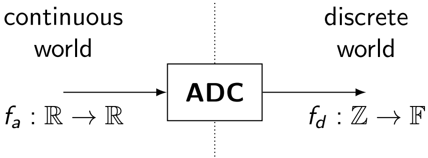

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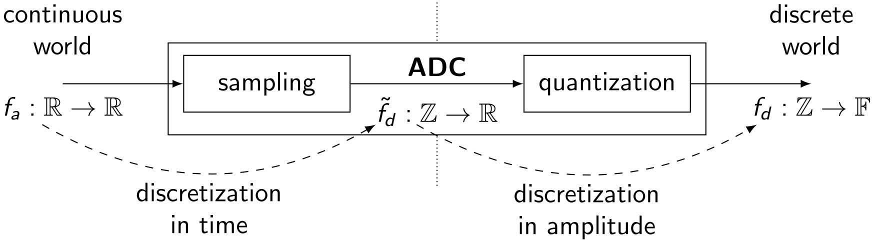
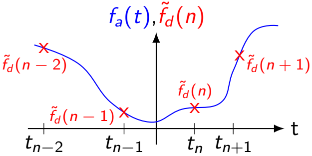


The problematic of sampling

The goal of sampling is to pick some values of the continuous signal f_a at particular sampling points $(t_n)_{n\in\mathbb{Z}}$ in order to create the sampled sequence $(\tilde{f}_d(n) = f_a(t_n))_{n\in\mathbb{Z}}$.



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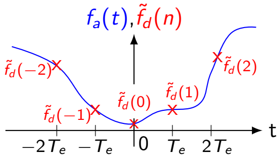

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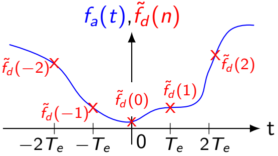

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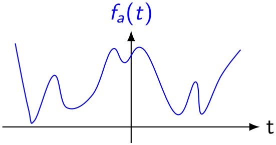

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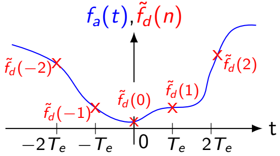

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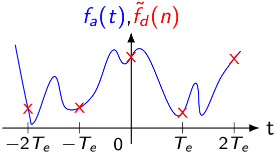

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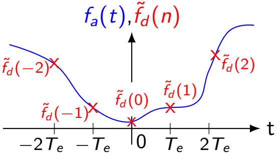

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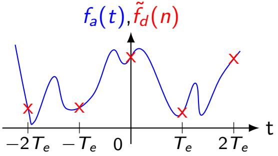

T_e too large $\rightarrow$ undersampling

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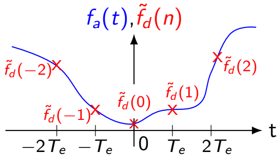

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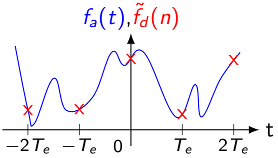

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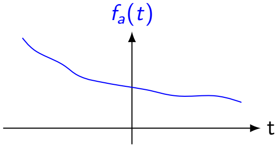

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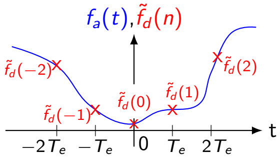

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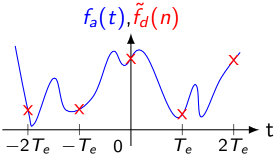

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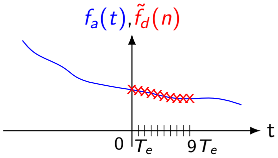

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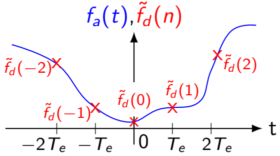

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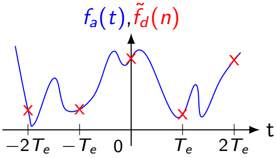

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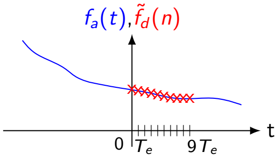

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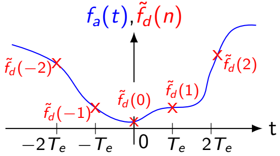

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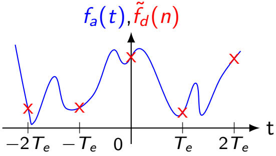
$\Rightarrow$ $\boxed{\text{What is the best frequency to sample a signal?}}$



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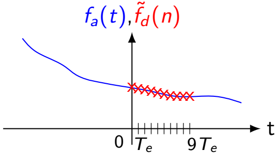

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The temporal representation is not convenient for that $\Rightarrow$ we need a new one!

How fast is a signal varying?

Need for a new representation...

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Ex: Consider the simple signal $x(t) = A_0 \cos(2\pi f_0 t)$

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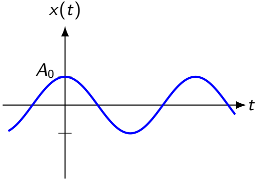

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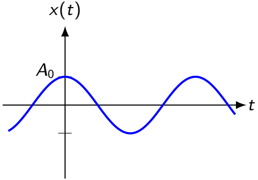
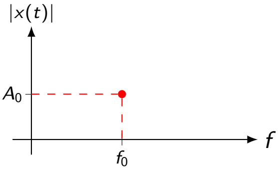


$\rightarrow$ frequency representation



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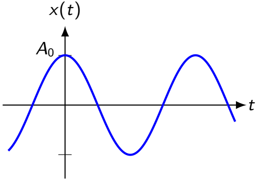

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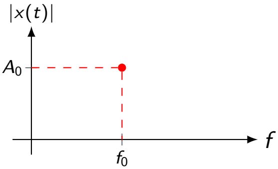

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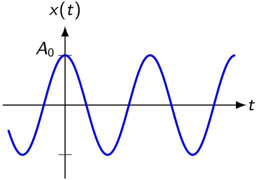

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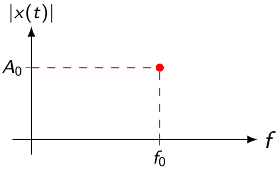

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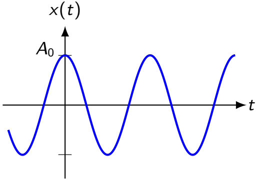

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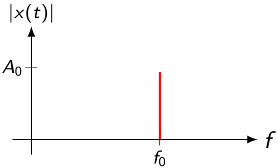

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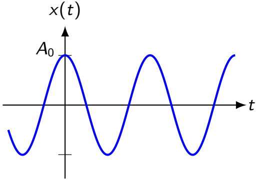

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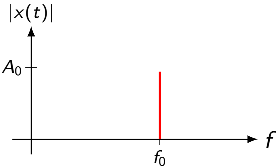

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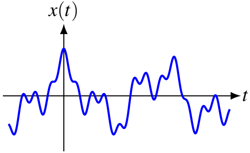

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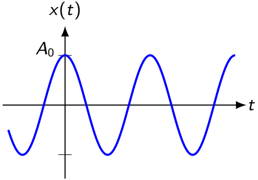

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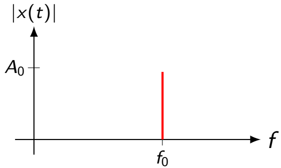

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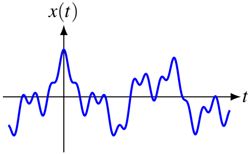

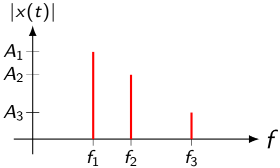

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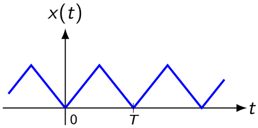
$\rightarrow$ Periodic signals



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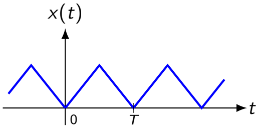

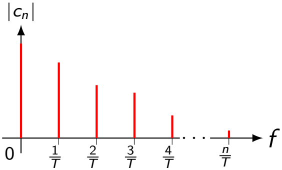

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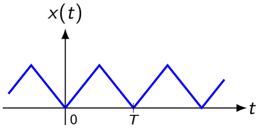
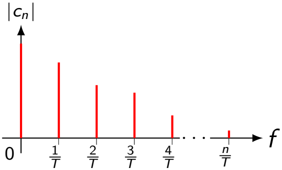


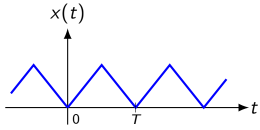
Fourier series decomposition

$$c_n = \frac{1}{T}\int_0^T x(t)e^{-2i\pi\frac{n}{T}t}dt$$



The frequency representation of sine/cosine waves (and any linear mixture of them) is pretty straightforward...
What about more general signals?

$\rightarrow$ Periodic signals



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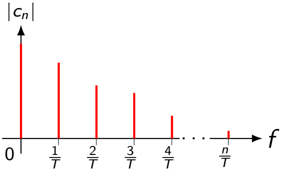
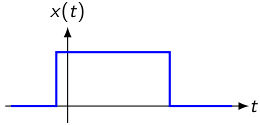


$\rightarrow$ All other (non-periodic) signals



How fast is a signal varying?

What about more complicated signals?

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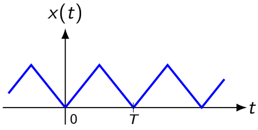

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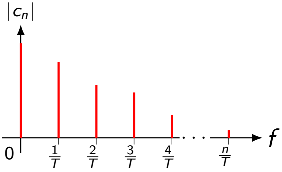

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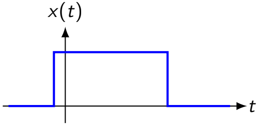

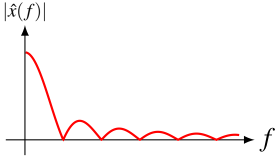

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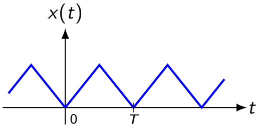

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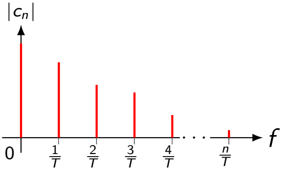

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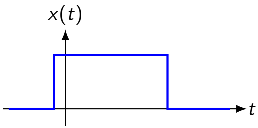
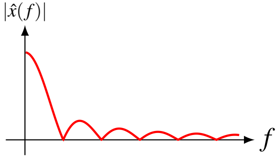


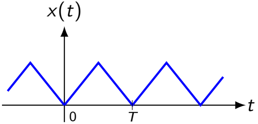
Fourier transform

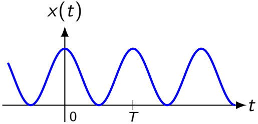
$$\hat{x}(f) = \int_{-\infty}^{+\infty} x(t)e^{-2i\pi ft}dt$$

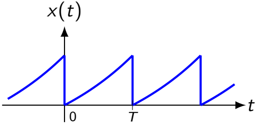


Superposition principle

Why does Fourier series decomposition only apply to periodic signals?







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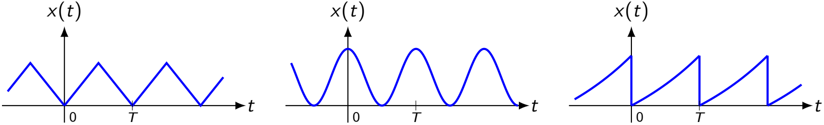

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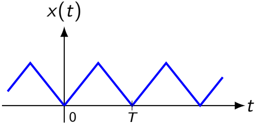

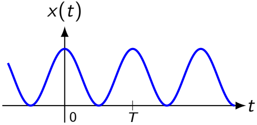

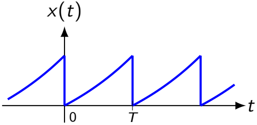

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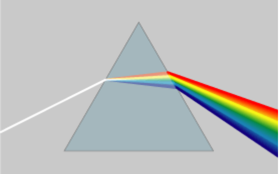
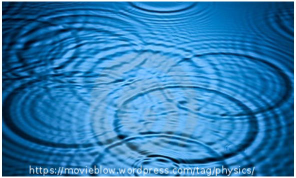
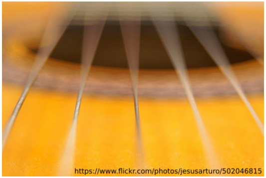

Light decomposition



Water ripples and interferences



Guitar strings vibrating

Superposition principle

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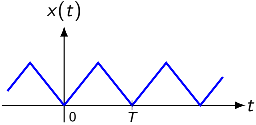

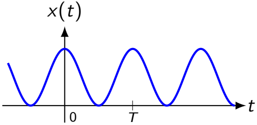

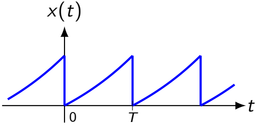

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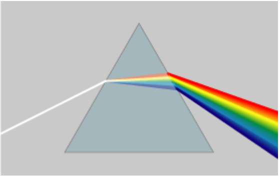

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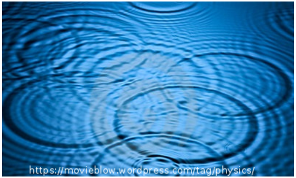

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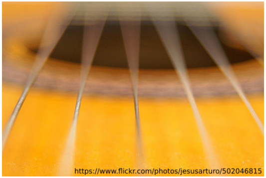

Guitar strings vibrating

$\Rightarrow$ This is the base idea of Fourier series decomposition, namely to express some potential complicated periodic function as a sum of much simpler cosine and sine waves.

Fourier series decomposition

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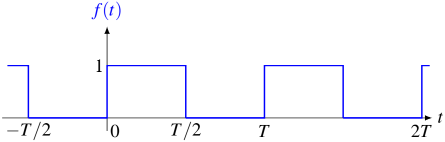

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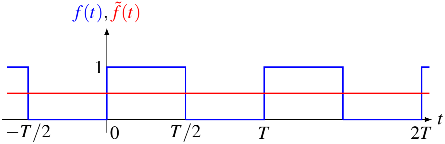

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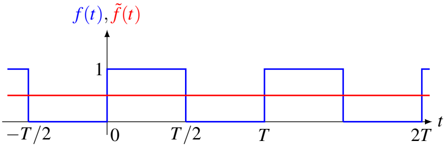

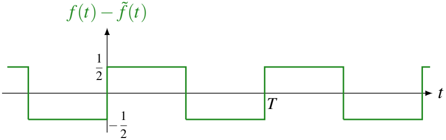

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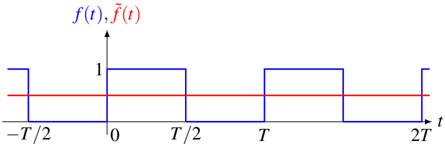

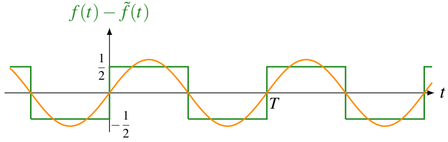

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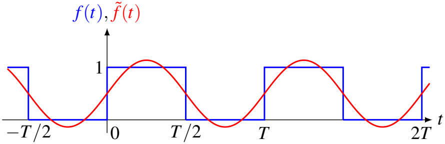

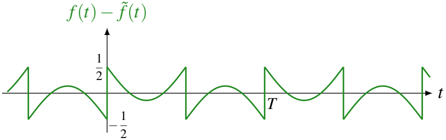

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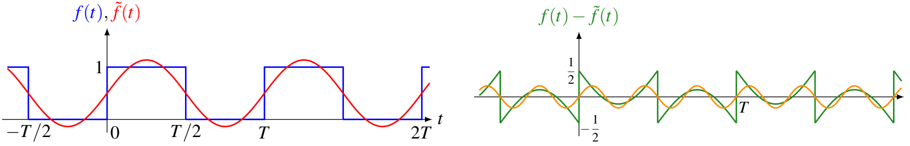

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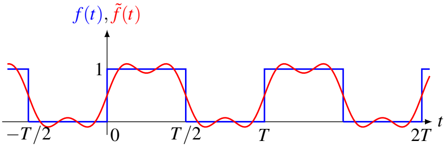

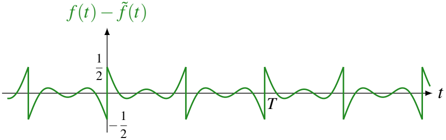

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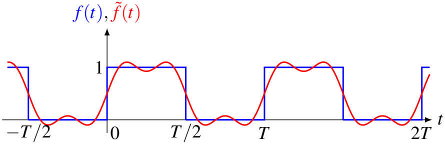

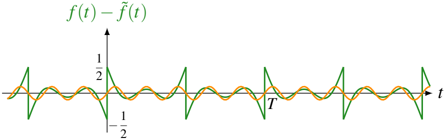

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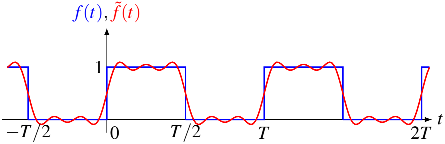

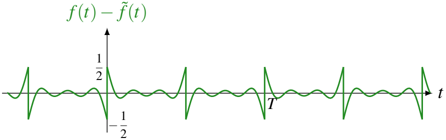

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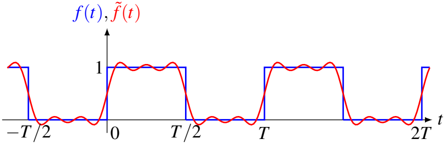

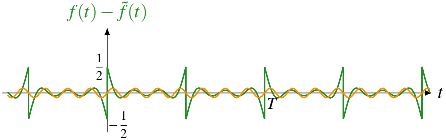

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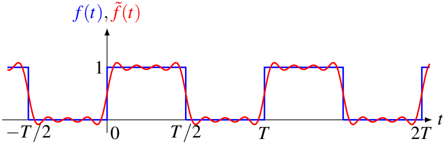

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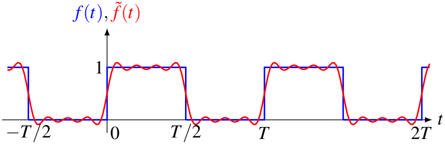

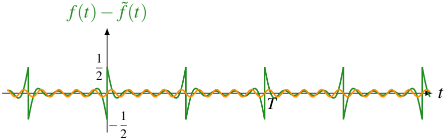

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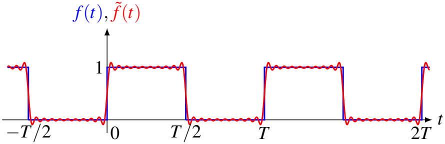

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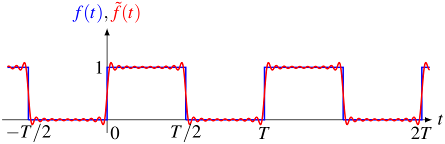

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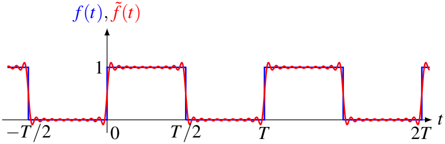

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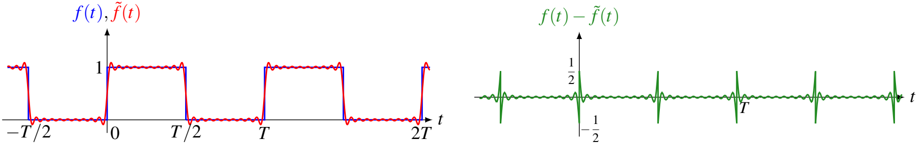

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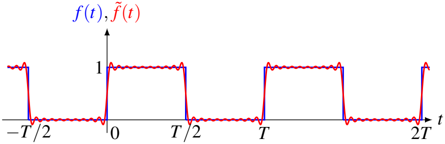

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$\Rightarrow$ *dot product* between the original function $f(t)$ and $\sin(2\pi \frac{n}{T} t)$.

The dot product

The *dot product* $\langle x, y\rangle$ between two elements $(x, y) \in E \times E$ of some space E measures the *similarity* between those two elements.

$\rightarrow$ $E \equiv \mathbb{R}^2$

Let $x = \binom{x_1}{x_2}$ and $y = \binom{y_1}{y_2}$ be two vectors in $\mathbb{R}^2$

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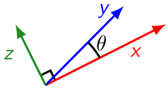
$$= \|x\|\|y\| \cos\theta$$



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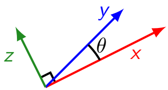

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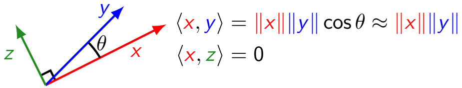

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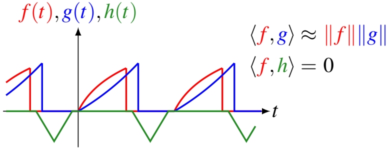
$\rightarrow$ $E \equiv \mathcal{L}^2([0,T])$ (square integrable functions over $[0,T]$).

Let f and g be two functions in $\mathcal{L}^2([0,T])$.

$$\Rightarrow \langle f, g\rangle = \frac{1}{T}\int_0^T f(t)\overline{g(t)}dt$$



Geometric illustration of Fourier series decomposition (1/2)

Decomposition of a vector over a basis

If (u_1, u_2) is the canonical basis of $\mathbb{R}^2$, then any vector $x \in \mathbb{R}^2$ can be decomposed as :
$x = x_1 u_1 + x_2 u_2$

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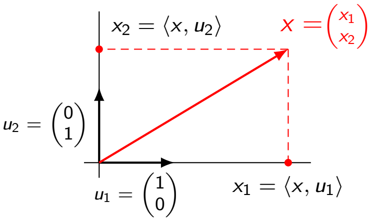

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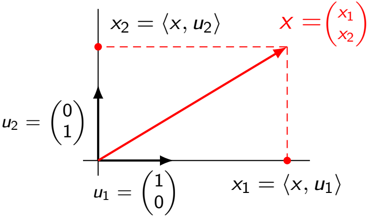

$\Rightarrow$ This can be generalized straightforwardly to $\mathbb{R}^n$:
If $(u_1, \dots, u_n)$ is an orthonormal basis of $\mathbb{R}^n$, then any $x \in \mathbb{R}^n$ can be expressed as

$$x = \sum_{i=1}^{n} \langle x, u_i \rangle u_i = \sum_{i=1}^{n} x_i u_i \quad \Leftrightarrow \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Geometric illustration of Fourier series decomposition (2/2)

The same decomposition goes for functions

$\mathcal{L}^2([0,T])$ is a space of functions (*i.e.*, of infinite dimension). But this decomposition remains valid, provided that we have an infinite orthonormal basis of $\mathcal{L}^2([0,T])$.

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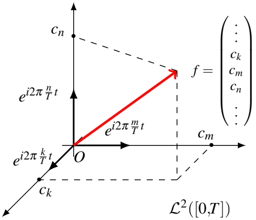
$$\boxed{f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{i2\pi\frac{n}{T}t}} \equiv \text{Fourier series decomposition of } f.$$



$c_n \equiv$ coordinate of f with respect to the basis function $e^{i2\pi\frac{n}{T}t}$.

Fourier series decomposition

Dirichlet theorem

Using the fact that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, one can rewrite

$$\boxed{f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{i2\pi\frac{n}{T}t} = a_0 + \sum_{n=1}^{+\infty} a_n \cos(2\pi\frac{n}{T}t) + b_n \sin(2\pi\frac{n}{T}t)}$$

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The Fourier series decomposition holds for any piecewise $\mathcal{C}^1$, $T-$periodic function f that is continuous on t.
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Fourier coefficients are given by:

- $\forall n \in \mathbb{Z},\ c_n = \frac{1}{T}\int_0^T f(t)e^{-i2\pi\frac{n}{T}t}dt$
- $\forall n \geq 1,\ a_n = \frac{2}{T}\int_0^T f(t)\cos(2\pi\frac{n}{T}t)dt$ and $a_0 = \frac{1}{T}\int_0^T f(t)dt$ (mean value of f).
- $\forall n \geq 1,\ b_n = \frac{2}{T}\int_0^T f(t)\sin(2\pi\frac{n}{T}t)dt$

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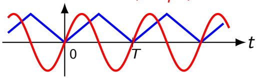

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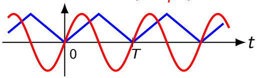

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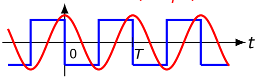

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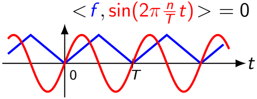

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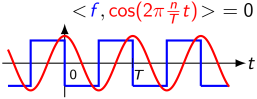

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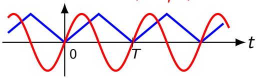

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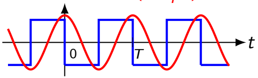

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$\rightarrow$ Parseval equality: $\displaystyle \frac{1}{T}\int_0^T |f(t)|^2 dt = \sum_{n=-\infty}^{+\infty} |c_n|^2 = a_0^2 + \frac{1}{2}\sum_{n=1}^{+\infty}(a_n^2 + b_n^2)$

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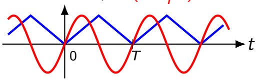

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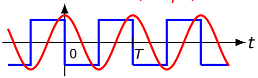

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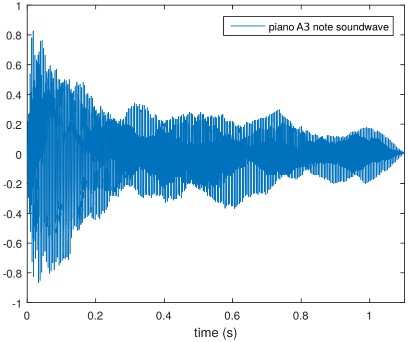
Example

Harmonic analysis of a signal

Let's consider a recording of a piano A3 note (frequency 220 Hz):



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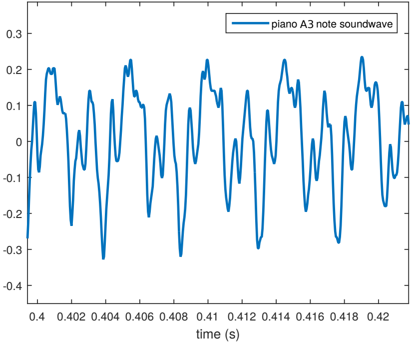

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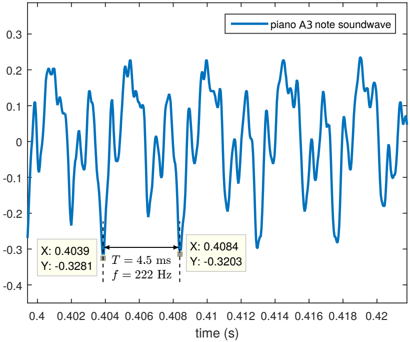

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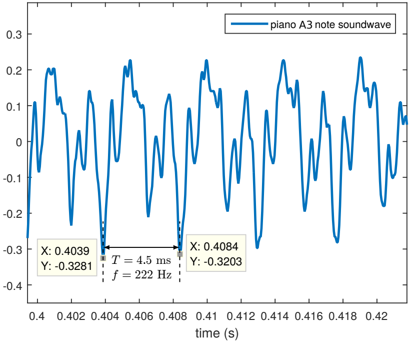

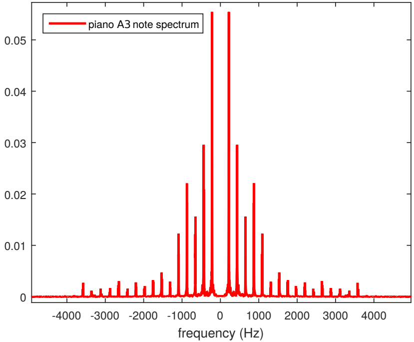

$\rightarrow$ The obtained spectrum is symmetric (since $c_n = \overline{c_{-n}}$).

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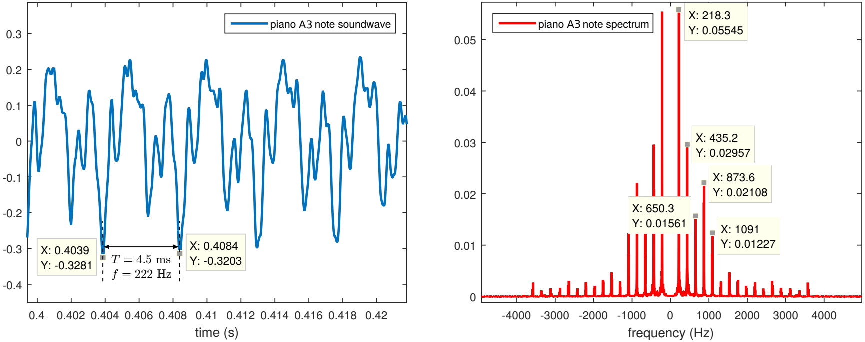
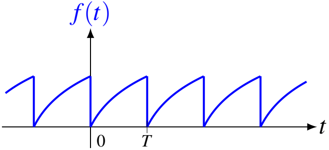
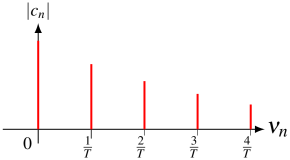


→ The obtained spectrum is symmetric (since $c_n = \overline{c_{-n}}$).
→ The fundamental frequency is $f = 218.3$ Hz.
→ The harmonics $2f$, $3f$, $4f$ and $5f$ have relatively large magnitudes.

The Fourier transform extends the Fourier series decomposition to non-periodic functions: Intuitively, the coefficient c_n is associated with frequency $\frac{n}{T} \Rightarrow$ the "gap" between two successive coefficients is $\Delta c = c_{n+1} - c_n = \frac{1}{T}$.





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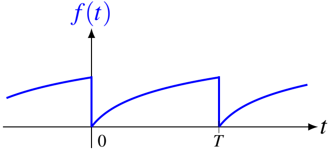

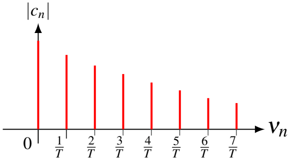

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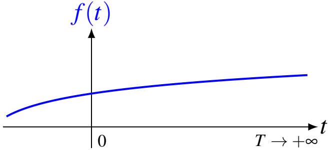
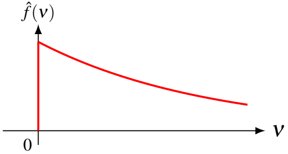
Thus, when $T \to +\infty$, the $T-$periodic function f "becomes" non-periodic, and $\Delta c \to 0$ $\Rightarrow$ a non periodic function f has a continuous spectrum.





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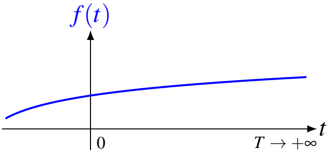
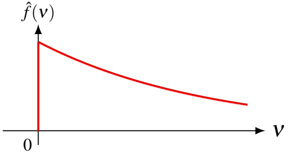
The Fourier transform of some non-periodic (and integrable) function f is defined as the complex-valued function $\hat{f} : \mathbb{R} \to \mathbb{C}$, $\boxed{\nu \mapsto \hat{f}(\nu) = \int_{-\infty}^{+\infty} f(t)e^{-i2\pi\nu t}dt}$





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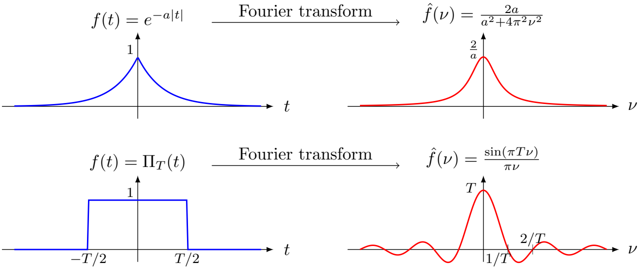

Some useful stuff about the Fourier transform

→ If f is real and even, then $\hat{f}$ is real and even.

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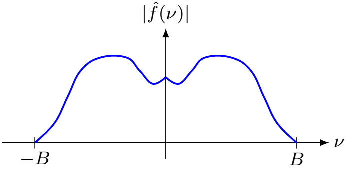
→ f is said to be bandlimited if $\exists B > 0$ s.t $|\hat{f}(\nu)| = 0 \ \forall |\nu| > B$.



Bernstein theorem:

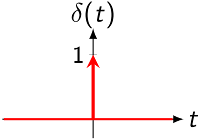
$$|f'(t)| \leq 2\pi B \int_{-B}^{B} |\hat{f}(\nu)| d\nu$$

Mathematical model of sampling (1/2)

Say hi! to Dirac

Dirac delta function is defined by $\delta : t \mapsto \begin{cases} +\infty & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$

and $\displaystyle\int_{-\infty}^{+\infty} \delta(t)dt = 1$



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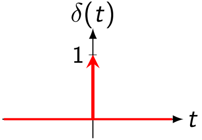

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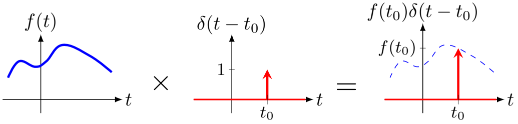

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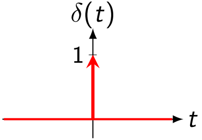

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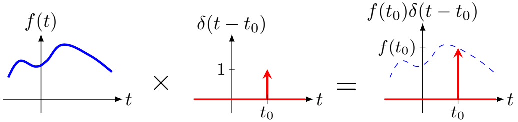

$\rightarrow (f * \delta)(t) = (\delta * f)(t) = f(t) \rightarrow \delta$ is the identity element for the convolution product.

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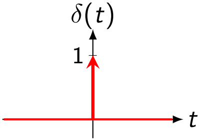

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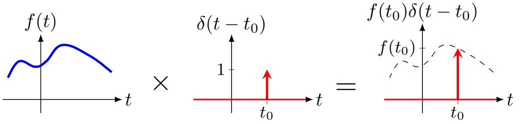

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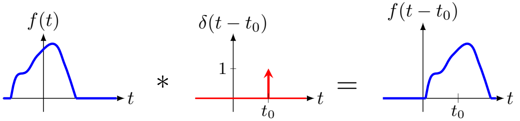

Dirac delta can be extended to the Dirac comb, also called sampling function with period T_e:

$$\text{Ш}_{T_e}(t) = \dots + \delta(t+T_e) + \delta(t) + \delta(t-T_e) + \dots$$
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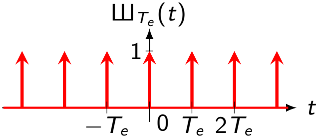

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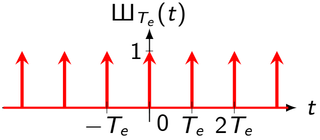

It allows to easily model the sampling operation:

$$\tilde{f}_d(n) = f_a(t) \times \text{Ш}_{T_e}(t)$$
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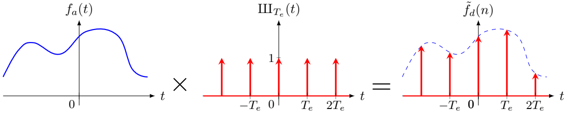

Mathematical model of sampling

Sampling with the Dirac comb

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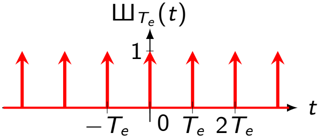

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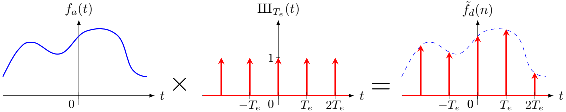

Besides, the Dirac comb maps to itself through the Fourier transform, and this property is the key to prove Shannon sampling theorem:

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\hat{Ш}_{T_e}(\nu) = \frac{1}{T_e} Ш_{\frac{1}{T_e}}(\nu) = \frac{1}{T_e} \sum_{n=-\infty}^{+\infty} \delta\left(\nu - \frac{n}{T_e}\right)
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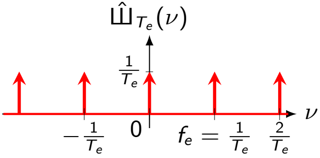

Shannon sampling theorem (1/2)

One can now express the spectrum of the sampled signal $\tilde{f}_d(n)$ using elementary properties of the Fourier transform and the Dirac comb:

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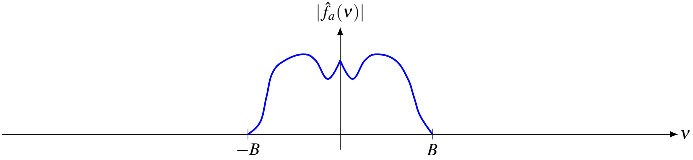

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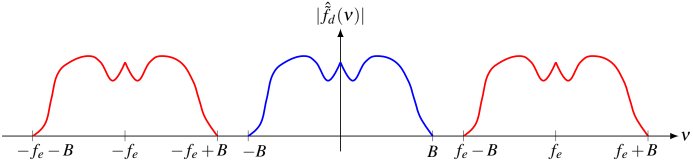

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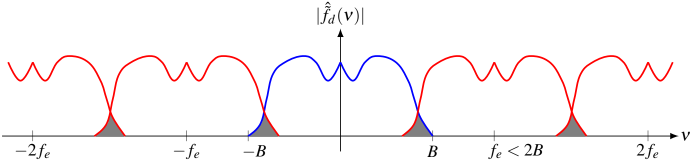

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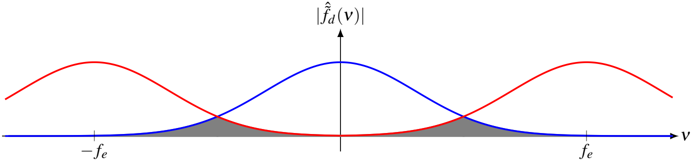
If f_a is not bandlimited, aliasing inevitably occurs.



Shannon sampling theorem (2/2)

Shannon sampling theorem

Also called *Nyquist-Shannon theorem*, *Whittaker–Shannon–Kotelnikov theorem*, *Whittaker–Nyquist–Kotelnikov–Shannon theorem* and *cardinal theorem of interpolation*...

It is possible to exactly recover a bandlimited signal f_a with frequency range $[-f_{\max}, f_{\max}]$ from its sampled sequence $\tilde{f}_d$ if the sampling rate f_e satisfies

$$\boxed{f_e \geq 2f_{\max}} \quad \text{(Nyquist condition)}$$

Shannon sampling theorem (2/2)

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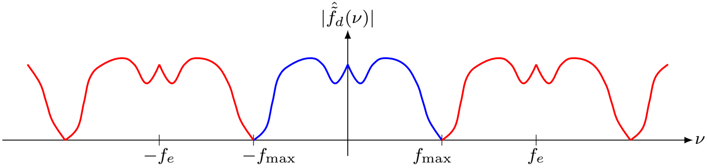
The reconstruction of f_a from $\tilde{f}_d$ derives from the application of a low-pass filter on $\hat{\tilde{f}}_d(\nu)$ with cutoff frequency $f_{\max}$.



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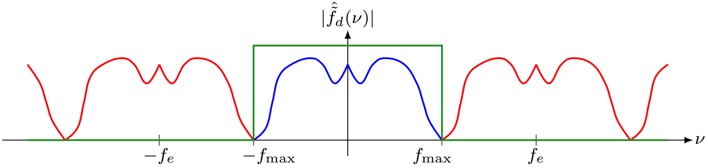

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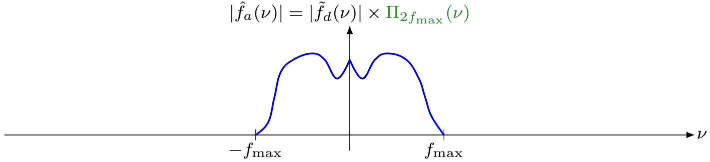
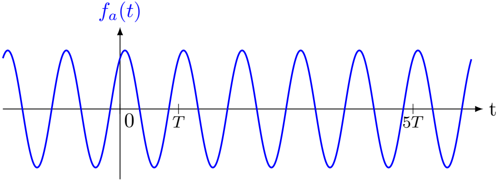


$\Rightarrow$ It yields Shannon interpolation formula: $f_a(t) = \displaystyle\sum_{n=-\infty}^{+\infty} \tilde{f}_d(n) \mathrm{sinc}\Big(\frac{\pi}{T_e}(t - nT_e)\Big)$

The aliasing effect

What if $f_e < 2f_{\max}$?

If Nyquist condition is not fulfilled, the overlap occuring in $|\hat{\tilde{f}}_d(\nu)|$ between the original spectrum $|\hat{f}_a(\nu)|$ and its replicates generates aliasing.
$\rightarrow$ the signal that is reconstructed from $(\tilde{f}_d(n))_{n\in\mathbb{Z}}$ is not f_a, but is such that its sampling at frequency f_e yields $(f_d(n))_{n\in\mathbb{Z}}$.



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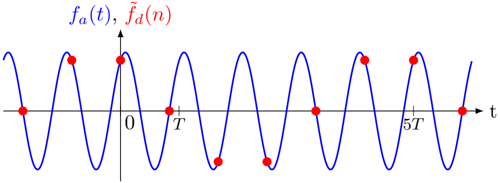

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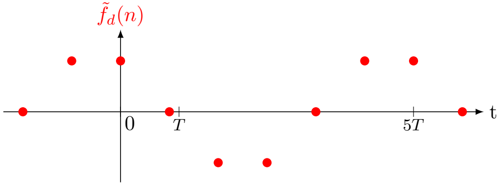

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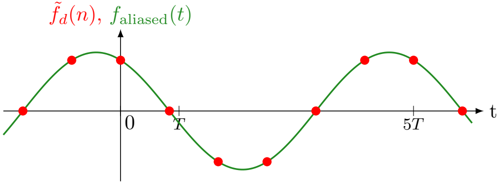

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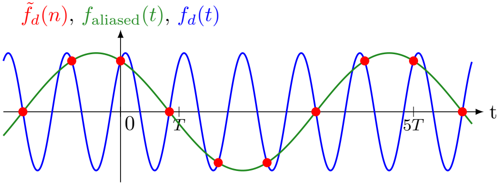

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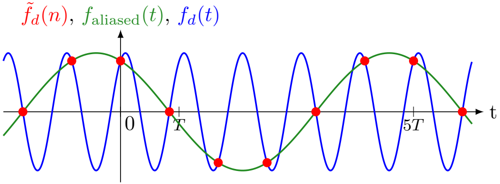
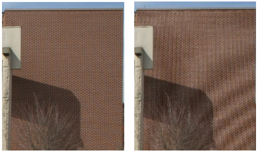


$\rightarrow$ Examples of aliasing in real life:

Moiré pattern:



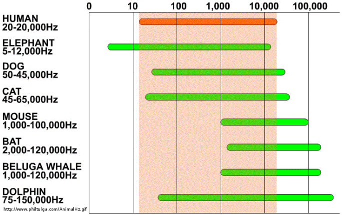
Stroboscopic effect:

Applications of Shannon sampling theorem

Humans can hear frequencies up to 20 kHz:

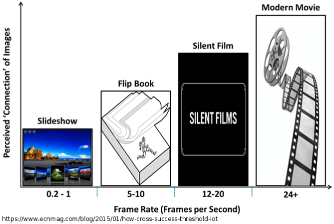
→ Sounds must be sampled at least at 40 kHz.

→ Sampling at 44.1 kHz in practice to account for an anti-aliasing lowpass filter.



The humain visual system perceives individual images for rates up to 10 to 12 images per second:

→ Standard video frame rates are 24, 25 and 30 frames per second.



The problematic of quantization

After the sampling stage, the input signal has become discrete with respect to its variable(s), but it is still continous with respect to its amplitude.

The goal of the quantization step is to map the values of the input sample sequence $\tilde{f}_d(n)$ to a discrete and finite set $\mathbb{F}$ (called dictionary), to create the final discrete sequence $f_d(n) = \mathcal{Q}(\tilde{f}_d(n)) : \mathbb{Z} \rightarrow \mathbb{F}$.

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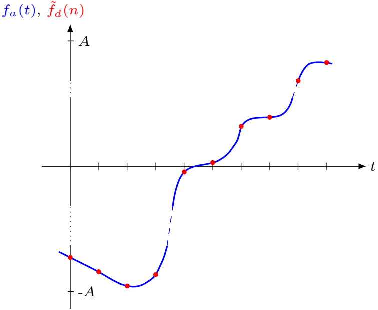

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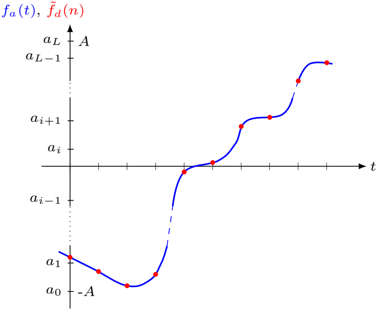

1. $[-A, A]$ is divided into L non overlapping intervals $[a_{i-1}, a_i] \to [-A, A] = \bigcup_{i=1}^{L} [a_{i-1}, a_i]$.

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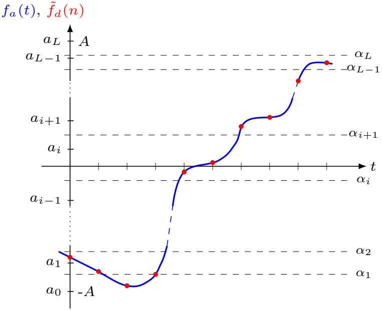

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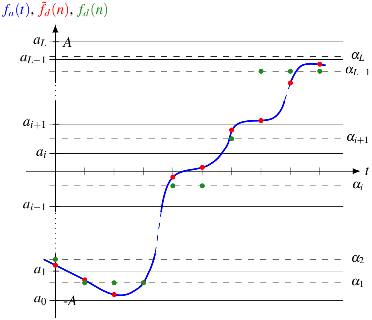

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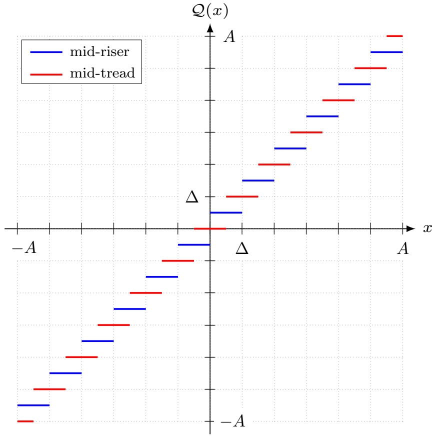
2. A quantization level α_i is chosen in each interval $[a_{i-1}, a_i]$.
3. Values $\tilde{f}_d(n)$ are rounded to the quantization level of the interval they fall in:

$$\boxed{\mathcal{Q}\left(\tilde{f}_d(n) \in [a_{i-1}, a_i]\right) = \alpha_i}.$$

$\Delta_i = a_i - a_{i-1} \rightarrow$ quantization step.

Uniform quantization

In general, the quantized value are encoded on b bits (*i.e.* $L = 2^b$) and the quantization is uniform: $\forall i, \Delta_i = \Delta = \frac{2A}{2^b} = \frac{A}{2^{b-1}}$.



Two main strategies:

- Mid-riser quantizer
  $\rightarrow \mathcal{Q}(x) = \Delta \left(\left\lfloor \frac{x}{\Delta} \right\rfloor + \frac{1}{2} \right)$
  - ✓ Is readily encodable on b bits.
  - ✗ $0, A$ and $-A$ are not quantized levels.
- Mid-tread quantizer
  $\rightarrow \mathcal{Q}(x) = \Delta \left\lfloor \frac{x}{\Delta} + \frac{1}{2} \right\rfloor$
  - ✓ $0, A$ and $-A$ are quantized levels.
  - ✗ Has an odd number of quantization levels.

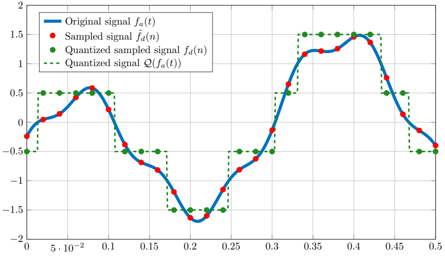
Quantization $\equiv$ rounding $\equiv$ irreversible operation $\equiv$ loss of information.

The induced distortion is called the quantization noise $\epsilon(n) = f_d(n) - \tilde{f}_d(n)$.

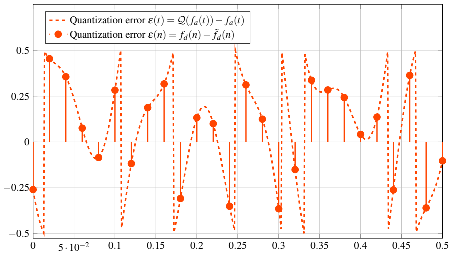
For a uniform quantization with step Δ, $|\epsilon(n)| \leq \frac{\Delta}{2} \Rightarrow \epsilon(n) \in [-\frac{\Delta}{2}, \frac{\Delta}{2}]$.

When Δ is small, $\epsilon(n)$ can be approximated by a uniform random variable in $[-\frac{\Delta}{2}, \frac{\Delta}{2}]$ with variance $\sigma_e^2 = \frac{\Delta^2}{12}$

$\Rightarrow \mathrm{SNR}_{\mathcal{Q}}(b) \propto 6.02b$ dB.



Quantization on 2 bits.



Quantization error ($\frac{\Delta}{2} = 0.5$).

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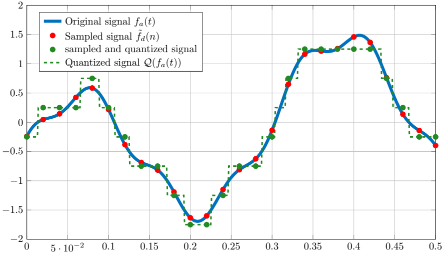
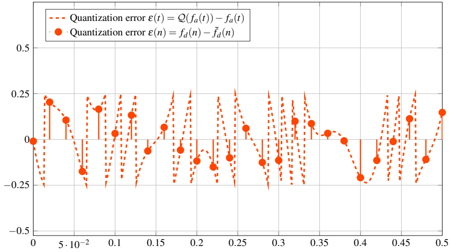


Quantization on 3 bits.



Quantization error ($\frac{\Delta}{2} = 0.25$).

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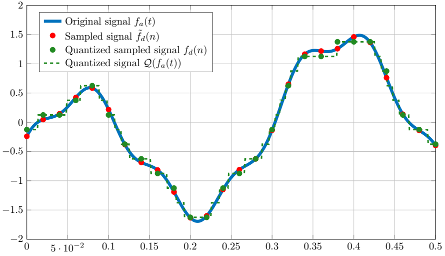
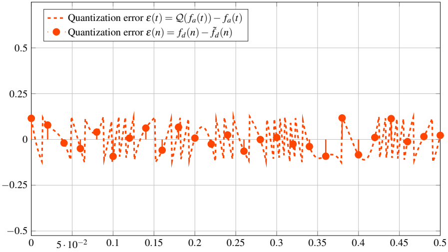


Quantization on 4 bits.



Quantization error ($\frac{\Delta}{2} = 0.125$).

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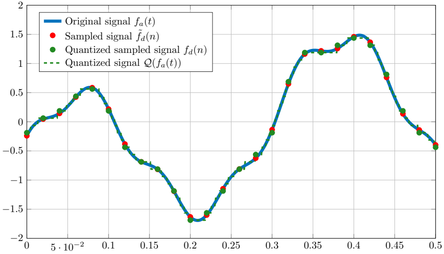
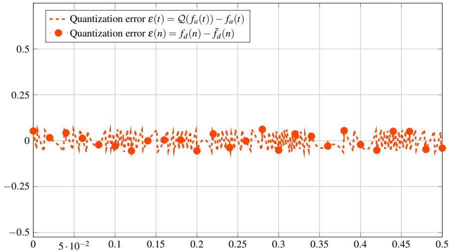


Quantization on 5 bits.



Quantization error ($\frac{\Delta}{2} = 0.0625$).