Programmatic Manipulation of Type Specifiers in Common Lisp

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Overview

1. Common Lisp Types
   - Native type specifiers
   - Type calculus with type specifiers

2. Reduced Ordered Binary Decision Diagrams (ROBDDs)
   - Representing CL types as ROBDDs
   - Reductions to accommodate CL subtypes
   - Type calculus using ROBDDs
   - Type checking and code generation with BDDs

3. Conclusion
   - Summary
   - Questions
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Types are sets. Subtypes are subsets. Intersecting types are intersecting sets. Disjoint types are disjoint sets.
Type specifiers are powerful and intuitive

Type specifiers can be extremely intuitive thanks to homoiconicity.

- **Simple**
  - integer

- Compound type specifiers
  - (satisfies oddp)
  - (and (or number string) (not (satisfies MY-FUN)))

- Specifiers for the empty type
  - nil
  - (and number string)
  - (and (satisfies evenp) (satisfies oddp))
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There are many type specifiers for the same type.
We can ask questions with CL type specifiers.

- **Type membership?** `(typep x T1)`

\[ x \in T_1 \]
We can ask questions with CL type specifiers.

- Type membership? (typep x T1)
- Type inclusion? (subtypep T1 T2)
We can ask questions with CL type specifiers.

- **Type membership?** \((\text{typep } x \ T_1)\)
- **Type inclusion?** \((\text{subtypep } T_1 \ T_2)\)
- **Type equivalence?** \((\text{and} \ (\text{subtypep } T_1 \ T_2) \ (\text{subtypep } T_2 \ T_1))\)

\[
(T_1 \subset T_2) \land (T_2 \subset T_1)
\]
We can ask questions with CL type specifiers.

- Type membership? \((\text{typep } x \ T_1)\)
- Type inclusion? \((\text{subtypep } T_1 \ T_2)\)
- Type equivalence? \((\text{and} (\text{subtypep } T_1 \ T_2) (\text{subtypep } T_2 \ T_1))\)
- Type disjointness? \((\text{subtypep} \ ('(\text{and} ,T_1 ,T_2) \ \text{nil})\)

\[ T_1 \cap T_2 \subseteq \emptyset \]
We can ask questions with CL type specifiers.

- Type membership? `(typep x T1)`
- Type inclusion? `(subtypep T1 T2)`
- Type equivalence? `(and (subtypep T1 T2) (subtypep T2 T1))`
- Type disjointness? `(subtypep '(and ,T1 ,T2) nil)`

Sometimes, `subtypep` returns *don’t know*. 
Type expressions can be barely human readable.

(setf T1 '(not (or (and fixnum unsigned-byte) 
                   (and number float) 
                   (and fixnum float))))

(setf T2 '(or (and fixnum 
               (not rational) 
               (or (and number (not float)) 
                   (not number))) 
           (and (not fixnum) 
                (or (and number (not float)) 
                    (not rational)))))
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(setf T1 '(not (or (and fixnum unsigned-byte) (and number float) (and fixnum float))))

(setf T2 '(or (and fixnum (not rational) (or (and number (not float)) (not number)))
               (and (not fixnum) (or (and number (not float)) (not rational)))))

The same type may be checked multiple times.
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The same type may be checked multiple times. We can do better.
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Type specifier viewed as a **Boolean expression of variables**

A CL type specifier has a dual in Boolean algebra notation.

Type specifier: \((\text{not } (\text{or } (\text{and } A \ C) \ (\text{and } B \ C) \ (\text{and } B \ D))))\)

Boolean Expression: \(\neg ((A \land C) \lor (B \land C) \lor (B \land D))\)
Type specifier viewed as a **Boolean expression of variables**

Forget about the CL type system for the moment, and just concentrate on Boolean algebra with binary variables.

**Boolean Expression:** \[ \neg ((A \land C) \lor (B \land C) \lor (B \land D)) \]
Type specifier viewed as a Boolean expression of variables

If we *order* the variables, then every Boolean expression has a unique truth table.

Boolean Expression: \( \neg ((A \land C) \lor (B \land C) \lor (B \land D)) \)
Type specifier viewed as a **Boolean expression of variables**

The truth table can be represented as an OBDD, ordered binary decision diagram. A **green arrow** a variable being true; a **red arrow** represents the variable being false.

**Boolean Expression:** \( \neg ((A \land C) \lor (B \land C) \lor (B \land D)) \)
Type specifier viewed as a **Boolean expression of variables**

Every path from root to leaf corresponds to one row of the truth table.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>( \neg ((A \land C) \lor (B \land C) \lor (B \land D)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>⊥</td>
<td>⊥</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>⊥</td>
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<th>¬((A ∧ C) ∨ (B ∧ C) ∨ (B ∧ D))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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</table>
Reduced Ordered Binary Decision Diagrams (ROBDDs)

Representing CL types as ROBDDs

Type specifier viewed as a Boolean expression of variables

4 variables $\Rightarrow 2^{4+1} - 1 = 31$ nodes

The graph size grows exponentially with number of variables.

We can do better.
Standard Rule 1: the *terminal* rule

There are 3 standard reduction rules. The terminal rule allows us to replace leaf nodes with *singleton objects*, NIL and T. Divides size by 2.
Standard Rule 2: the *deletion* rule

The deletion rule allows us to remove nodes which have the same red (false) and green (true) pointer.
Reducing to 11 nodes
More reduction

The deletion rule can be applied multiple times.
Standard Rule 3: the *merging* rule

The merging rule allows us to merge structurally congruent nodes, *i.e.*, with same children, and same label.
More congruent nodes

The merging rule can be applied multiple times.
Reduced Ordered Binary Decision Diagrams (ROBDDs)  

ROBDD: Reduced ordered binary decision diagram

Started with 31 nodes, we can represent the CL type specifier with only 8 nodes.

```
(not (or (and A C)
          (and B C)
          (and B D)))
```

Standard algorithm to serialize to a canonical disjunctive form.

```
(or (and A (not B) (not C))
    (and A B (not C) (not D))
    (and (not A) B (not C) (not D))
    (and (not A) (not B)))
```
Reduced Ordered Binary Decision Diagrams (ROBDDs)

Representing CL types as ROBDDs

**ROBDD: Reduced ordered binary decision diagram**

This serialization is in no sense minimum in form.
Standard ROBDD reduction rules are insufficient for CL type system.

\[(\text{and number (not string)}) = \text{number}\] are equivalent types, but the BDDs are different!
Brief Recap

We would like to use ORBDDs to programmatically represent and manipulate CL types.

- We have used the ORBDD developed for Boolean algebra of binary variables,
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- We unfortunately lack unique ORBDD representations for equivalent CL types.
- We find that it does not quite work for reasoning about CL types.
- A solution is needed.
- We introduce a 4th reduction rule: the *subtype rule*. Our contribution.
The types **number** and **string** are disjoint, therefore, **string ⊂ number**.

<table>
<thead>
<tr>
<th>Child to search</th>
<th>Relation</th>
<th>Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P ).green</td>
<td>( P \subset C )</td>
<td>( C \to C).green</td>
</tr>
<tr>
<td>( P ).green</td>
<td>( P \subset \overline{C} )</td>
<td>( C \to C).red</td>
</tr>
<tr>
<td>( P ).red</td>
<td>( \overline{P} \subset C )</td>
<td>( C \to C).green</td>
</tr>
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Subtype rule (4), CL type system compatibility

The types `number` and `string` are disjoint; therefore, `string \subset number`.

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As before, we can ask questions with ROBDDs.
As before, we can ask questions with ROBDDs.

Questions
Are two types the same? Or disjoint? Or is one a subtype of the other?

Functions
bdd-and, bdd-or, bdd-and-not.
Are the two types the same? No, BDDs are different.

\[
\text{(not (or (and A C) (and B C) (and B D)))}
\]

\[
(\text{or (and A (not C) (or (and B (not D)) (not B))) (and (not A) (or (and B (not C) (not D)) (not D))))}
\]
Are two types disjoint? No, the intersection is non-nil.

\[
\begin{align*}
\text{(setq } T_1 \text{ (bdd ' (and (not (and (not A) D))) (not (or (and A C) (and B C) (and B D))))))} \\
\text{(setq } T_2 \text{ (bdd ' (or (and A (not C) (or (and B (not D)) (not B))) (and (not A) (or (and B (not C) (not D)) (not D))))))))
\end{align*}
\]
Is one a subtype of the other? Yes. $T_1 \subset T_2$. 

(bdd-and-not T2 T1) 

(bdd-and-not T1 T2)
Run-time calls to `bdd-type-p`

```lisp
(defun bdd-type-p (obj bdd)
  (etypecase bdd
    (bdd-false nil)
    (bdd-true t)
    (bdd-node
      (bdd-type-p obj
        (if (typep obj (bdd-label bdd))
          (bdd-left bdd)
          (bdd-right bdd))))))

Guarantees that each base-type is checked maximum of once.
```
Compile time call to \texttt{bdd-typep}, via compiler-macro

\begin{verbatim}
(bdd-typep X ' (or (and sequence (not array)) number
(and (not sequence) array)))
\end{verbatim}
Compile time call to \texttt{bdd-typep}, via compiler-macro

\begin{verbatim}
(bdd-typep X \texttt{(or (and sequence (not array))
number
(and (not sequence) array)))

(funcall (lambda (obj)
 (block nil
 (tagbody
 1 (if (typep obj 'array)
     (go 2)
     (go 3))
 2 (return (not (typep obj 'sequence))))
 3 (if (typep obj 'number)
     (return t)
     (go 4))
 4 (return (typep obj 'sequence))))))
\end{verbatim}

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Donald Knuth’s new toy.

Binary decision diagrams (BDDs) are wonderful, and the more I play with them the more I love them. For fifteen months I’ve been like a child with a new toy, being able now to solve problems that I never imagined would be tractable... I suspect that many readers will have the same experience ... there will always be more to learn about such a fertile subject. [Donald Knuth, Art of Computer Science, Volume 4]
Summary

- Native CL type specifiers are
  - Powerful and intuitive
  - But may suffer performance issues
  - Missing capability (subtypep)

- ROBDDs offer an interesting alternative
  - We have extended Standard ROBDD theory to CL types
  - Shown type calculus operations, equality, intersection, relative complement, etc
  - Demonstrated efficient compile time code generation for type checking.
  - Competitive performance

- Lots more work to do.

- For more information see the LRDE website:
  - https://www.lrde.epita.fr/wiki/User:Jnewton
Questions?
ROBDD: Reduced Ordered Binary Design Diagrams

Having as few nodes as possible has advantages in:

- **Correctness** in presence of subtypes,
- **Memory** allocation,
- Execution **time** of graph-traversal related operations, and
- Generated **code size** (as we’ll see later).
Possible ROBDD sizes for 4 variables

Of the $2^4 = 65,536$ different Boolean functions of 4 variables, various sizes of reduced BDDs are possible. **Worst case size** is 32 nodes. **Average size** is approximately 20 nodes.
Distributions for 2 to 5 variables

Distribution of ROBDD size over all possible Boolean functions of N variables.
Expected and worst case ROBDD size

- "Worst case size"
- "Average size"

Number of variables vs. BDD size graph.
**FIRST TRY:** Expands to the following. $O(2^n)$ code size.

$O(n)$ execution time.

If the type specifier is known at compile time.

```lisp
(defun call (lambda (obj)
  (if (typep obj 'array)
      (if (typep obj 'sequence)
          nil
          t)
      (if (typep obj 'number)
          t
          (if (typep obj 'sequence)
              t
              nil))))

X)

We can do better.
```
**Conclusion**

Questions

BETTER: $O(2^{n^2})$ code size. $O(n)$ execution time.\(^1\)

```lisp
(funcall (lambda (obj)
    (labels ((#:f1 ()
      (typep obj 'sequence))
    (#:f2 ()
      (or (typep obj 'number)
          (#:f1)))
    (#:f3 ()
      (not (typep obj 'sequence)))
    (#:f4 ()
      (if (typep obj 'array)
          (#:f3)
          (#:f2))))
  (#:f4)))
```

\(^1\) $O(2^{n^2})$ is a non-rigorous estimate.
Experimental problem: thoroughly partition a set of types
Maximal Disjoint Type Decomposition

\[(\text{bit} \quad \text{float} \quad \text{fixnum} \quad \text{number} \quad \text{rational} \quad \text{unsigned-byte})\]

\[\longrightarrow\]

\[(\text{bit} \quad \text{float} \quad (\text{and} \quad \text{fixnum} \quad \text{unsigned-byte} \quad (\text{not} \quad \text{bit})))\]

\[(\text{and} \quad \text{fixnum} \quad (\text{not} \quad \text{unsigned-byte}))\]

\[(\text{and} \quad \text{number} \quad (\text{not} \quad \text{float}) \quad (\text{not} \quad \text{rational}))\]

\[(\text{and} \quad \text{rational} \quad (\text{not} \quad \text{fixnum}) \quad (\text{not} \quad \text{unsigned-byte}))\]

\[(\text{and} \quad \text{unsigned-byte} \quad (\text{not} \quad \text{fixnum}))\]
Combinations of number and condition

![Graph]

**DECOMPOSE-TYPES**
**DECOMPOSE-TYPES-GRAPH**
**BDD-DECOMPOSE-TYPES**
**DECOMPOSE-TYPES-BDD-GRAPH**
Subtypes of fixnum: (member ... )
Type specifier summary

- Easy and intuitive (thanks to homoiconicity)
- Run-time calls to subtypep and typep
- Issues of performance and correctness of subtypep and typep
Subtypes

(and (not number) (not string))
Caveat of subtypep

Sometimes subtypep returns *don’t know*. Sometimes for good reasons. Sometimes not.

\[
\text{CL–USER> } (\text{subtypep '}(\text{satisfies oddp})(\text{satisfies evenp})) > \text{NIL, NIL}
\]

\[
\text{CL–USER> } (\text{subtypep 'arithmetic-error '(not cell-error)}) > \text{NIL, NIL}
\]