Leader Election in a Synchronous Ring

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https://www.lrde.epita.fr/~renault/teaching/algorep/

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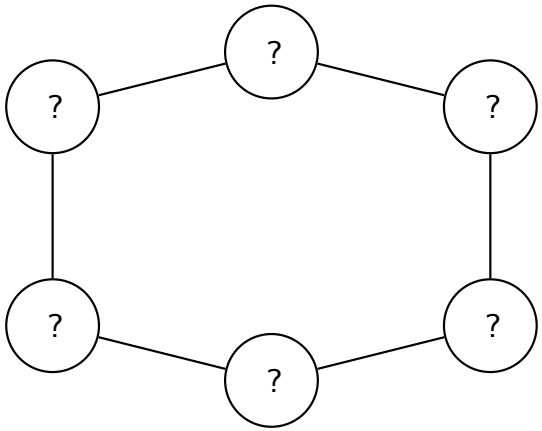
Problem Statement

- The network digraph is a ring with n nodes
- All processes are identical
- Each process can only communicate with clockwise neighbour and counterclockwise neighbour

One process outputs *"I'm the leader"* while the other process output *"I'm not the leader"*



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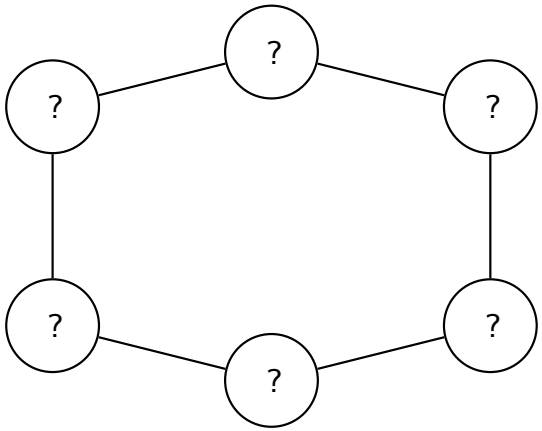

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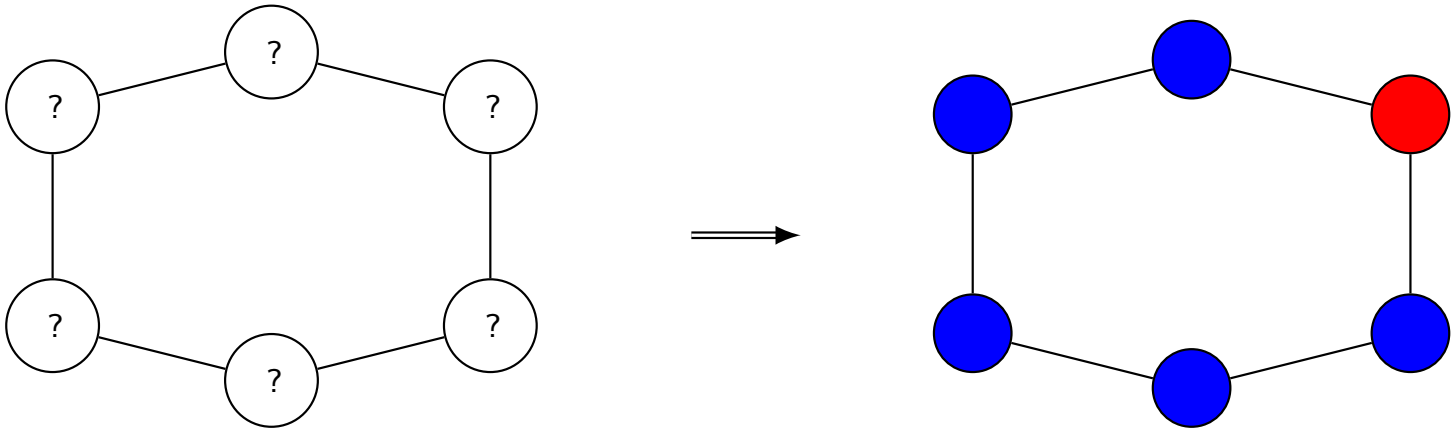

Impossibility Result for Identical Processes

Theorem

Let S be a system of n processes, $n > 1$, arranged in a bidirectionnal ring. If all the processes are identical then S does not solve the leader-election problem.

Sketch of Proof

1. Suppose there is a system S that solves this problem

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4. Then if a process reaches a state where it considers to be the leader, all the other processes do so.

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3. By induction on the number r of rounds, all the processes are in identical states immediately after r rounds.
4. Then if a process reaches a state where it considers to be the leader, all the other processes do so.
5. But this violates the uniqueness requirement

Problem Statement Revisited

- The network digraph is a ring with n nodes
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- The network digraph is a ring with n nodes
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- Each process can only communicate with clockwise neighbour and counterclockwise neighbour

Two kind of algorithms solving the leader election problem exist :

- *Comparison-based : UIDs are only used in comparisons*
- *Non-Comparison-based : UIDs may be used for computation*

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LCR Algorithm

- Tribute to LeLann[1977] algorithm
  Optimized later by Chang & Roberts [1979]
- Unidirectionnal Ring
- The size of the ring is unknown to the processes
- Comparison-based Algorithm
- It elects the process with the maximum UID

LCR Algorithm : Informal

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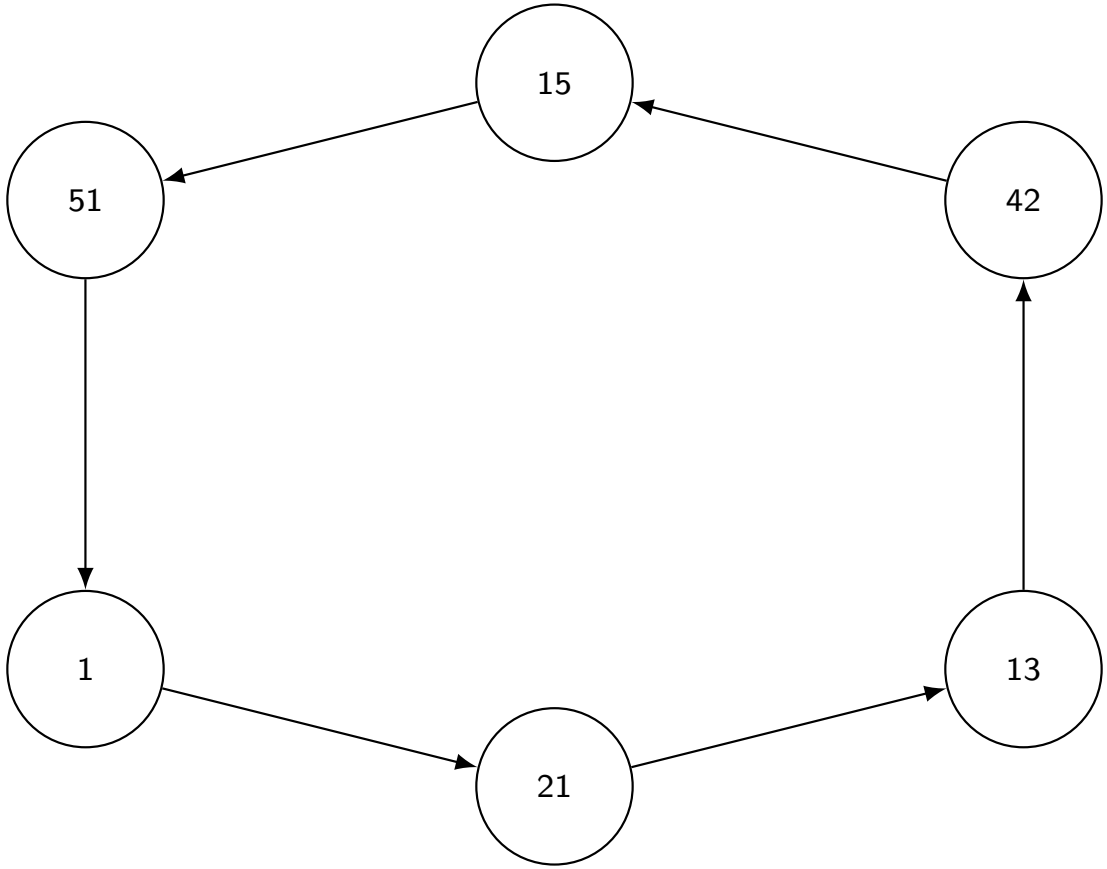
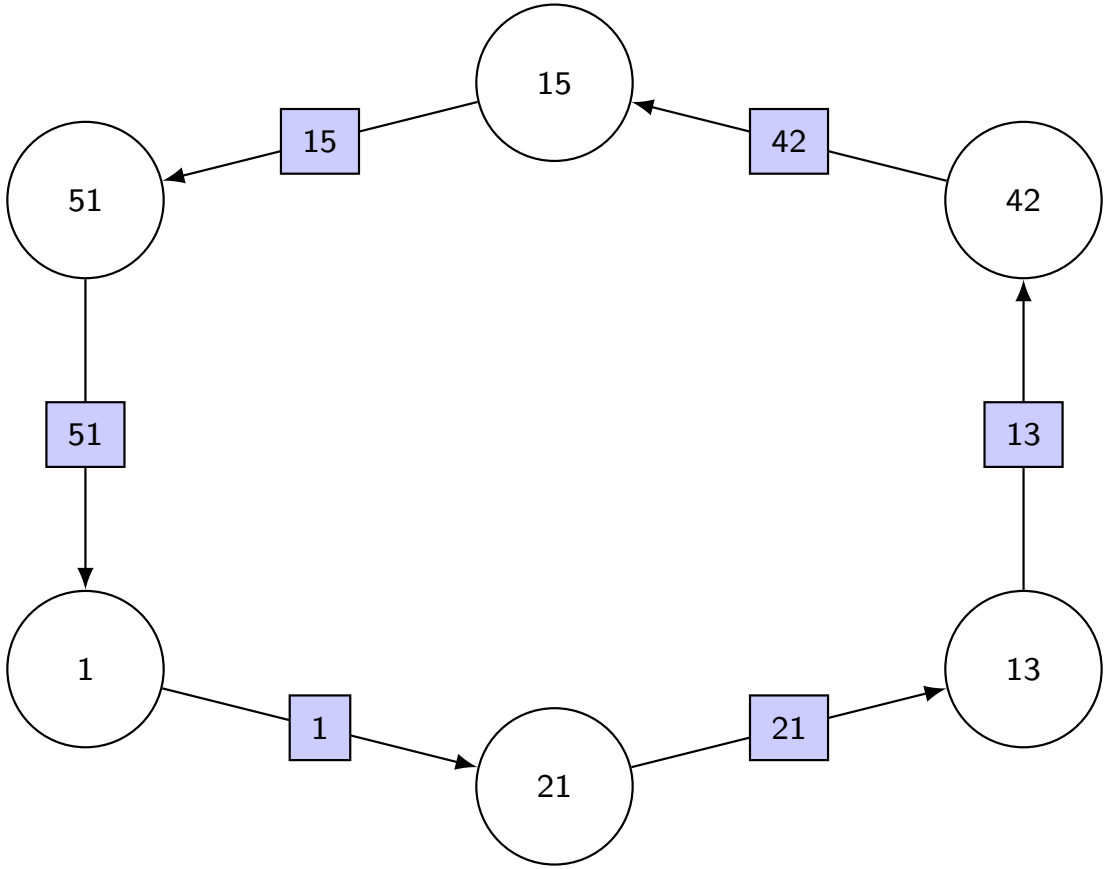
   - If the incoming UID is smaller, then it discards it
   - If it is equal, then the process declares itself the leader

Example



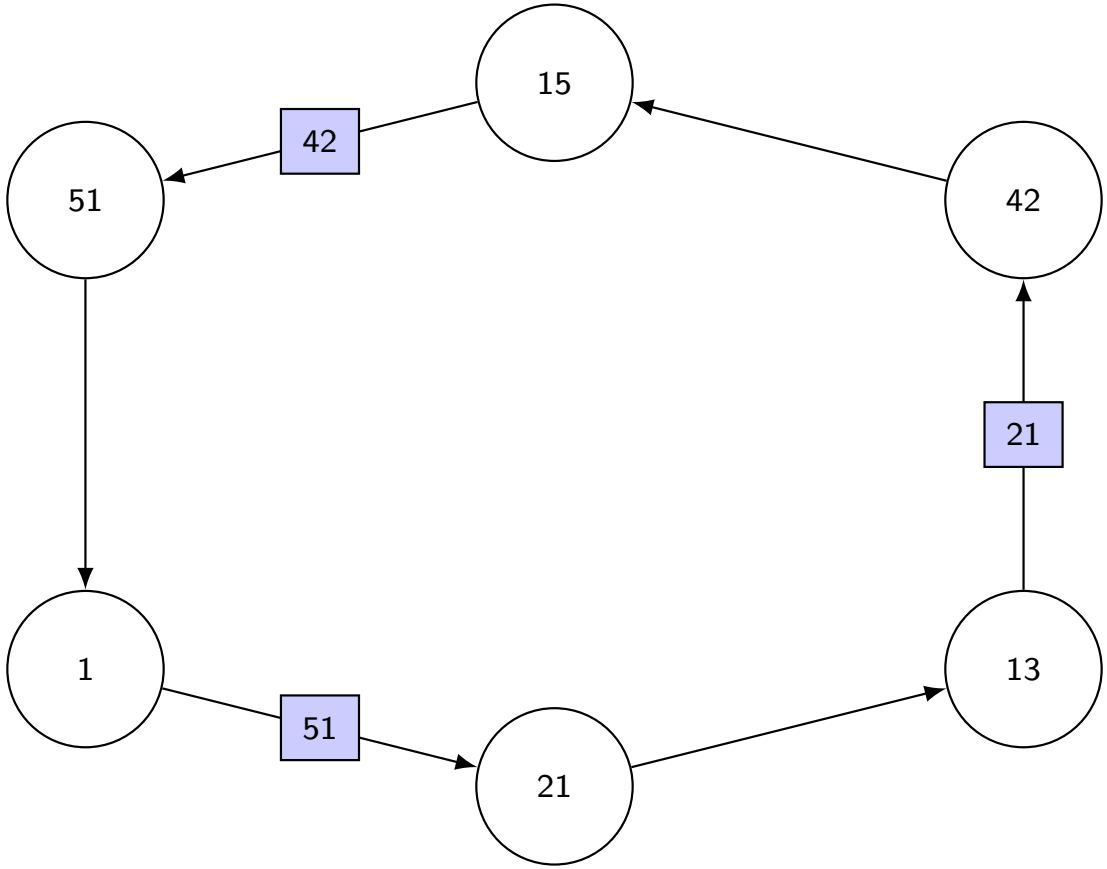
Initial State

Example



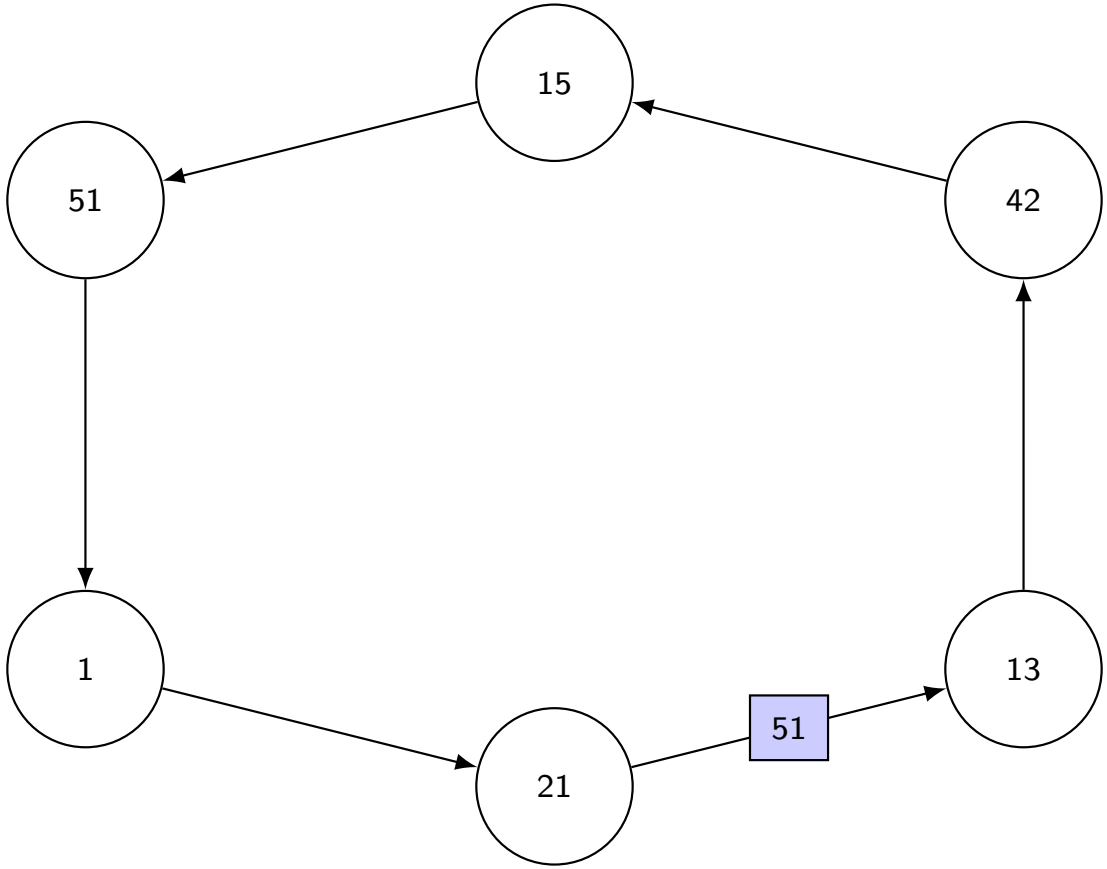
Round 1

Example



Round 2

Example



Round 3

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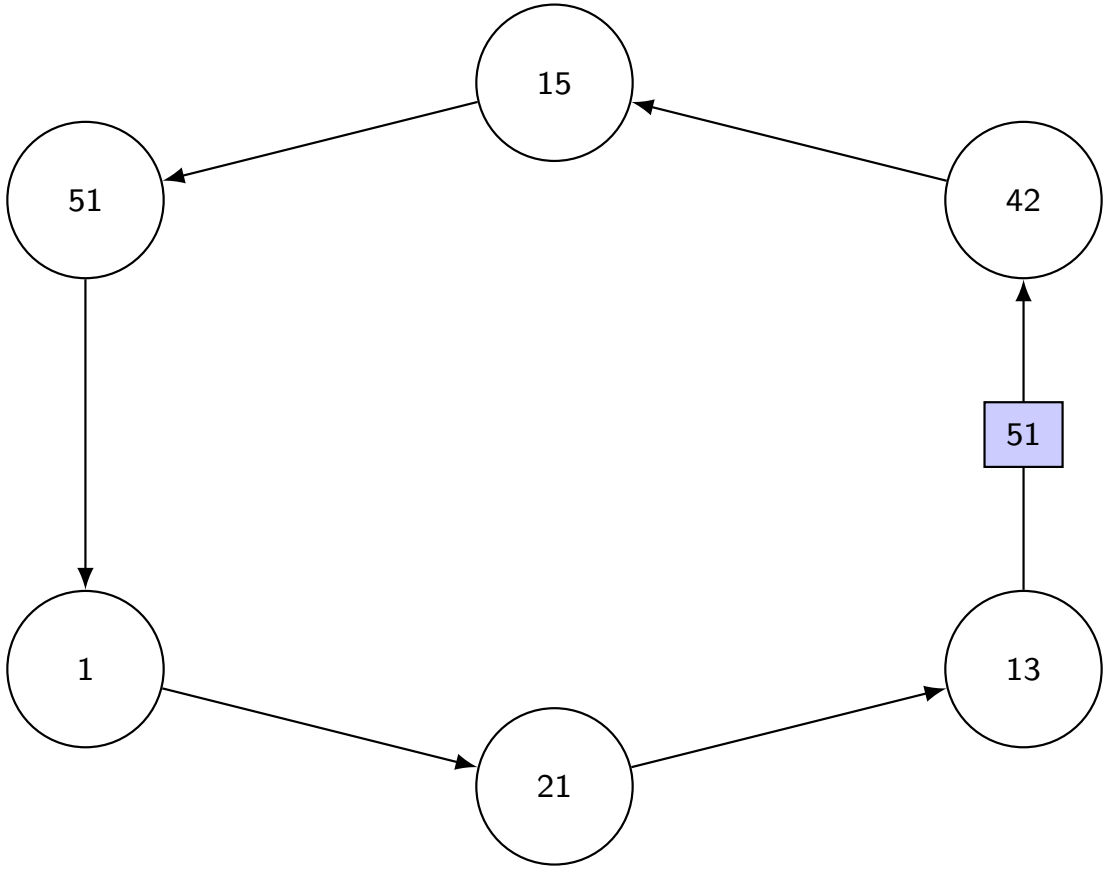
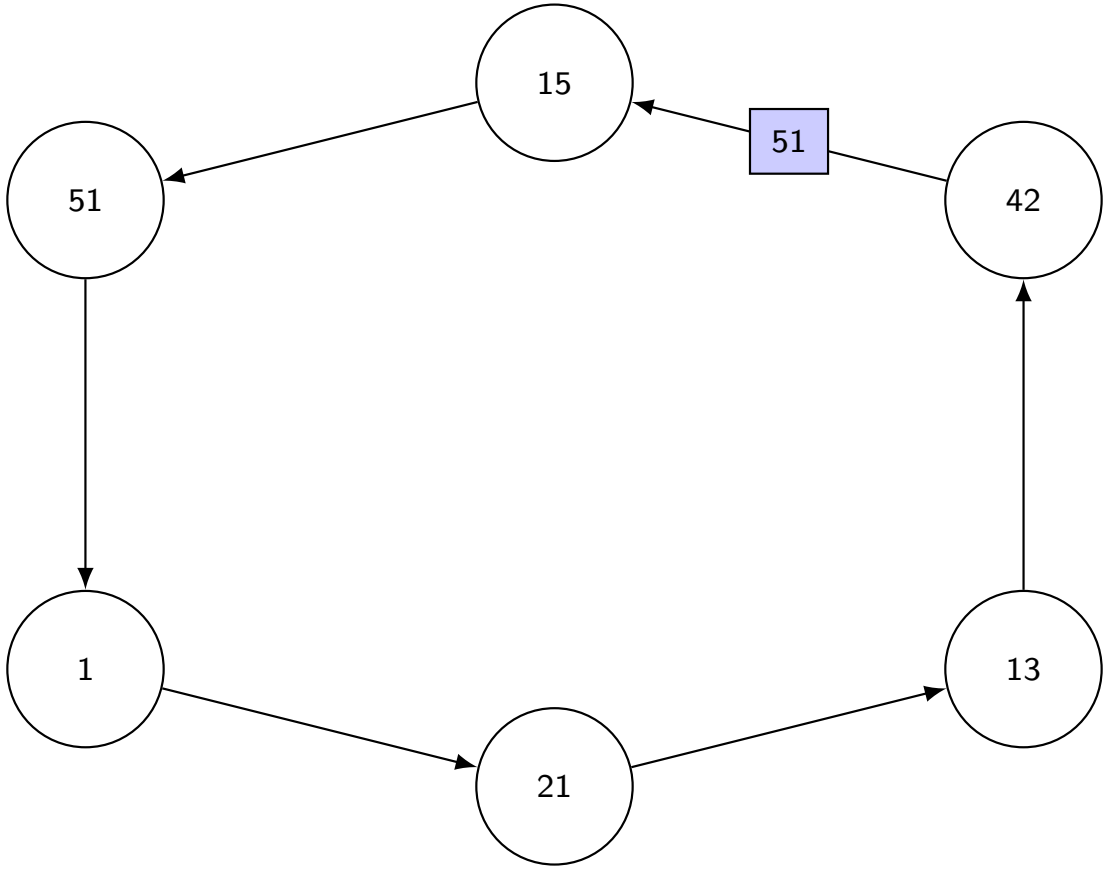


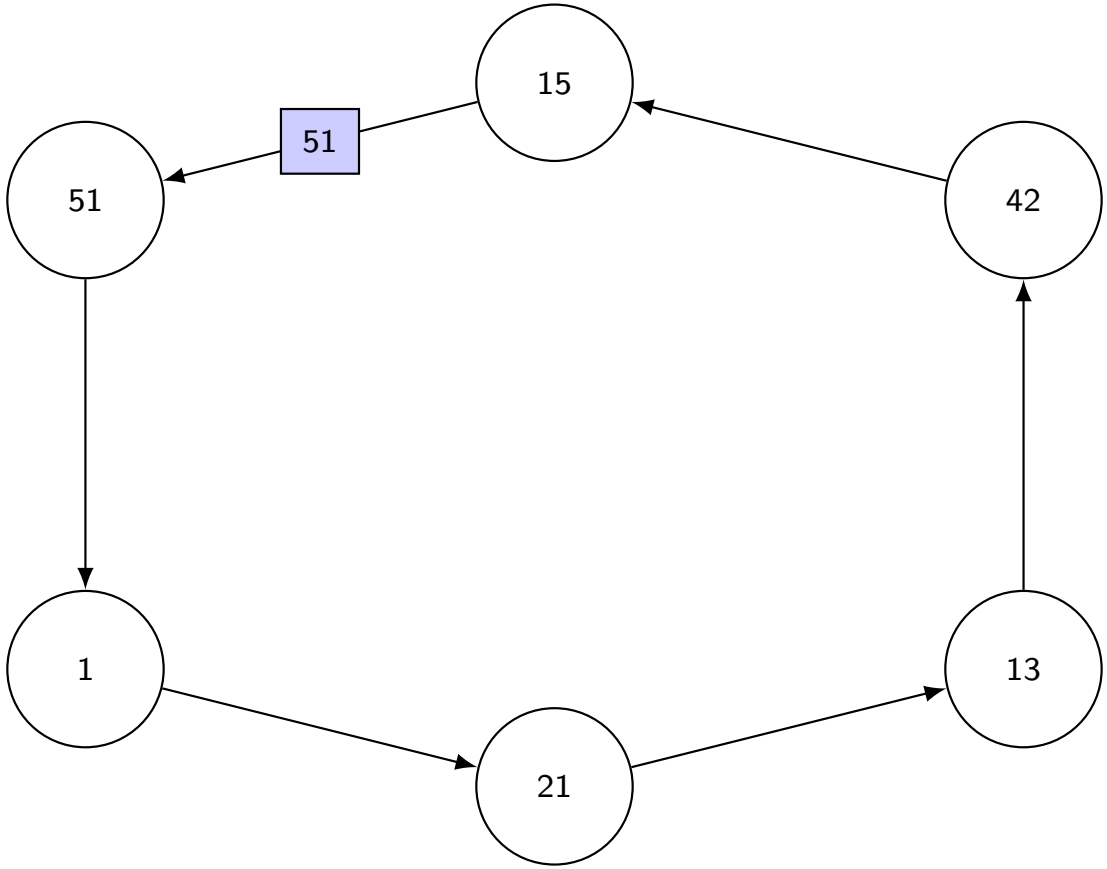
Round 4

Example



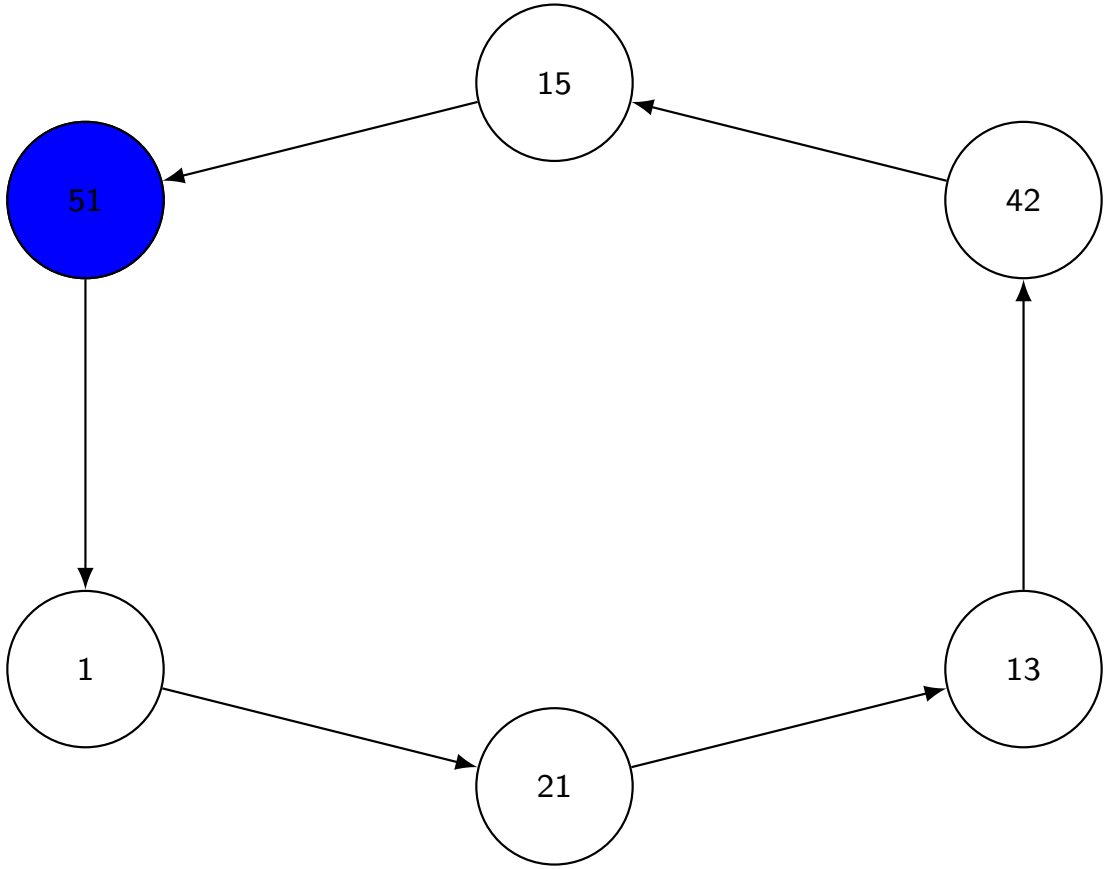
Round 5

Example



Round 7

Example



Election Successful

Complexity

- Best Case : *UIDs are sorted by increasing order*
  - n rounds
  - $O(n)$ messages
- Worst Case : *UIDs are sorted by decreasing order*
  - n rounds
  - $O(n^2)$ messages

When a node has been elected, n rounds and n messages are required to ensure the halting of the system.

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Problem

The communication complexity of LCR algorithm is high !

We want to minimize the number of messages to avoid network congestion.

HS Algorithm

- Tribute to Hirshberg & Sainclair[1980] algorithm
- Bididirectionnal Ring
- The size of the ring is unknown to the processes
- Comparison-based Algorithm
- It elects the process with the maximum UID

HS Algorithm : Informal

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   - If both tokens come back safely, process i starts a new phase

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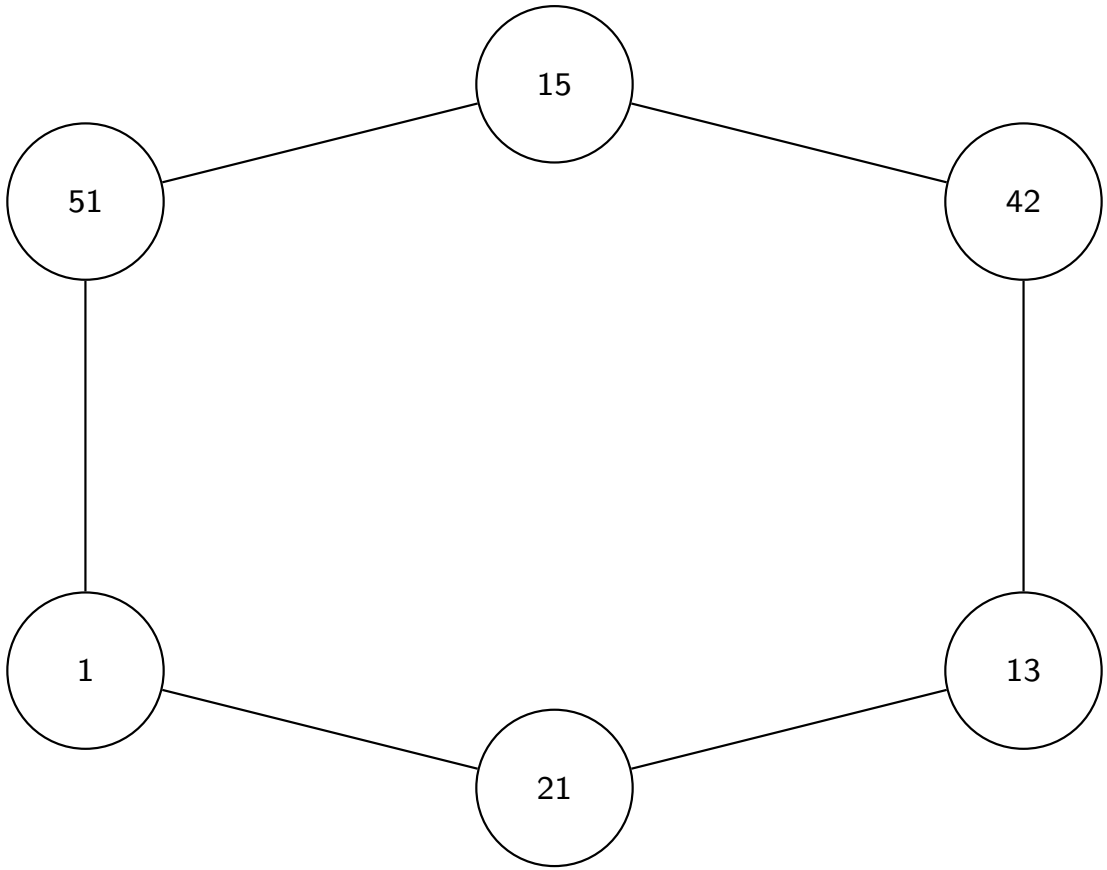
   - If both tokens come back safely, process i starts a new phase
   - Otherwise the process considers itself as a **non-leader**

Example



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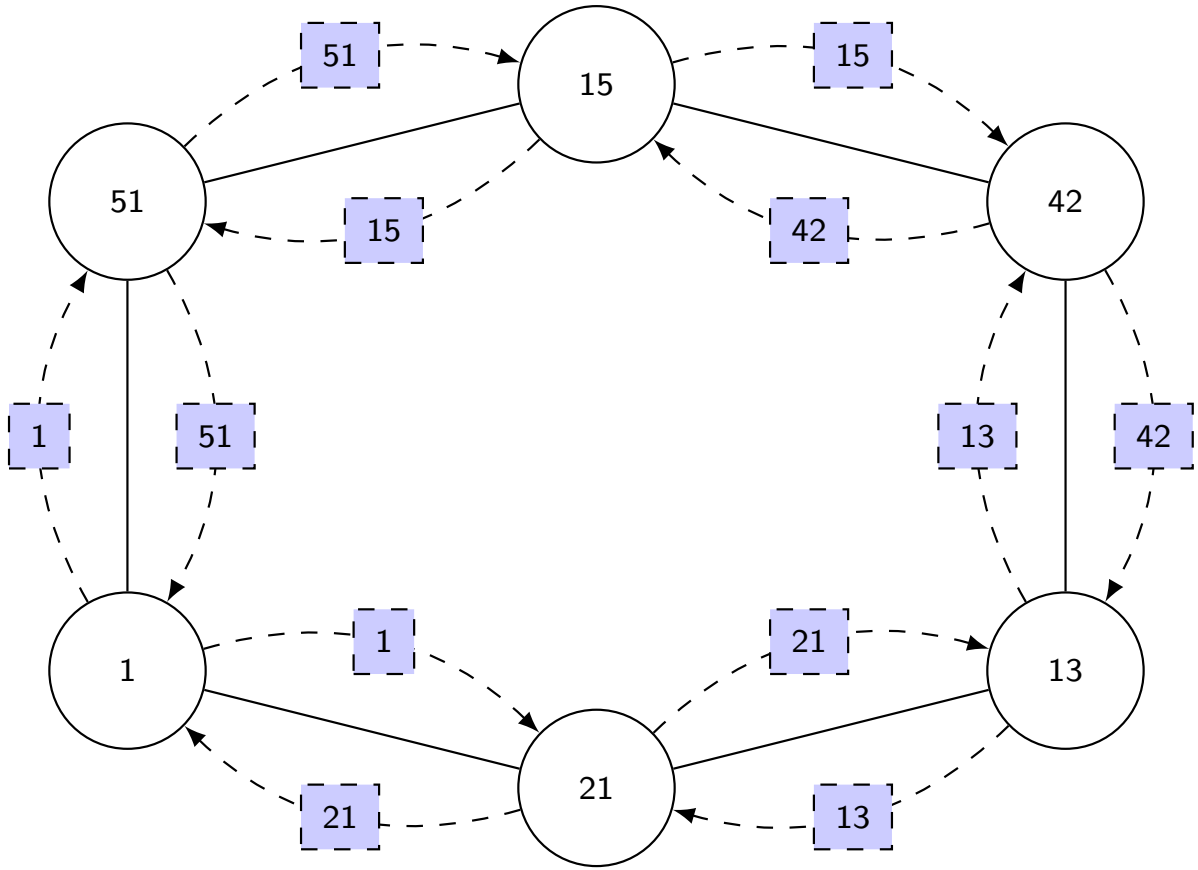

Round 1, phase 1, distance = 1

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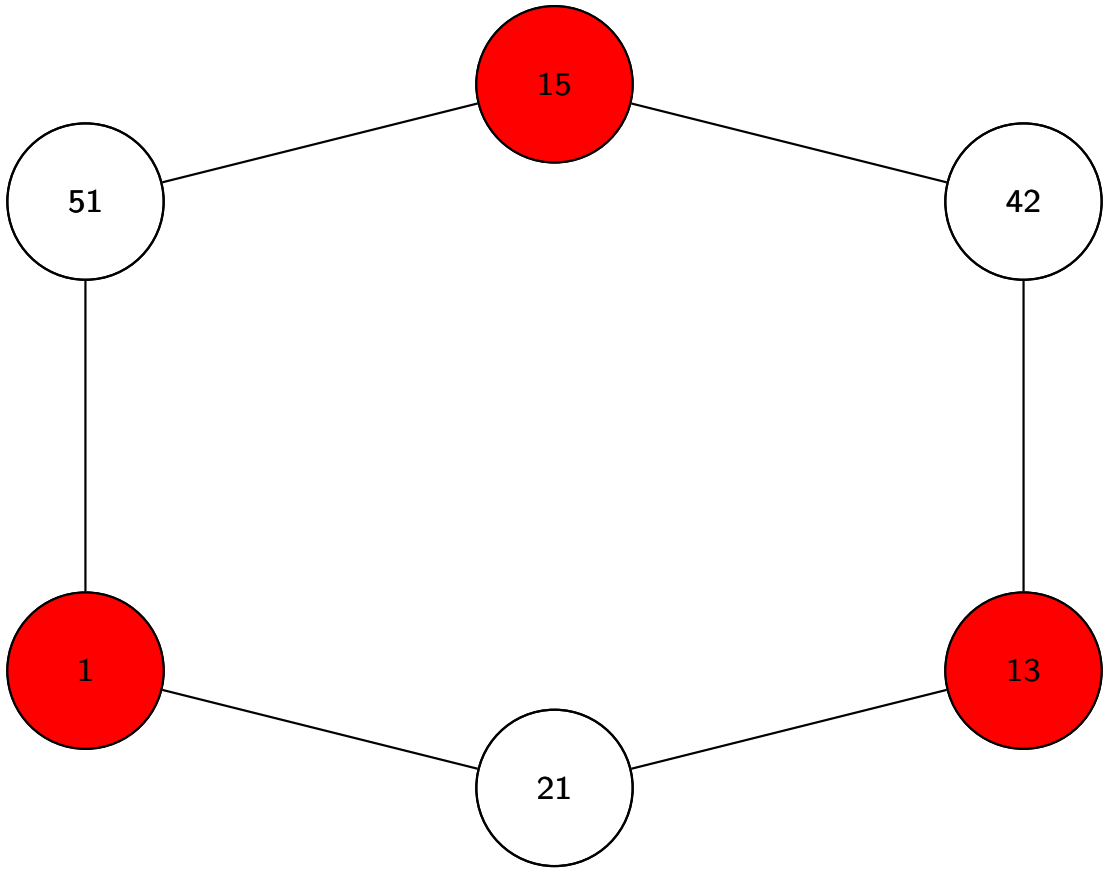
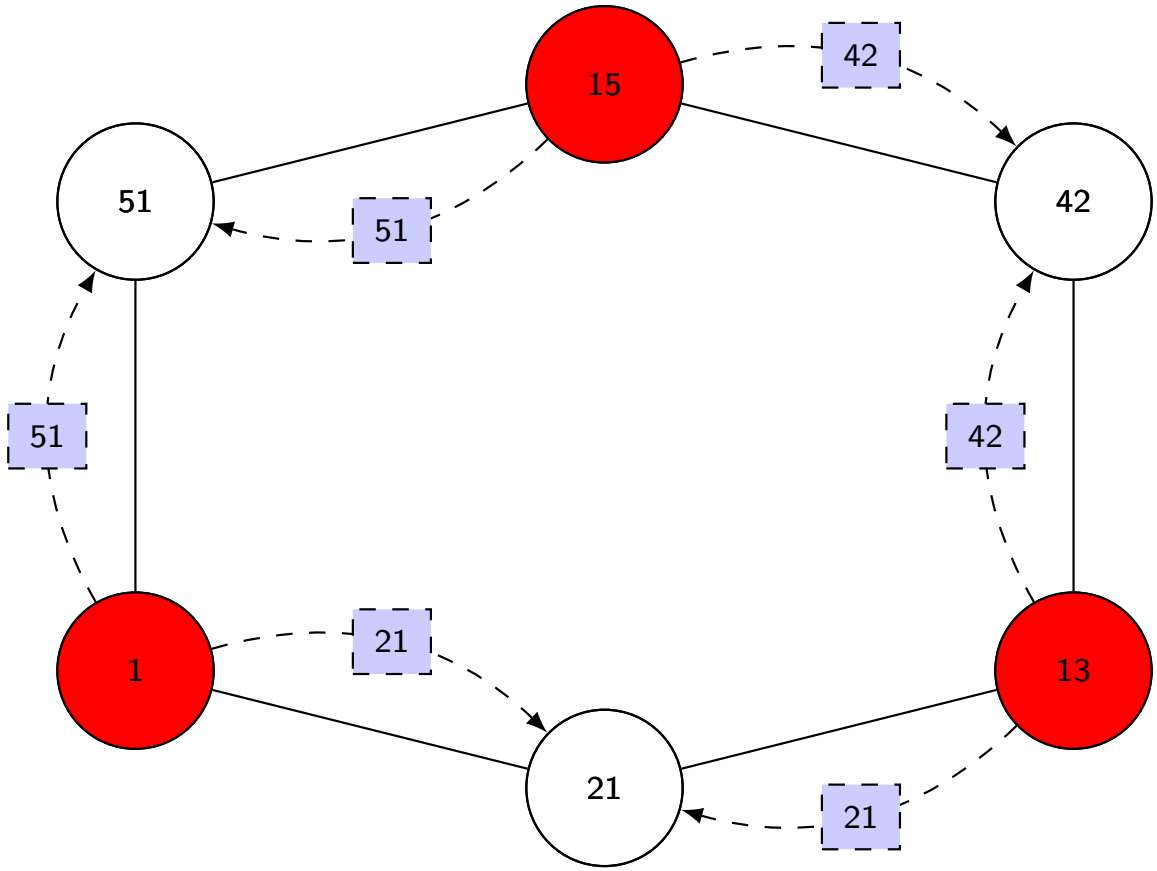


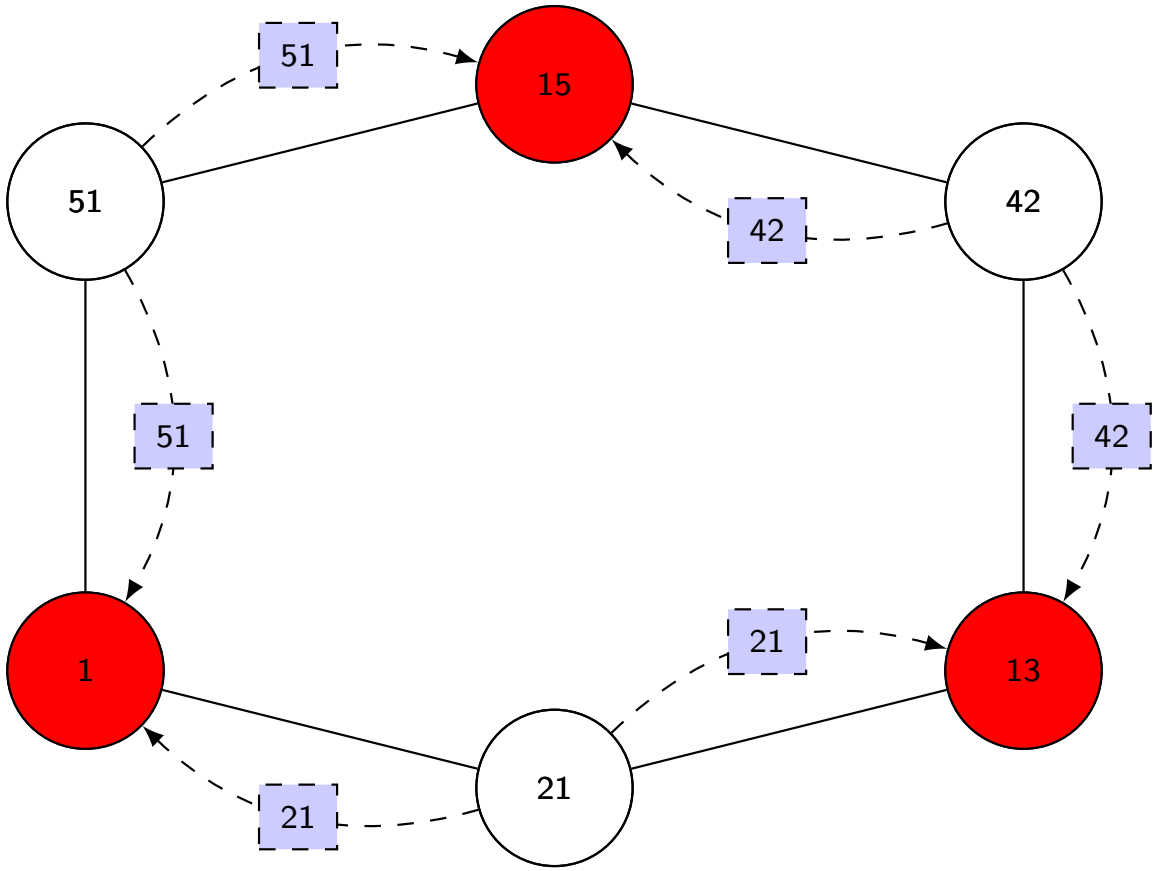
Round 1, distance = 1, Discarded messages

Example



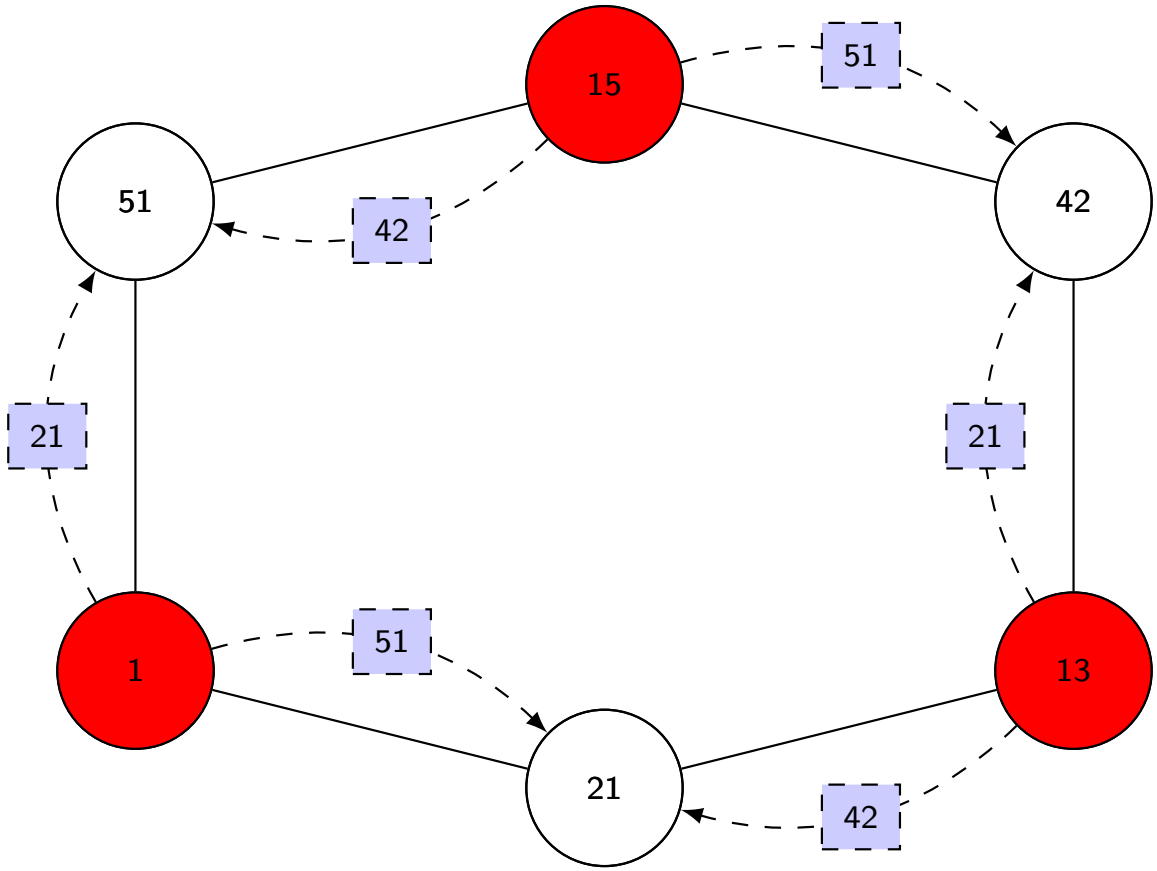
Round 2, phase 1, distance = 1

Example



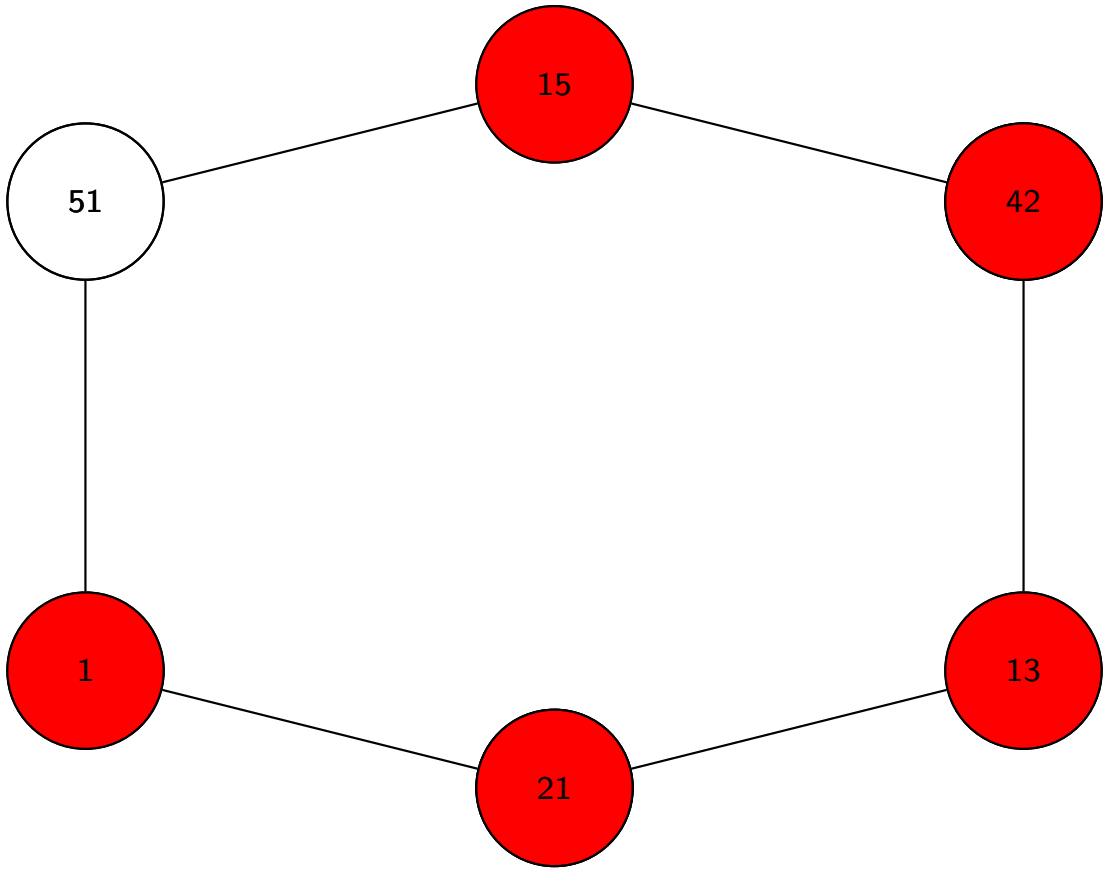
Round 2, phase 2, distance = 2

Example



Round 2, phase 2, distance = 2

Example



Messages then go back to 51, and a last phase is started so that 51 can detect it is the leader

Communication Complexity 1/2

1. **Phase 0** : every process sends a message in both directions
   $4 \times n$ messages

2. **Phase** ℓ : for $\ell > 0$ a process sends a token if it receive exactly two tokens in phase $\ell - 1$, i.e. it has not been defeated in phase $2^{\ell-1}$. This implies that within any group of $2^{\ell-1} + 1$ consecutive processes at most one will initiate tokens in phase ℓ.
   There is $\lfloor \frac{n}{2^{\ell-1}+1} \rfloor$ process that initiates tokens at phase ℓ.

   At phase ℓ the number of messages is $4(2^{\ell}(\lfloor \frac{n}{2^{\ell-1}+1} \rfloor)) \leq 8n$

Communication Complexity 2/2

How many phase are executed before a leader is elected ?

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The number of messages is at most $8n(1 + \lceil \log n \rceil)$

Time Complexity

- The time complexity for phase ℓ is $2^{\ell+1}$
  The complexity of all but the final phase is $2 \times 2^{\log n}$
- In the final phase takes n since tokens only travel outbound
- The final complexity is at most $3n$ (if n is power of 2) $5n$ otherwise.

HS Summary

Time Complexity

$O(n)$ (dominated by last phase)

Message Complexity

$O(n \log n)$

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TimeSlice Algorithm

- Unidirectionnal Ring
- UIDs are positive integer
- Deeper use of synchrony (especially non-arrival of a message) than HS or LCR
- Non-comparison-based Algorithm
- n is known in advance
- It elects the process with the minimum UID

TimeSlice Algorithm : Informal

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4. When a process receive a non-null token

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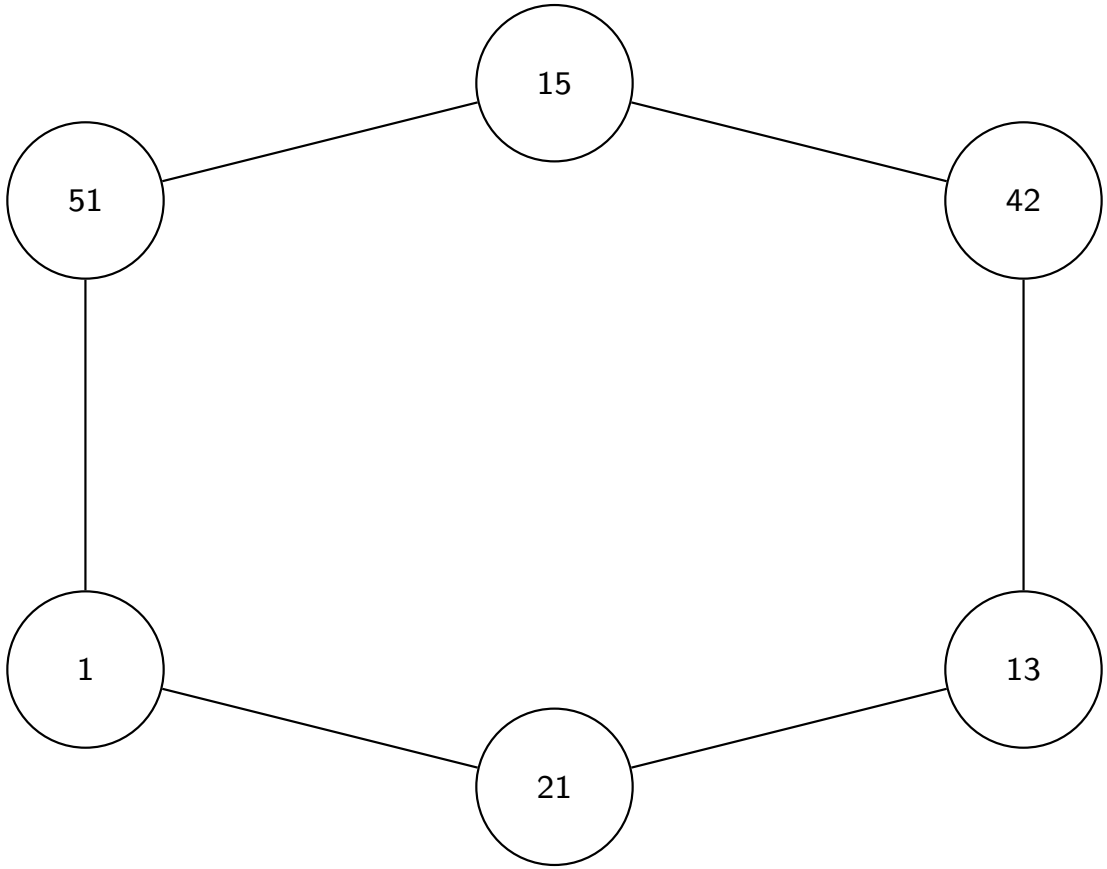
4. When a process receive a non-null token
   - if token_{uid} is equal to process'UID then the process declares itself as the leader
   - Otherwise, it declares itself as a non-leader and relay the token

Example



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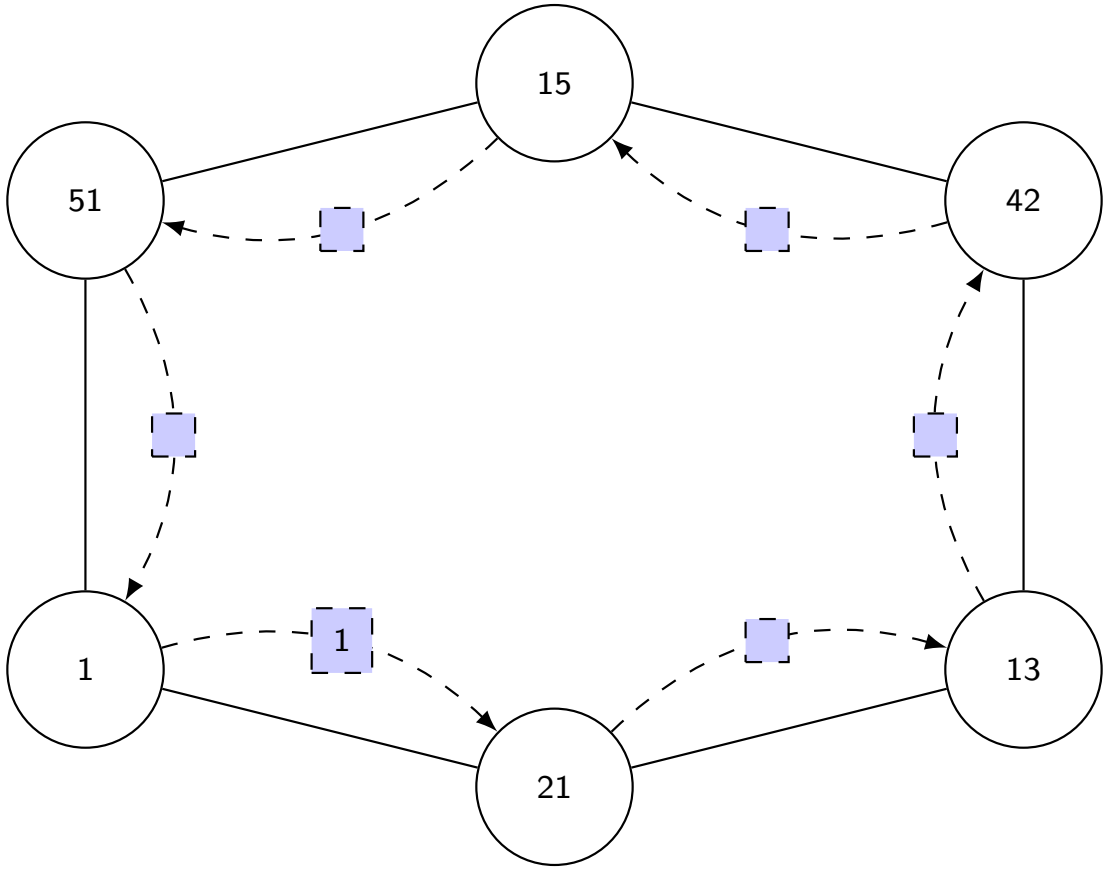
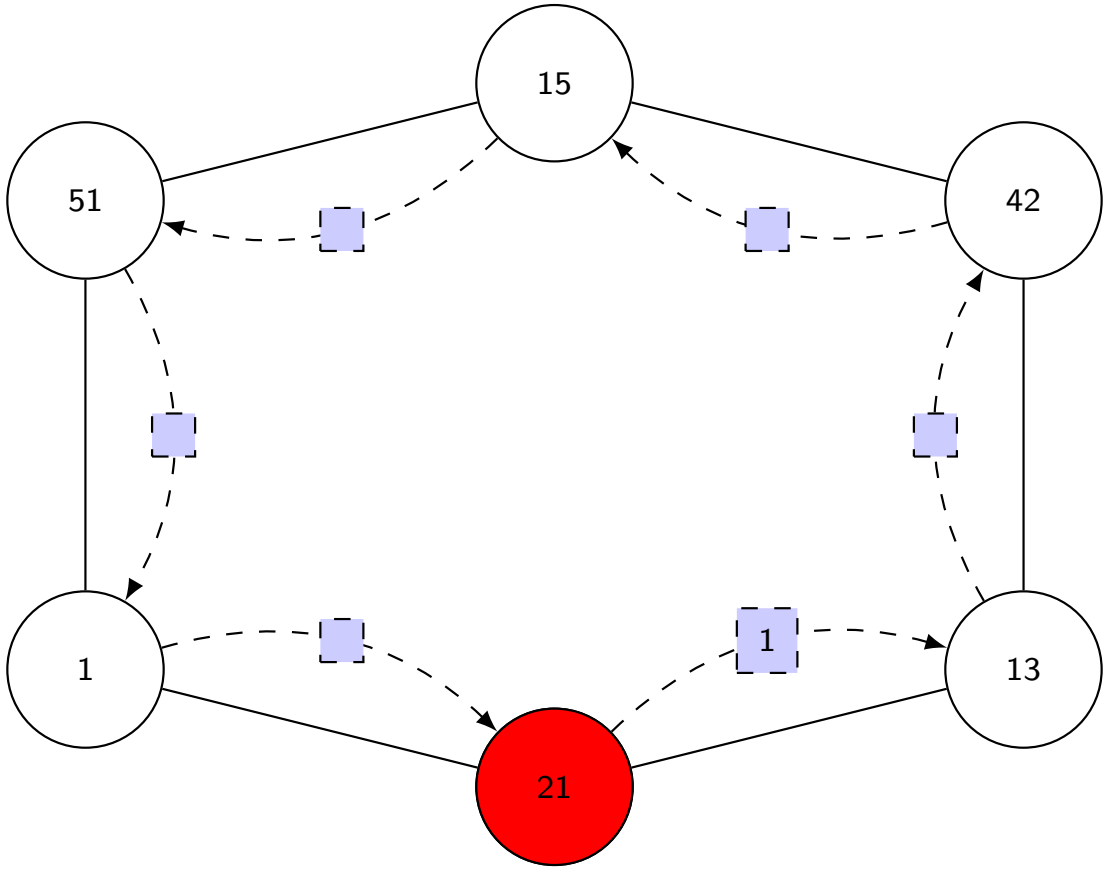


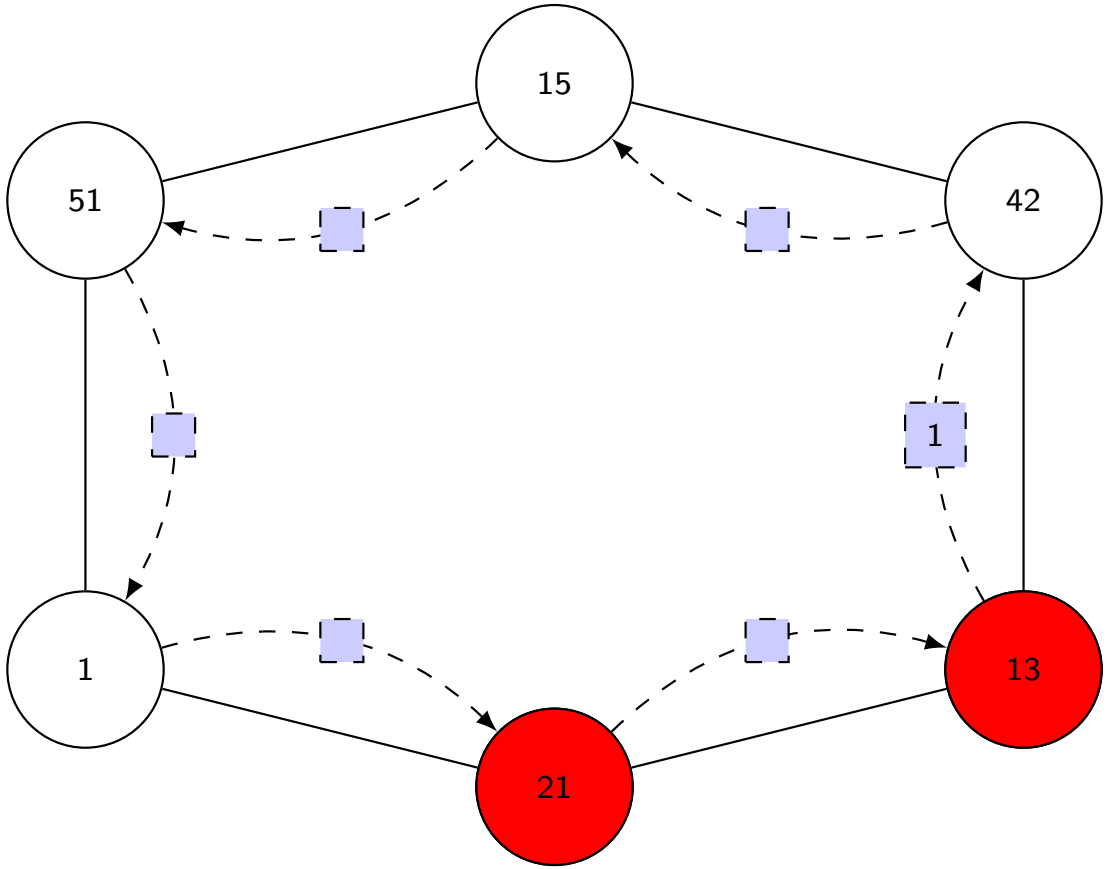
Round 1, phase 1

Example



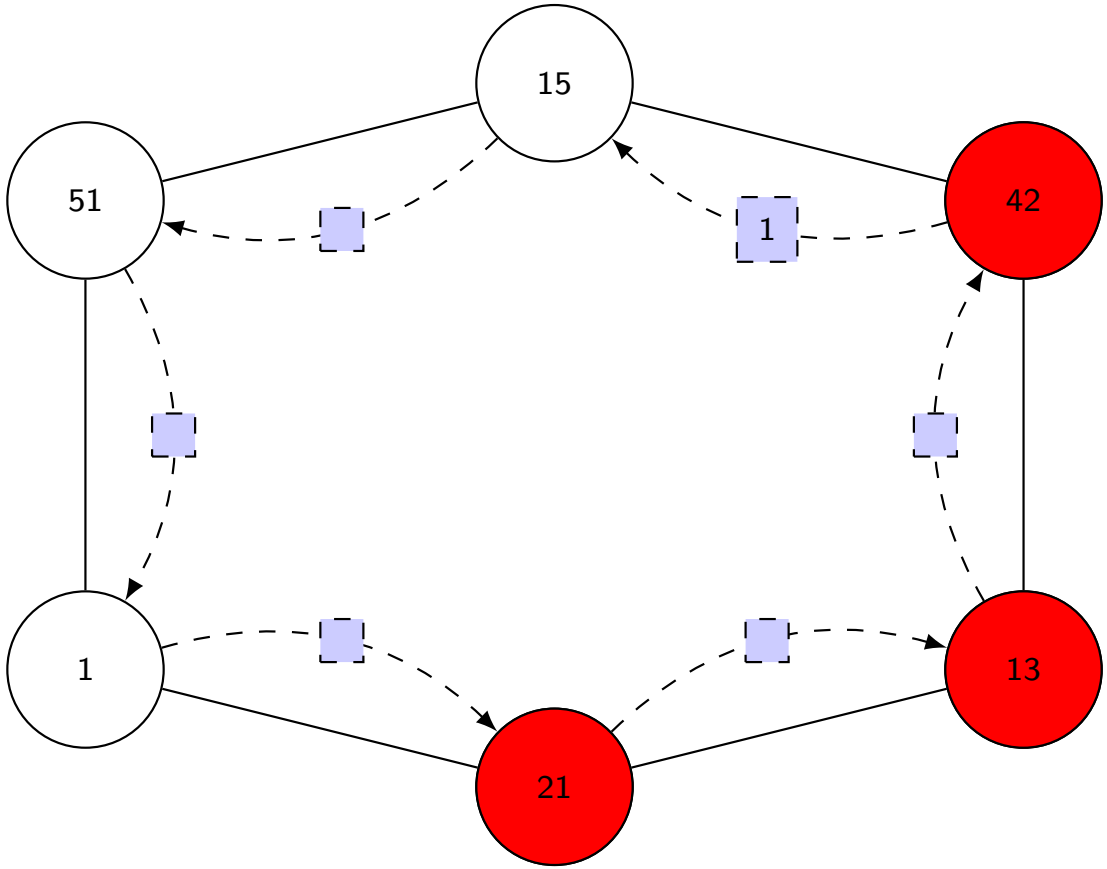
Round 2, phase 1

Example



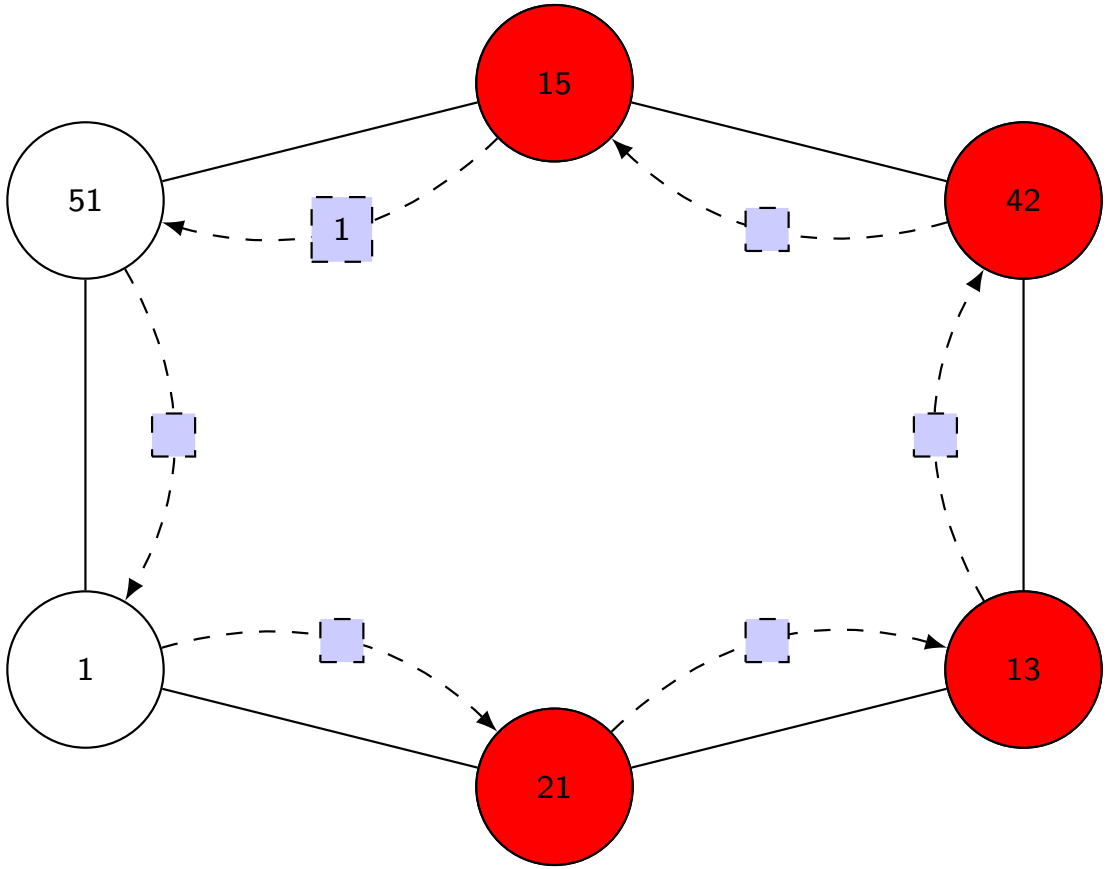
Round 3, phase 1

Example



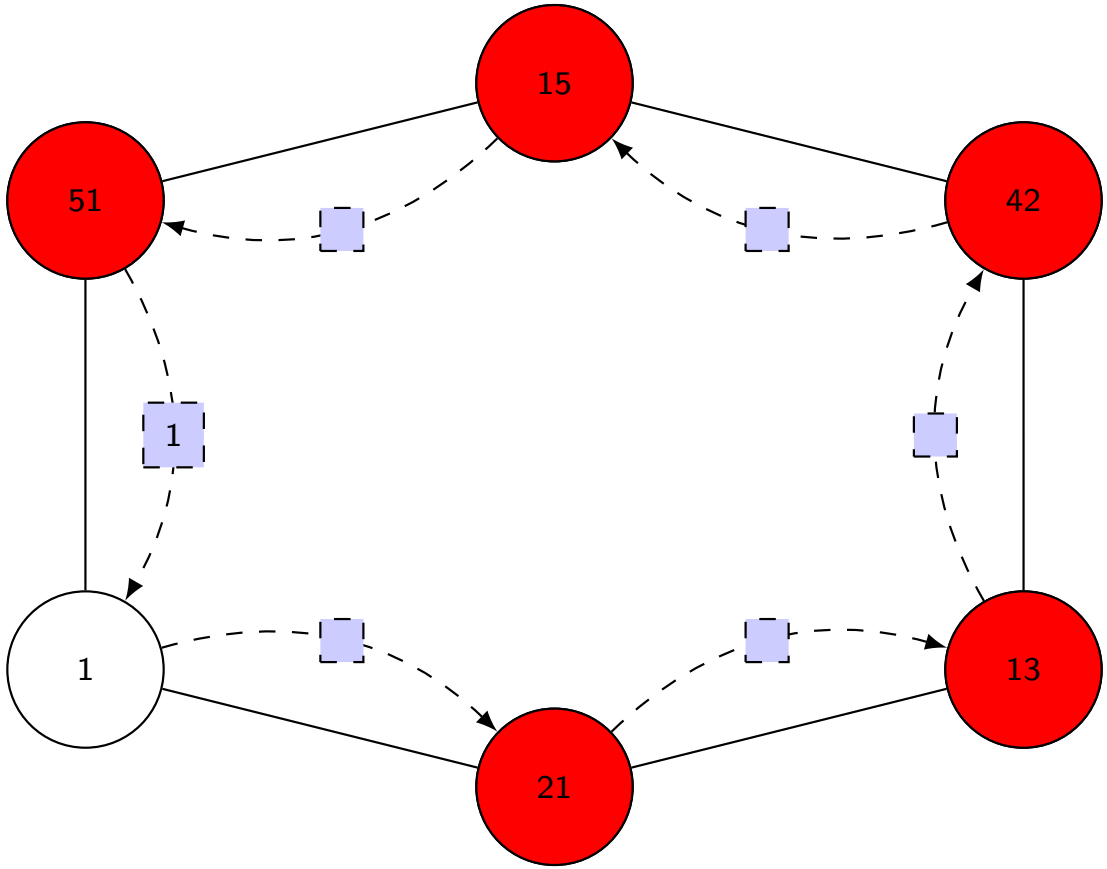
Round 4, phase 1

Example



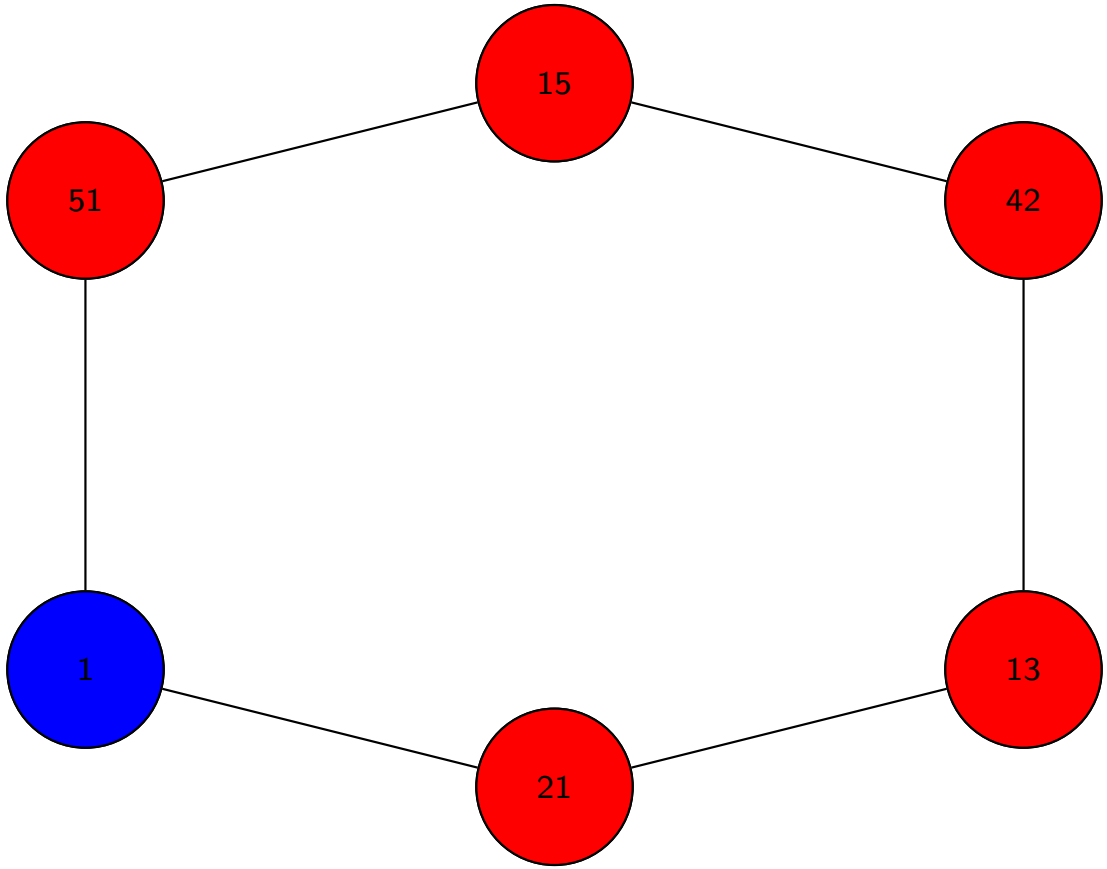
Round 5, phase 1

Example



Round 6, phase 1

Example



Complexity

- Communication : $O(n)$
- Time : $n \times UID_{min}$

Limitations

- Small ring networks
- UIDs from small positive integers
- Huge running time

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Lower Bounds

Comparison-based

The best case is $\Omega(n \log n)$ messages.

Non-Comparison-based

$O(n)$ messages can be reached but only at the cost of large time complexity (Ramsey Theorem).