Algorithms for Synchronous Networks

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https://www.lrde.epita.fr/~renault/teaching/algorep/

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More complex algorithms

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In this lecture we consider any strongly connected network digraph with n nodes.

1 Leader Election

2 Breadth-First Search

3 Shortest Path

4 Minimum Spanning Trees

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Flooding Algorithm : Informal

Pre-requisite :

- Processes have UIDs and use them only for comparison
- Processes known graph diameter D, i.e. $\forall i, j \; max \; distance(i, j)$

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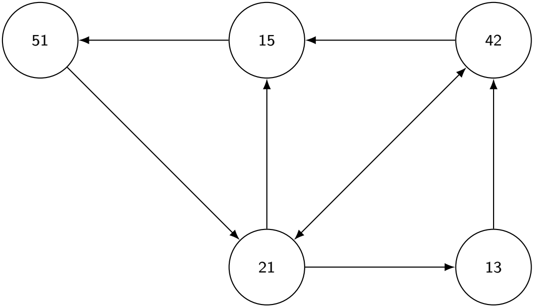
- After D rounds each process knows if it is the leader

Flooding Algorithm : Example



Initial State, $D = 3$, $n = 5$

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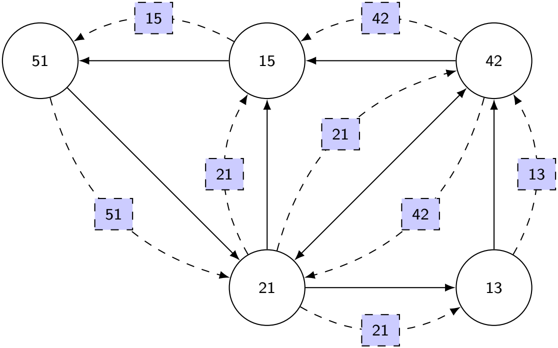
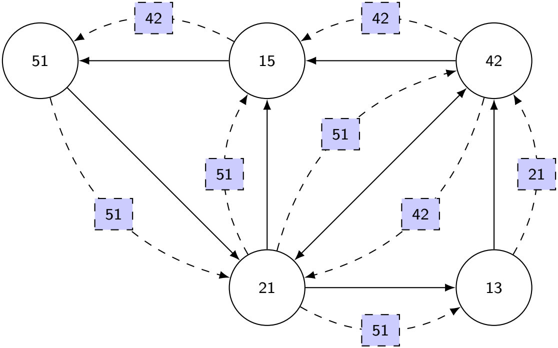


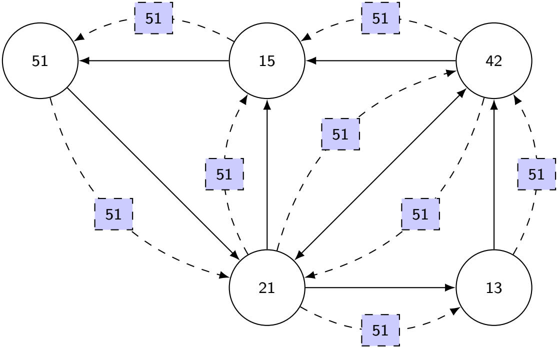
Round 1

Flooding Algorithm : Example



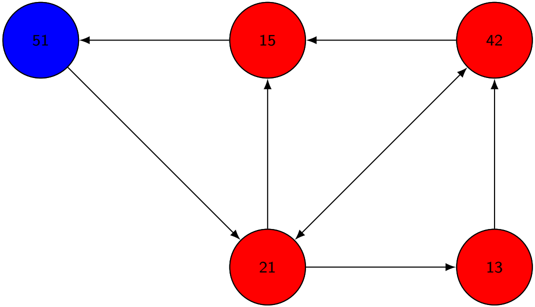
Round 2

Flooding Algorithm : Example



Round 3

Flooding Algorithm : Example



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- Communication Complexity : $D \times |E|$, with $|E|$ the number of edges in the network

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Upper Bound on the diameter

The algorithm still works if we consider D' as an upper bound of the diameter D

Improvements

- Process can send their max UID value only when they first learn about it
- When learning a new value from a bidirectionnal edge, there is no need to propagate back the information

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How to build a leader election without knowing the diameter ?
In other words, how to detect the diameter ?

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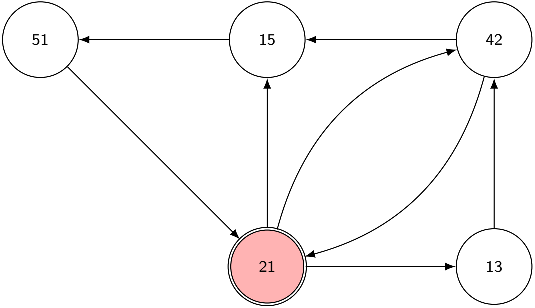
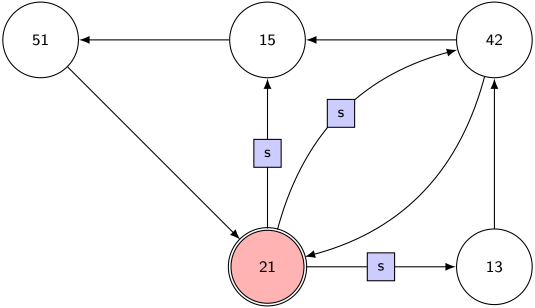


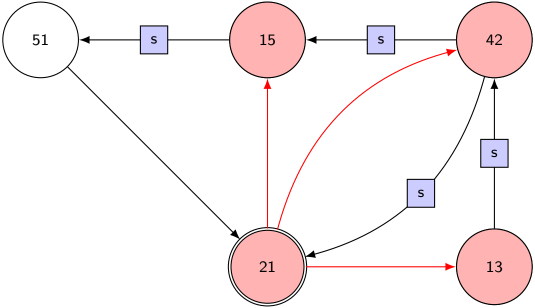
Initial State

Example



Round 0

Example



Round 1

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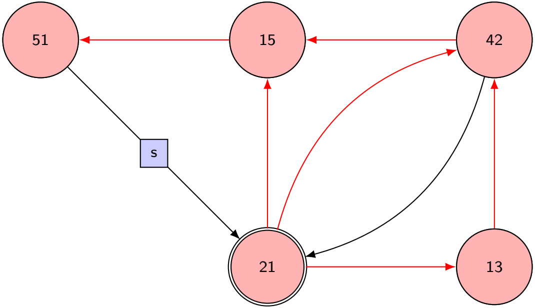
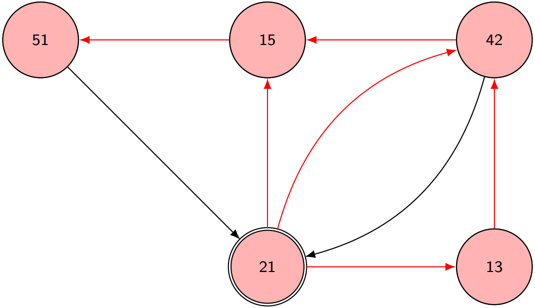


Round 2

Example



Complexity

- Time : D rounds, with D the diameter of the network
- Number of messages $|E|$, with $|E|$ the number of edges

Broadcast (and Piggybacking)

If a process wants to broadcast an information *m* :

- Initiate an execution of the Breadth-First Search algorithm
- Add to the *search* message the information *m*
- Other processes propagates *m* with their own *search* messages

All process get eventually notified !

Children Pointers

Sometimes knowing its children is important !
(Reuse the spanning tree)

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Bi-directional communications :

- Easy : When choosing a parent, notify it
- Overhead : One message per son/child link

Uni-directional communications :

- Tricky since messages can be send via indirect routes
- Messages should carry the UID of the sender and information about *parent/non-parent*

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Homework

What is the complexity with these new messages ?

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Complexity

- **undirected graph** : Time :$O(|D|)$; messages $O(n \times |D|)$
- **directed graph** :
  - $O(D)$ time : many BFS can run in parallel
  - $O(n \times |D|)$ messages : messages can be collapsed together

Breadth-First Search : Computing diameter

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Each process construct the tree and use the termination acknowledgement to propagate distance from leaf to root

1 Leader Election

2 Breadth-First Search

3 Shortest Path

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Problem Statement

- Generalisation of the BFS problem
- Communications can be uni-directional or bi-directional
- Each edge is associated to a weight
- The weight of a path is defined as the weight of each edge along the path

How to find the shortest path between two nodes ?

In other words, with some i_0 designated process,
what is the minimal distance of a node ?

Bellman-Ford : Informal

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- When dist changed, each node update its parent field
- After $n - 1$ rounds *dist* contains the shortest distance, and the field *parent* contains the parent in the shortest path tree

Example

Complexity

- Time : $n-1$
- Message : $(n-1) \times |E|$, with $|E|$ the number of edges in the network

A false assumption is to believe that
the algorithm stabilize after $|D|$ rounds

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Problem Statement

How to find the minimum/maximum spanning tree ?

A minimum-weight spanning tree minimizes the total cost for any source process to communicate with all the other process in the network.

Let us assume that n is known

Some Definitions

Consider a graph $G = (V, E)$

Spanning Forest

A forest that consists of undirected edges of E such that every vertex V of G is covered.

Spanning Tree

A spanning forest of G such with every edges connected

Basic (non-distributed) construction

1. Start we a trivial spanning forest with n nodes and no edges
2. Repeatedly
   - Select a component C in the forest
   - Chose arbitrary outgoing edge e from C having minimum weight among all outgoing edges of C
   - Combine C with the component at the other end of e
   - Include e in the new combined-component
3. Stop when there is only one component

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Cycle Problem

If all edges have distinct weights there is exactly one MST for G

Distributed Algorithm : Intuition

Assume every edge have different weights, and undirected graph

- Tribute to Gallager, Humblet and Spira
- The algorithm builds the components in *levels*
- For each k, the level k components constitute a spanning forest that is a sub-graph of the MST
- Each level k component have at least 2^k nodes

Distributed Algorithm : Informal

1. Level 0 : components consisting of individual nodes and no edges
2. To get level $k + 1$:
   - Each component search along its spanning tree the outgoing edge with the minimum weight
   - The leader of each component communicate with the selected component to mark the edge as being in the new tree
   - A new leader must be chosen for the new component and propagate through the new component
3. Stop when there only one component

To detect if two nodes are in the same component
tests messages are sent along edges

Complexity

- **Time** : $O(n \log(n))$
  - At most $\log n$ level
  - Each level takes $O(n)$

- **Communication** : $O((n + |E|) \times \log(n))$
  - $O(n)$ messages sent per level
  - $O(|E|)$ additional messages to find local minimum weights per level

Communication can be reduced to $O(n \log(n) + |E|)$ by using a more careful strategy to find minimum weight edges.
Idea : remember edges going into the same component.

Remarks

Non-unique edge weight

We can define a lexicographic order using UID of processes

Leader Election

When building the MST a leader is elected naturally !