How to Make nD Functions Digitally Well-Composed in a Self-dual Way

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Abstract. Latecki *et al.* introduced the notion of 2D and 3D well-composed images, *i.e.*, a class of images free from the "connectivities paradox" of digital topology. Unfortunately natural and synthetic images are not *a priori* well-composed. In this paper we extend the notion of "digital well-composedness" to nD sets, integer-valued functions (gray-level images), and interval-valued maps. We also prove that the digital well-composedness implies the equivalence of connectivities of the level set components in nD. Contrasting with a previous result stating that it is not possible to obtain a discrete nD self-dual digitally well-composed function with a local interpolation, we then propose and prove a self-dual discrete (non-local) interpolation method whose result is always a digitally well-composed function. This method is based on a sub-part of a quasi-linear algorithm that computes the morphological tree of shapes.

Keywords: Well-composed functions · Equivalence of connectivities · Cubical grid · Digital topology · Interpolation · Self-duality

1 Introduction

Connectivities paradox is a well-documented issue in digital topology [7]: a connected component has to have a different connectivity whether it belongs to the background or to the foreground. Well-composed images have been introduced in 2D [9] and 3D [8] to solve that issue, as one of their main properties is the equivalence of connectivities. Intuitively, a major interest of well-composed image is to have functions (values) defined independently from the underlying space structure (graph). Indeed, life is easier if the connectivity of upper and lower level sets are the same. This is especially true when considering self-duality (recall that a transform φ is self-dual iff $\varphi(-u) = -\varphi(u)$, where φ acts on the space of functions): peaks and valleys are not processed with the same connectivity, and as a consequence the self-duality property is not "perfectly pure". The companion paper [4] discusses at length these questions, which have been largely ignored in the literature on self-duality.

Given that sequences of 3D images become more and more frequent, notably in the medical imaging field and in material sciences, it is important to extend well-composedness in 4D, and more generally in nD. However, extending the notion to higher dimension is not straightforward. The main objective of this paper is twofold.

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- First, we review and study some possible extensions of the well-composedness concept: based on the equivalence of connectivity, based on the continuous framework, based on the combinatorial definition of n-surfaces, and, most importantly in this paper, the digital well-composedness, based on some critical configurations. We prove in particular that digital well-composedness implies the equivalence of connectivities.
- Second, we propose a non-local interpolation producing, from a gray-level image defined on the nD cubical grid, an interpolated digital well-composed nD image, with $n \geq 2$. Recall that, in the same setting, we have proved [1] that, under some usual constraints, *no* local self-dual interpolation method can succeed in making nD digital well-composed functions for $n \geq 3$. Last, let us mention that another approach, based on changing the image values, has been investigated in [2]

The outline of this paper is the following. Section 2 presents several notions of well-composed sets and functions; as a side-effect, it also provides a disambiguation of the multiple definitions of "well-composedness". In Section 3, we propose an extension to nD of the notion of digital well-composedness, precisely on nD digitally well-composed sets, functions, and interval-valued maps. Section 4 studies a front propagation algorithm $\mathfrak{FP}$, and states that, when an nD interval-valued map U is digitally well-composed, then $\mathfrak{FP}(U)$ is digitally well-composed. In Section 5 we explain how practically we can turn an nD integer-valued function u into an nD well-composed function. Last we conclude in Section 6 and gives some perspectives of our work.

Note that, due to limited space, the proofs of theorems and propositions will be given in an extended version of this paper, available on the Internet from http://hal.archives-ouvertes.fr.

2 About Well-Composedness

2.1 Well-Composed Sets

An important result of the paper is the clarification of the terminology and of the various approaches dealing with well-composedness on cubical grids. There exist four different definitions:

- the *well-composedness based on the "equivalence of connectivities"* (or EoC well-composedness), EWC for short, which is the seminal definition of "well-composed 2D sets";
- the *digital well-composedness*, DWC for short, which relies on the definition of critical configurations (explained later in this paper);
- the *continuous well-composedness*, CWC for short, which relies on the continuous framework;
- the *Alexandrov well-composedness*, AWC for short, which relies on the combinatorial definition by Evako *et al.* [3] of n-surface.

As said in the introduction, digital topology is well known to force the practitioners to use a pair of connectivities [7], *e.g.* in 2D both c_4 and c_8. To avoid

Table 1. Different "flavors" of well-composedness: their definition is emphasized (underlined), and the relations between them are depicted. The bottom line is dedicated to their nD generalizations where $n \geq 2$. Additionally, note that the relation "EWC $\Leftarrow$ DWC" in nD comes from this present paper.

2D case:	**EWC** [9]	$\Longleftrightarrow$	DWC	$\Longleftrightarrow$	AWC	$\Longleftrightarrow$	CWC
3D case:	EWC	$\Longleftarrow$	DWC	$\Longleftrightarrow$	AWC	$\Longleftrightarrow$	**CWC** [8]
nD case:	EWC	$\Longleftarrow$	**DWC** (this paper)	$\Leftarrow ? \Rightarrow$	**AWC** [12]	$\Leftarrow ? \Rightarrow$	**CWC** [10]

this issue, Latecki, Eckhardt and Rosenfeld have introduced the notion of *well-composed* sets and gray-level images in [9] for the 2D case, and shortly afterwards, in [8] for the 3D case. Let us recall these seminal definitions.

Definition 1 (2D well-composed sets, 2D EWC). *A 2D digital set $X \in \mathbb{Z}^2$ is weakly well-composed if any 8-component of X is a 4-component. X is well-composed if both X and its complement $\mathbb{Z}^2 \setminus X$ are weakly well-composed.*

Starting from this definition, denoted by 2D EWC in Table 1, the seminal paper [9] shows that it is equivalent to the digital well-composedness, and to the continuous well-composedness. Though, a definition of the 3D well-composedness based on the notion of the "equivalence of connectivities" (3D EWC) does not lead to any interesting topological properties. So, in [8], Latecki introduced the "continuous well-composedness" (CWC) setting:

Definition 2 (3D well-composed sets, 3D CWC). *A 3D digital set $X \in \mathbb{Z}^3$ is well-composed if the boundary of its 3D continuous analog is a 2D manifold.*

As depicted in Table 1, both in 2D and in 3D, the digital well-composedness, the Alexandrov well-composedness, and the continuous well-composedness are equivalent (this property is used in [6]). A major difference between 2D and 3D comes from the fact that, in 3D and in any greater dimension, we lose the property "EWC $\Rightarrow$ CWC". Indeed, the 3D set $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\middle|\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$ satisfies the "equivalence of connectivities" property—this set is actually both a 26-component and a 6-component—but it is not digitally well-composed.

The continuous well-composedness has been extended in nD by Latecki (Definition 4 in [10]). Yet, to the best of our knowledge, neither Latecki nor any other author have expanded on this definition. One reason might be that it is difficult to handle from a computational point of view. In contrast, the combinatorial notion of n-surface [3] seems highly adapted to model boundaries of subsets of $\mathbb{Z}^n$ (or of any subdivision of this space). Following these ideas, a definition of the Alexandrov well-composedness (AWC) has been proposed in [12].

A third approach, based on some forbidden critical configurations, is called the digital well-composedness (DWC), and is the focus of this paper. For practical purposes such as self-duality, the "well-composedness based on the equivalence of connectivities" (EWC) is the property we are looking for. It is a global property, while DWC is a local one. In this paper, we define the generalization to nD of the digital well-composedness (Section 3) and we show (Section 3.2) that "DWC $\Rightarrow$ EWC in nD". Hence, DWC is a practical way to check/enforce EWC.

Studying whether or not the continuous well-composedness, the digital well-composedness and the Alexandrov well-composedness are equivalent in nD for $n > 3$ is beyond the scope of the paper (in nD, we thus have "AWC $\Leftarrow$**?**$\Rightarrow$ CWC" and "DWC $\Leftarrow$**?**$\Rightarrow$ AWC" in Table 1).

2.2 2D Well-Composed Functions

Let us denote by $\mathbb{V}$ a finite ordered set of values; $\mathbb{V}$ can be a finite subset of $\mathbb{Z}$ or of $\mathbb{Z}/2$.

Given a 2D image $u : \mathcal{D} \subseteq \mathbb{Z}^2 \to \mathbb{V}$ and $\lambda \in \mathbb{V}$, the *upper and lower threshold sets* of u are respectively defined by $[u \geq \lambda] = \{\, x \in \mathcal{D} \mid u(x) \geq \lambda \,\}$ and $[u \leq \lambda] = \{\, x \in \mathcal{D} \mid u(x) \leq \lambda \,\}$. The *strict* threshold sets are obtained when replacing $\geq$ by $>$ and $\leq$ by $<$.

Definition 3 (2D well-composed functions). *A function u is* well-composed *iff all its upper threshold sets are well-composed.*

Note that relying on *lower* threshold sets (instead of upper ones) leads to an equivalent definition. Note that using strict threshold sets (instead of large ones) also leads to an equivalent definition; indeed, in digital topology with a finite set of values, we have: $\exists\, \epsilon \in \mathbb{R}$ such that $\forall\, \lambda,\ [\, u \geq \lambda \,] = [\, u > \lambda - \epsilon \,]$ and $[\, u \leq \lambda \,] = [\, u < \lambda + \epsilon \,]$.

A characterization of well-composed 2D images is that, any sub-part of 2×2 pixels of u valued as $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ shall satisfy $\text{intvl}(a, d) \cap \text{intvl}(b, c) \neq \emptyset$, where $\text{intvl}(v_1, v_2) = [\![\min(v_1, v_2), \max(v_1, v_2)]\!]$.

3 Digital Well-Composedness in nD

This section presents a generalization to nD of the notions of digitally well-composed sets and functions, and an extension of digital well-composedness to interval-valued maps.

3.1 Notations

In the following, we consider sets and functions (typically gray-level images) defined on $(\mathbb{Z}/s)^n$, where $s \in \{1, 2\}$. When $s = 1$ we have the original space $\mathbb{Z}^n$, whereas with $s = 2$ we have a single subdivision of the original space (every coordinates of $z \in \left(\frac{\mathbb{Z}}{2}\right)^n$ are multiple of $\frac{1}{2}$). Practically, the definition domain will always be limited to an hyperrectangle $\mathcal{D} \subset (\mathbb{Z}/s)^n$. Let us denote by $\mathbb{B}_s$ the canonical basis of $(\mathbb{Z}/s)^n$. Given a point $z \in (\mathbb{Z}/s)^n$ and a subset $\mathcal{F} = \{f_1, \ldots, f_k\} \subseteq \mathbb{B}_s$ with $2 \leq k \leq n$, a *block* associated with z and $\mathcal{F}$ is defined as:

$$S(z, \mathcal{F}) \ = \ \{\, z + \sum_{i=1}^{k} \lambda_i f_i \mid \lambda_i \in \{0, \frac{1}{s}\}, \forall\, i \in [\![1, k]\!] \,\}.$$

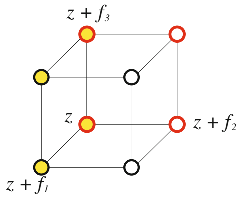


Fig. 1. A block of dimension 3 associated with z contains 8 points of $(\mathbb{Z}/s)^n$. Two blocks of dimension 2 associated with z are also depicted, each containing 4 points; their points are respectively filled in yellow and contoured in red. $(z + f_2,\ z + f_3)$ and $(z + f_1,\ z + f_2 + f_3)$ are two pairs of antagonist points, respectively for the red 2D block and for the 3D block.

Remark that a block $S(z, \mathcal{F}) \subset (\mathbb{Z}/s)^n$ actually belongs to a subspace of dimension k. In the following, we will thus say *a block S of dimension k*, meaning that we consider a block $S(z, \mathcal{F})$ such as $\text{card}(\mathcal{F}) = k$ and whatever z. Figure 1 depicts some blocks.

Given a block $S \subset (\mathbb{Z}/s)^n$ of dimension k, and $p, p' \in S$, we say that p and p' are *antagonist in S* iff they maximize the distance L^1 between two points in S. Obviously an antagonist to a given point $p \in S$ exists and is unique; it is denoted by $\text{antag}_S(p)$. We are now able to generalize the definition of critical configurations to any dimension $n \geq 2$.

3.2 Digitally Well-Composed nD Sets

A *primary critical configuration* of dimension k in $(\mathbb{Z}/s)^n$, with $2 \leq k \leq n$, is any set $\{p, \text{antag}_S(p)\}$ with S being a block of dimension k. A *secondary critical configuration* of dimension k in $(\mathbb{Z}/s)^n$ is any set $S \setminus \{p, \text{antag}_S(p)\}$ with S being a block of dimension k.

Definition 4 (nD DWC sets). *A set $X \subseteq (\mathbb{Z}/s)^n$ is* digitally well-composed *iff, for any $k \in [\![2, n]\!]$, and for any block S of dimension k, $X \cap S$ is neither a primary nor a secondary critical configuration.*

Notice that the definition of digital well-composedness is *self-dual*: any set $X \subseteq (\mathbb{Z}/s)^n$ is digitally well-composed iff $(\mathbb{Z}/s)^n \setminus X$ is digitally well-composed. An image is *a priori* not digitally well-composed; for instance, the classical gray-level "Lena" image contains 38039 critical configurations.

Let us now present a major result about digital well-composedness for nD sets.

Theorem 1 (Existence of a $2n$-path between antagonist points). *A set $X \subseteq (\mathbb{Z}/s)^n$ is digitally well-composed iff, for any block S and for any couple of points $(p,\ \text{antag}_S(p))$ of $X \cap S$, resp. $S \setminus X$, there exists a $2n$-path between them in $X \cap S$, resp. $S \setminus X$.*

Definition 5 (nD EWC sets). *A set $X \subseteq (\mathbb{Z}/s)^n$ is* well-composed based on the equivalence of connectivities (EWC) *iff the set of $(3^n - 1)$-components* *of X, resp. of $(\mathbb{Z}/s)^n \setminus X$, is equal to the set of $2n$-components of X, resp. of $(\mathbb{Z}/s)^n \setminus X$.*

As corollary, we have the equivalence of all the classical connectivities on a cubical grid for a digitally well-composed set:

Corollary 1 (nD DWC $\Rightarrow$ nD EWC). *If a set $X \subseteq (\mathbb{Z}/s)^n$ is digitally well-composed (DWC), then X is well-composed based on the equivalence of connectivities (EWC).*

3.3 Digitally Well-Composed nD Functions

The generalization of digital well-composedness from nD sets to nD functions is the same as before.

Definition 6 (nD DWC functions). *Given $u : \mathcal{D} \subseteq (\mathbb{Z}/s)^n \to \mathbb{V}$ (a gray-level image), u is* digitally well-composed *iff all its upper threshold sets are digitally well-composed.*

Following the characterizations of 2D and 3D well-composed gray-level images given respectively by Latecki in [9] and by the authors in [1], we can now express a characterization of nD digitally well-composed images. Let us recall that the span operator is defined on $V \subset \mathbb{V}$ by $\text{span}(V) = [\![\min(V), \max(V)]\!] \in \mathbb{I}_\mathbb{V}$, where $\mathbb{I}_\mathbb{V}$ denotes the set of intervals on $\mathbb{V}$.

Property 1 (Characterization of nD DWC functions). *Given a gray-level image $u : \mathcal{D} \subseteq (\mathbb{Z}/s)^n \to \mathbb{V}$, u is digitally well-composed iff, for any block S of dimension k and for any couple of points (p, p') with $p' = \text{antag}_S(p)$, we have:*

$$\text{intvl}(u(p), u(p')) \;\cap\; \text{span}\{\, u(p'') \;\big|\; p'' \in S \setminus \{p, p'\} \,\} \;\neq\; \emptyset.$$

3.4 Digitally Well-Composed nD Interval-Valued Maps

We call *interval-valued map* a map defined on $\mathcal{D} \subseteq (\mathbb{Z}/s)^n \to \mathbb{I}_\mathbb{V}$. Given an interval-valued map U, we define two functions on $\mathcal{D} \to \mathbb{V}$, its lower bound $\lfloor U \rfloor$ and its upper bound $\lceil U \rceil$, such as $\forall\, z \in \mathcal{D}$, $U(z) = [\![\lfloor U \rfloor(z), \lceil U \rceil(z)]\!]$.

Remark that interval-valued maps are just a particular case of set-valued maps defined on $\mathcal{D} \to 2^\mathbb{V}$ (also denoted by $\mathcal{D} \rightsquigarrow \mathbb{V}$). The threshold sets of set-valued maps have been defined in [12,5]; let us recall their definitions, and derive a simple characterization.

$$[U \trianglerighteq \lambda] \;=\; \{\, z \in \mathcal{D} \;\big|\; \exists\, v \in U(z),\, v \geq \lambda \,\},\ \text{and}\ [U \triangleleft \lambda] \;=\; \mathcal{D} \setminus [U \trianglerighteq \lambda],$$
$$[U \trianglelefteq \lambda] \;=\; \{\, z \in \mathcal{D} \;\big|\; \exists\, v \in U(z),\, v \leq \lambda \,\},\ \text{and}\ [U \triangleright \lambda] \;=\; \mathcal{D} \setminus [U \trianglelefteq \lambda],\ \forall\, \lambda \in \mathbb{V}.$$

Definition 7 (nD DWC interval-valued maps). *An interval-valued map U is digitally well-composed iff all its threshold sets are digitally well-composed.*

Property 2 (Characterization of nD DWC interval-valued maps). *An nD interval-valued map $U : \mathcal{D} \subseteq (\mathbb{Z}/s)^n \to \mathbb{I}_\mathbb{V}$ is digitally well-composed iff both $\lceil U \rceil$ and $\lfloor U \rfloor$ are nD digitally well-composed functions (defined on $\mathcal{D} \to \mathbb{V}$).*

4 A Study of a Front Propagation Algorithm

In this section, we study a front propagation algorithm that takes a major role in transforming any nD function into a digitally well-composed function.

4.1 Origin of the Front Propagation Algorithm

The front propagation algorithm studied in this section is related to the algorithm, proposed in [5], which computes in quasi-linear time the morphological tree of shapes of a nD image. Schematically, the tree of shapes computation algorithm is composed of 4 steps:

$$u \xrightarrow{immersion} U \xrightarrow{sort} (u^\flat, \mathcal{R}) \xrightarrow{union-find} \mathcal{T}(u^\flat) \xrightarrow{emersion} \mathcal{T}(u).$$

The input is an integer-valued image u, defined on the nD cubical grid. First an immersion step creates an interval-valued map U, defined on a larger space $\mathcal{K}$. A front propagation step, based on a hierarchical queue, takes U and produces two outputs: an image $u^\flat$ and an array $\mathcal{R}$ containing the elements of $\mathcal{K}$. In this array, the elements are sorted so that the next step, an union-find-based tree computation, produces $\mathcal{T}(u^\flat)$ the tree of shapes of $u^\flat$. Actually $u^\flat|_{\mathbb{Z}^n} = u$ and $\mathcal{T}(u^\flat)|_{\mathbb{Z}^n} = \mathcal{T}(u)$. The last step, the emersion, removes from $\mathcal{T}(u^\flat)$ all the elements of $\mathcal{K} \setminus \mathbb{Z}^n$, and also performs a canonicalization of the tree. So $\mathcal{T}(u)$, the tree of shapes of u, is obtained [5].

The front propagation step (highlighted in red in the schematic description) acts as a *flattening* of an interval-valued map U into a function $u^\flat$, because we have $\forall\, z,\ u^\flat(z) \in U(z)$ [5]. In the following, we will denote by $\mathfrak{FP}$ both the front propagation algorithm (the part highlighted in red above) and the mathematical operator $\mathfrak{FP} : U \mapsto u^\flat$.

Last, let us give two important remarks. **1.** We are going to reuse the front propagation algorithm $\mathfrak{FP}$, yet in a *very different* way than it is used in the tree of shapes computation algorithm (see later in Section 5). Indeed, its input U will be different (both the structure and the values of U will be different), and its purpose also will be different (flattening versus sorting). **2.** Actually, the front propagation algorithm is *just a part* of the solution that we present to make nD functions digitally well-composed.

4.2 Brief Explanation of the Front Propagation Algorithm

Let us now explain shortly the $\mathfrak{FP}$ algorithm, which is recalled in Algorithm 1 (see [5] for the original version). This algorithm uses a classical front propagation on the definition domain of U. This propagation is based on a hierarchical queue, denoted by Q, the current level being denoted by ℓ. There are two notable differences with the well-known hierarchical-queue-based propagation. First the values of U are interval-valued so we have to decide at which (single-valued) level to enqueue the domain points. The solution is to enqueue a point h at the value of the interval $U(h)$ that is the closest to ℓ (see the procedure PRIORITY_PUSH). The image $u^\flat$ actually stores the enqueuing level of the points. Second, when

```
PRIORITY_PUSH(Q, h, U, ℓ)
/* modifies Q */
begin
    [lower, upper] ← U(h)
    if lower > ℓ then
        ℓ' ← lower
    end
    else if upper < ℓ then
        ℓ' ← upper
    end
    else
        ℓ' ← ℓ
    end
    PUSH(Q[ℓ'], h)
end

PRIORITY_POP(Q, ℓ) : H
/* modifies Q, and sometimes ℓ */
begin
    if Q[ℓ] is empty then
        ℓ' ← level next to ℓ such as Q[ℓ']
        is not empty
        ℓ ← ℓ'
    end
    return POP(Q[ℓ])
end
```

```
𝔉𝔓(U) : Image
/* computes u♭ */
begin
    for all h do
        deja_vu(h) ← false
    end
    PUSH(Q[ℓ∞], p∞)
    deja_vu(p∞) ← true
    ℓ ← ℓ∞   /* start from root level */
    while Q is not empty do
        h ← PRIORITY_POP(Q, ℓ)
        u♭(h) ← ℓ
        for all
        n ∈ 𝒩(h) such as deja_vu(n) =
        false do
            PRIORITY_PUSH(Q, n, U, ℓ)
            deja_vu(n) ← true
        end
    end
    return u♭
end
```

Algorithm 1: Computation of the function $u^\flat$ from an interval-valued map U. Left: the routines PRIORITY_PUSH and PRIORITY_POP handle a hierarchical queue. Right: the queue-based front propagation algorithm $\mathfrak{FP}$.

the queue at the current level, $Q[\ell]$, is empty (and when the hierarchical queue Q is not yet empty), we shall decide what is the next current level. We have the choice of taking the next level, either less or greater than ℓ, such that the queue at that level is not empty (see the procedure PRIORITY_POP). Practically, choosing going up or down the levels does not change the resulting image $u^\flat$. The neighborhood $\mathcal{N}$ used by the propagation corresponds to the $2n$-connectivity.

Such as in [5], the initialization of the front propagation relies on the definition of a point, p_∞ (first point enqueued), and of a value $\ell_\infty \in U(p_\infty)$, which is the initial value of the current level ℓ. Similarly to the case of the tree of shapes computation, p_∞ is taken in the outer boundary of the definition domain of U. The initial level ℓ_∞ is set to the median value of the points belonging to the inner boundary of the definition domain of U; more precisely, when the interval-valued U is constructed from an integer-valued function u, ℓ_∞ is computed from the values of the inner boundary of u. Using the median operator ensures that ℓ_∞ is set in a self-dual way: schematically $\ell_\infty(-u) = -\ell_\infty(u)$. An example is given later in Section 5.1.

Last, let us mention that a run of the $\mathfrak{FP}$ algorithm on a simple interval-valued map U is illustrated in the extended version of this paper, available on http://hal.archives-ouvertes.fr.

4.3 Properties of the Front Propagation Algorithm

The front propagation algorithm has three main properties: it is deterministic, it is self-dual, and its output is an nD digitally well-composed function if its input is an nD digitally well-composed interval-valued map.

Proposition 1 ($\mathfrak{FP}$ is deterministic). *Once given p_∞ and ℓ_∞, the front propagation algorithm $\mathfrak{FP}$ (Algorithm 1) is deterministic with respect to its input, the nD interval-valued map U.*

Proposition 2 ($\mathfrak{FP}$ is self-dual). *For any nD interval-valued map U, and whatever p_∞ and $\ell_\infty \in U(p_\infty)$ now considered as parameters, we have: $\mathfrak{FP}_{(p_\infty,\,\ell_\infty)}(U) \,=\, -\,\mathfrak{FP}_{(p_\infty,\,-\,\ell_\infty)}(-\,U)$, so $\mathfrak{FP}$ is* self-dual.

Actually, the front propagation algorithm features some continuity properties due to the fact that the front propagation is spatially coherent, and due to the way the hierarchical queue is handled [5]. Consequently, we have for $\mathfrak{FP}$ the following strong result.

Theorem 2 ($\mathfrak{FP}(U)$ is DWC if U is DWC). *If the nD interval-valued map $U : \mathcal{D} \subset \left(\frac{\mathbb{Z}}{2}\right)^n \to \mathbb{I}_\mathbb{Z}$ is digitally well-composed, the resulting nD function $\mathfrak{FP}(U)$ is digitally well-composed.*

5 Making an nD Function Digitally Well-Composed

In this section we present a method to make any nD integer-valued function u (typically a gray-level image) digitally well-composed. This method is composed of two steps; the first one is an interpolation of u that gives an interval-valued map U_{DWC}, and the second one is the flattening $\mathfrak{FP}$ that gives the resulting single-valued function u_{DWC}:

$$u : \mathcal{D} \subset \mathbb{Z}^n \to \mathbb{Z} \quad \xmapsto{\text{interpolation}} \quad U_{\text{DWC}} : \mathcal{D}_2 \subset \left(\frac{\mathbb{Z}}{2}\right)^n \to \mathbb{I}_\mathbb{Z} \quad \xmapsto{\text{flattening}} \quad u_{\text{DWC}} : \mathcal{D}_2 \to \frac{\mathbb{Z}}{2}.$$

We are looking for an interpolation method that turns any function u into a digitally well-composed map U_{DWC}, so that eventually $u_{\text{DWC}} = \mathfrak{FP}(U_{\text{DWC}})$ is a digitally well-composed function (thanks to Theorem 2).

5.1 Interpolation

Let us consider a function (gray-level image) $u : \mathcal{D} \subset \mathbb{Z}^n \to \mathbb{V}$. We subdivide the space $\mathbb{Z}^n$ into $(\mathbb{Z}/2)^n$, and define a new map on $(\mathbb{Z}/2)^n$. The new definition domain is $\mathcal{D}_2 \subset (\mathbb{Z}/2)^n$ where $\mathcal{D}_2$ is the smallest hyperrectangle such as $\mathcal{D} \subset \mathcal{D}_2$. A sensible property of this new map is to be equal to u on $\mathcal{D}$. The values of this new map over $\mathcal{D}_2 \setminus \mathcal{D}$ are obtained by *locally* interpolating the values of u. With

9	11	15
7	1	13
3	5	3

(a) u

{8}	{8}	{8}	{8}	{8}	{8}	{8}
{8}	**{9}**	⟦9, 11⟧	**{11}**	⟦11, 15⟧	**{15}**	{8}
{8}	⟦7, 9⟧	⟦1, 11⟧	⟦1, 11⟧	⟦1, 15⟧	⟦13, 15⟧	{8}
{8}	**{7}**	⟦1, 7⟧	**{1}**	⟦1, 13⟧	**{13}**	{8}
{8}	⟦3, 7⟧	⟦1, 7⟧	⟦1, 5⟧	⟦3, 13⟧	⟦3, 13⟧	{8}
{8}	**{3}**	⟦3, 5⟧	**{5}**	⟦3, 5⟧	**{3}**	{8}
{8}	{8}	{8}	{8}	{8}	{8}	{8}

(b) $\mathcal{I}_{\text{span}}(u)$

Fig. 2. (a): A simple 2D integer-valued function. (b): Its span-based interpolation (the central part of 5×5 points of $(\mathbb{Z}/2)^n$ with values in $\mathbb{I}_\mathbb{Z}$); the external border (in gray) is required when passing $\mathcal{I}_{\text{span}}(u)$ to the $\mathfrak{FP}$ algorithm (see text).

$B = \{-\frac{1}{2}, 0, \frac{1}{2}\}^n$, B_z the translation of B by z, and "op" an operator on subsets of $\mathbb{V}$, we can define the interpolating map:

$$\forall\, z \in \mathcal{D}_2,\quad (\mathcal{I}_{\text{op}}(u))\,(z) \;=\; \begin{cases} \text{op}\{\, u(z)\,\} & \text{if } z \in \mathcal{D}, \\ \text{op}\{\, u(z'),\ z' \in B_z \cap \mathcal{D}\,\} & \text{otherwise.} \end{cases}$$

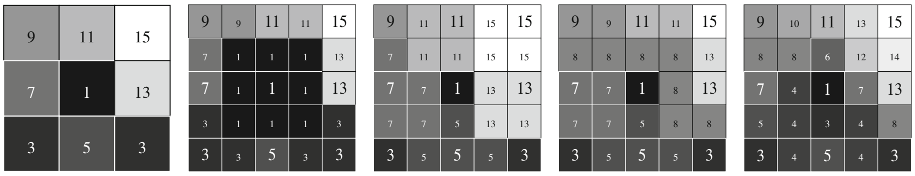
The following proposition, which could also be derived from [11], follows easily.

Proposition 3 **($\mathcal{I}_{\min}$ and $\mathcal{I}_{\max}$ give dual DWC functions).** *For any $u : \mathcal{D} \subset \mathbb{Z}^n \to \mathbb{Z}$, the nD integer-valued functions $\mathcal{I}_{\min}(u)$ and $\mathcal{I}_{\max}(u)$ are digitally well-composed, and the interpolation operators $\mathcal{I}_{\min}$ and $\mathcal{I}_{\max}$ are dual (they verify $\forall\, u,\ \mathcal{I}_{\min}(u) = -\,\mathcal{I}_{\max}(-\,u)$).*

Since we have $\forall\, V \subset \mathbb{V}$, $\text{span}(V) = [\![\, \min(V), \max(V)\,]\!]$, whatever an nD function u, the interval-valued map $\mathcal{I}_{\text{span}}(u)$ is such as $\lfloor \mathcal{I}_{\text{span}}(u) \rfloor = \mathcal{I}_{\min}(u)$ and $\lceil \mathcal{I}_{\text{span}}(u) \rceil = \mathcal{I}_{\max}(u)$. Since these two functions are digitally well-composed, the interval-valued map $\mathcal{I}_{\text{span}}(u)$ is digitally well-composed (thanks to Property 2). So we have:

Proposition 4 **($\mathcal{I}_{\text{span}}$ is self-dual and gives DWC maps).** *For any $u : \mathcal{D} \subset \mathbb{Z}^n \to \mathbb{Z}$, the nD interval-valued function $\mathcal{I}_{\text{span}}(u) : \mathcal{D}_2 \subset (\mathbb{Z}/2)^n \to \mathbb{I}_\mathbb{Z}$ is digitally well-composed, and the interpolation operator $\mathcal{I}_{\text{span}}$ is self-dual (it verifies $\forall\, u,\ \mathcal{I}_{\text{span}}(u) = -\,\mathcal{I}_{\text{span}}(-\,u)$).*

An example of the span-based interpolation is depicted in Figure 2. The outer/external boundary of the definition domain $\mathcal{D}_2$ of $\mathcal{I}_{\text{span}}(u)$ is displayed in gray. This boundary is filled with a single value $\ell_\infty(u)$, which is actually the median value of the set of values of the inner/internal boundary of the definition domain of u (see Section 4.2). We have: $\ell_\infty(u) = \text{med}\{\, 3, 3, 5, 7, 9, 11, 13, 15\,\} = 8$. When we take $U_{\text{DWC}} = \mathcal{I}_{\text{span}}(u)$ as input to the $\mathfrak{FP}$ algorithm, p_∞ can be any point of the outer boundary. This way, which is similar to [5], we ensure that the propagation starts with the external boundary of U_{DWC}, and that all the points of the internal boundary are enqueued. Having $\ell_\infty(-u) = -\ell_\infty(u)$ guarantees that "$\mathcal{I}_{\text{span}}$ with an outer boundary added" remains self-dual w.r.t. u.



(a) u **(b)** $\mathcal{I}_{\min}(u)$ **(c)** $\mathcal{I}_{\max}(u)$ **(d)** u_{DWC} **(e)** $\mathcal{I}_{\text{med}}(u)$

Fig. 3. Given an integer-valued function u, depicted in (a), we have the dual interpolations depicted in (b) and (c), and the self-dual digitally well-composed interpolation $u_{\text{DWC}} = (\mathfrak{FP} \circ \mathcal{I}_{\text{span}})(u)$ depicted in (d). Remark that $\mathcal{I}_{\text{span}}(u)$ is depicted in Figure 2b. The rightmost sub-figure (e) depicts a *local* self-dual interpolation based on the median operator, $\mathcal{I}_{\text{med}}(u)$; this interpolation is digitally well-composed in 2D but not in nD with $n > 2$ [1]. That contrasts with $\mathfrak{FP} \circ \mathcal{I}_{\text{span}}$, which is a *non-local* interpolation being digitally well-composed in any dimension.

5.2 Considering $\mathfrak{FP} \circ \mathcal{I}_{\text{span}}$ as a Solution

Now, let us consider $u_{\text{DWC}} = (\mathfrak{FP} \circ \mathcal{I}_{\text{span}})(u)$.

Proposition 5 ($\mathfrak{FP} \circ \mathcal{I}_{\text{span}}$ is an nD self-dual DWC interpolation). *Given any nD integer-valued function (gray-level image) $u : \mathcal{D} \subset \mathbb{Z}^n \longrightarrow \mathbb{Z}$, the nD function $(\mathfrak{FP} \circ \mathcal{I}_{\text{span}})(u) : \mathcal{D}_2 \subset (\mathbb{Z}/2)^n \longrightarrow \mathbb{Z}/2$ is a self-dual interpolation of u which is digitally well-composed.*

u_{DWC} is digitally well-composed. We know that $\mathcal{I}_{\text{span}}(u)$ is a digitally well-composed map (see Proposition 4), and that $\mathfrak{FP}$ transforms such a map into a digitally well-composed function (see Theorem 2). Thus u_{DWC} is a digitally well-composed function.

u_{DWC} is an interpolation of u. Since the interpolation $\mathcal{I}_{\text{span}}(u)$ satisfies $\forall z \in \mathcal{D}$, $\mathcal{I}_{\text{span}}(z) = \{u(z)\}$, and since $\mathfrak{FP} : U \to u^\flat$ is such that $\forall z \in \mathcal{D}_2$, $u^\flat(z) \in U(z)$, we can deduce that $u_{\text{DWC}|_{\mathcal{D}}} = u$. In addition, the values of u_{DWC} in $\mathcal{D}_2 \setminus \mathcal{D}$ are set "in-between" the ones of u because they belong to their span. Thus u_{DWC} is effectively an interpolation of u.

$-u_{\text{DWC}}$ is obtained from $-u$. Both $\mathcal{I}_{\text{span}}$ and $\mathfrak{FP}$ are self-dual (see respectively Propositions 4 and 2), so we have $(\mathfrak{FP} \circ \mathcal{I}_{\text{span}})(-u) = -(\mathfrak{FP} \circ \mathcal{I}_{\text{span}})(u)$. The transform $\mathfrak{FP} \circ \mathcal{I}_{\text{span}}$ is therefore self-dual.

As a conclusion, $\mathfrak{FP} \circ \mathcal{I}_{\text{span}}$ is an interpolation method that turns any nD integer-valued function into a DWC function in a self-dual way. Remark that it is a *non-local* interpolation method because the interpolated values are set by the propagation of a front in $\mathfrak{FP}$. An illustration is given by Figure 3.

6 Conclusion

In this paper, we studied several possible extension of the well-composedness concept. In particular, in the framework of digital topology, we prove that digital well-composedness implies the equivalence of connectivities. Based on this study, we have shown that a part of the quasi-linear tree-of-shapes computation algorithm produces an interpolated well-composed image. We think that this result is remarkable, contrasting with the fact that it is not possible to obtain a well-composed image in nD with a local self-dual interpolation [1]. Future work will build on this framework. In particular, the relationships between DWC, AWC, and CWC deserve an in-depth study.

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