A QUASI-LINEAR ALGORITHM TO COMPUTE THE TREE OF SHAPES OF nD IMAGES

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ISMM, Uppsala, May 2013

DUAL TREES

TWO SETS OF CUTS/THRESHOLDS

Given a nD image $u : \mathbb{Z}^n \to \mathbb{Z}$,

$$\begin{array}{ll} \text{lower cuts:} & [\, u < \lambda \,] = \{\, x \in X \mid u(x) < \lambda \,\} \\ \text{upper cuts:} & [\, u \geq \lambda \,] = \{\, x \in X \mid u(x) \geq \lambda \,\} \end{array}$$

Dual trees

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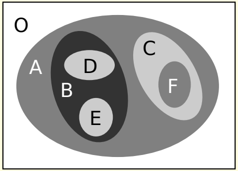
A couple of dual trees

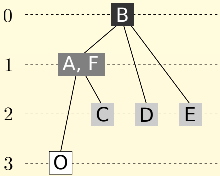
$$\begin{array}{lll} \leadsto & \text{min-tree:} & \mathcal{T}_{<}(u) \;=\; \{\,\Gamma \in \mathcal{CC}_{c_{2n}}([\,u < \lambda\,])\,\}_{\lambda} \\ \leadsto & \text{max-tree:} & \mathcal{T}_{\geq}(u) \;=\; \{\,\Gamma \in \mathcal{CC}_{c_{3^n-1}}([\,u \geq \lambda\,])\,\}_{\lambda} \end{array}$$

A Schematic Example

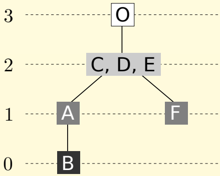
image



max-tree



min-tree



Dealing with trees

Some nice properties

- a tree is a versatile structure
- it is straightforward to deal with them
- and dual trees are easy to compute

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Connected operators

Those operators are powerful:

- they filter (simplify) images while preserving (i.e., do not shift) contours
- some of them are based on component trees

Examples:

- algebraic openings and closings
- levelings
- ...

DUALITY V. SELF-DUALITY

Different ideas:

Duality v. self-duality

Different ideas:

- we can try to be self-dual with two trees, yet...
  - we get some information redundancy between those trees
  - we have to juggle with two structures

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  - when we do not want to make such an assumption

(the "object-background" paradigm is limited...)

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  - when we do not want to make such an assumption

    (the "object-background" paradigm is limited...)

- "object contours belong to the set of level lines"
  - so there are many applications...

One self-dual tree

Shapes

With the cavity-fill-in operator Sat:

$$\begin{aligned} \mathcal{S}_<(u) &= \{\, \mathrm{Sat}_{c_{3^n-1}}(\Gamma);\ \Gamma \in \mathcal{T}_<(u) \,\} \\ \mathcal{S}_\geq(u) &= \{\, \mathrm{Sat}_{c_{2n}}(\Gamma);\ \Gamma \in \mathcal{T}_\geq(u) \,\} \end{aligned}$$

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$$

A self-dual tree

$\rightsquigarrow$ tree of shapes: $\mathfrak{S}(u) = \mathcal{S}_{<}(u) \cup \mathcal{S}_{\geq}(u)$

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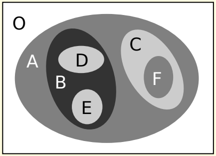
$\rightsquigarrow$ tree of shapes: $\quad \mathfrak{S}(u) \;=\; \mathcal{S}_<(u) \cup \mathcal{S}_\geq(u)$

Property

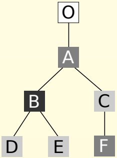
we have $\mathfrak{S}(-u) \;=\; \mathfrak{S}(u)$ (whereas $\mathcal{T}_\geq(-u) \;=\; \mathcal{T}_<(u)$)

Schematic example

image



tree of shapes



Schematic example

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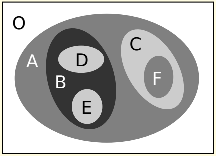

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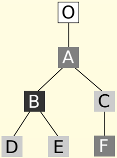
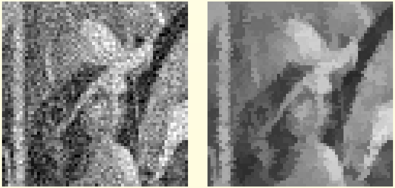


Alt. definitions of shapes

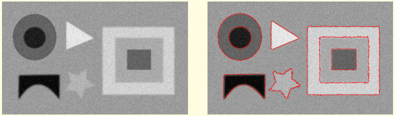
the shapes are

- the cavities of upper and lower cuts
- the interior regions of level lines.

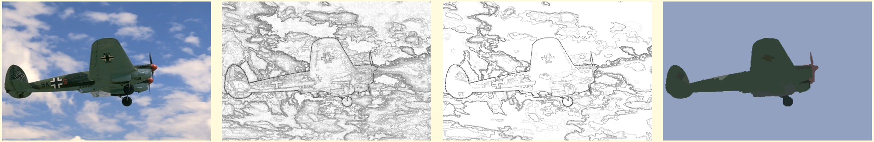
APPLICATIONS 1/2



grain filter ($\approx (\phi\gamma + \gamma\phi)/2$)



object detection



contour saliency $\rightsquigarrow$ extinction $\rightsquigarrow$ (hierarchical) segmentation

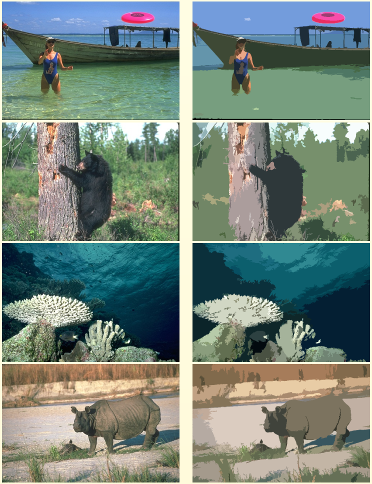
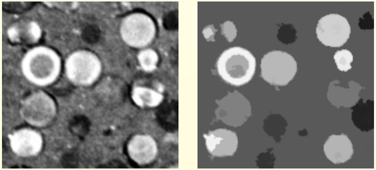


image simplification



morphological “shaping”

local feature detection

Now, how to...

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Several issues:

- their time complexity is $O(N^2)$... $\leftarrow$ that is bad
- they are unusable for nD images (limited to 2D images) $\leftarrow$ that is a pity
- they are hard to implement

$\rightsquigarrow$ so eventually they are merely used!

A two-pass algorithm

To compute the max-tree or the min-tree, we have a two-pass algorithm:

- Najman & Couprie, *Quasi-Linear Algorithm for the Component Tree*, IS&T/SPIE Symp. on E.I., 2004
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The two steps

1. sort the pixels in the *descending tree order*
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THE TWO STEPS

1. sort the pixels in the *descending tree order*
2. in the *reverse order*, rely on the Union-Find algorithm to compute the tree

Algorithm properties:

- quasi-linear time complexity for low quantized data
- works on nD images
- very easy to implement

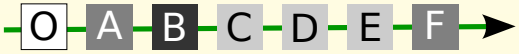
A 1ST KEY IDEA

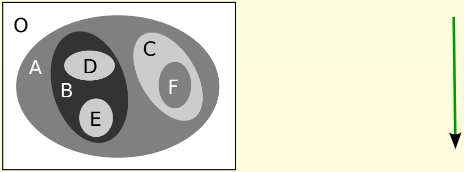
FIRST KEY IDEA

if we succeed in sorting the pixels such as descending the tree of shapes,
then we have a simple and efficient algorithm.

Computing the tree of shapes: 2nd pass

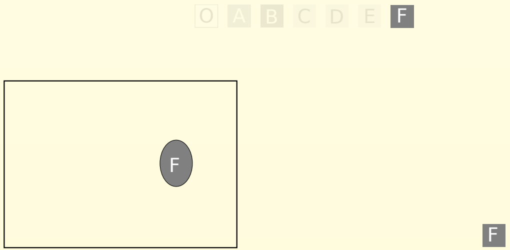
sort :





Now let us browse the pixels in the reserve sort order...

Computing the tree of shapes: 2nd pass



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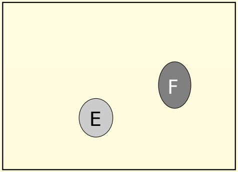

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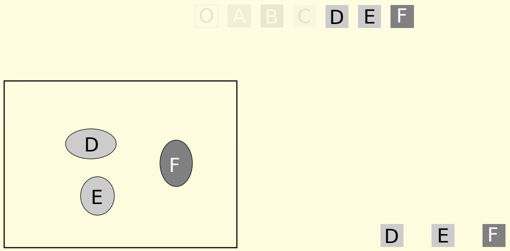

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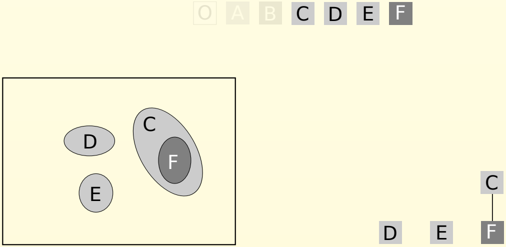
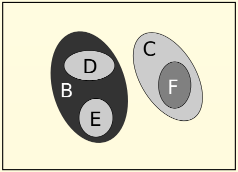
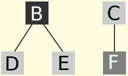


Computing the tree of shapes: 2nd pass

O A B C D E F





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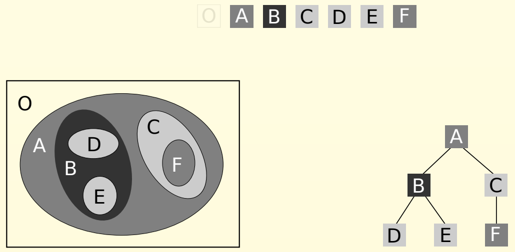

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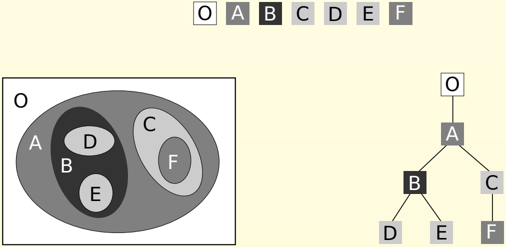

Done for the tree construction (2nd pass)!

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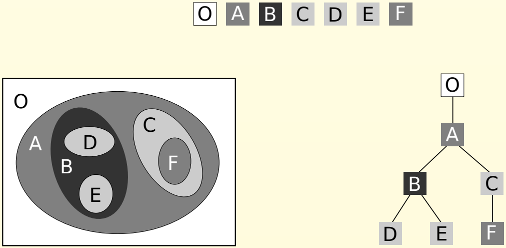

Done for the tree construction (2nd pass)!

Now we have to know how to sort pixels (1st pass).

Towards the 2nd key idea

About the 1st pass

sorting the pixels means progress "continuously"
both in *image space*[1] and in *value space*[2]

(starting from the image boundary, i.e., the root node)

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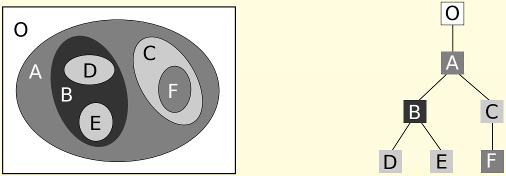
[1] through a spatially consistent growing
[2] jumping from a gray level to the "next" one (either upper or lower)
⇒ we can use a *propagation front* and a *hierarchical queue*

That seems easy... (unfortunately we'll see that is not!)

Running the 1st pass

sort : O A B C D E F



To sort pixels we start from the image boundary...

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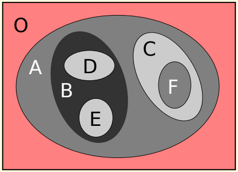

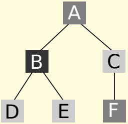

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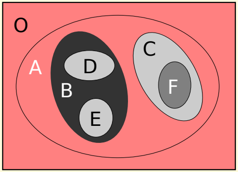

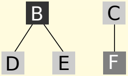

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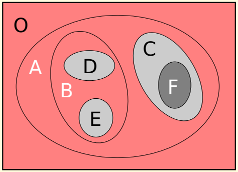

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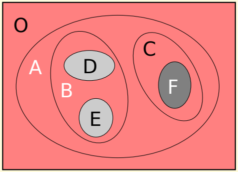

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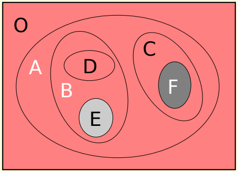

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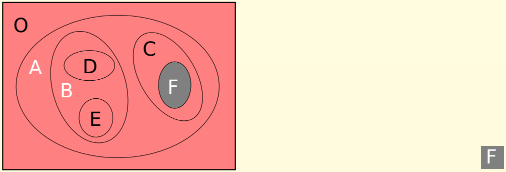

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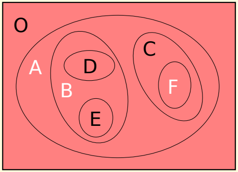
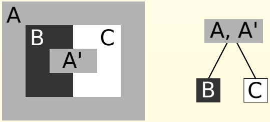


Done! Indeed, sorting pixels seems easy...

Towards the 2nd key idea



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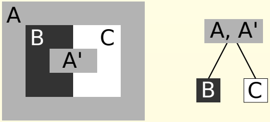
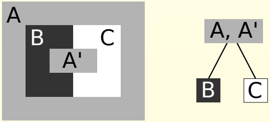
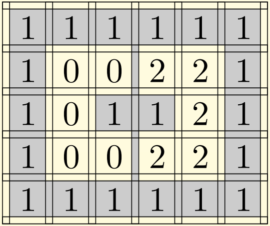


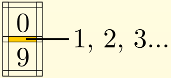
1	1	1	1	1	1
1	0	0	2	2	1
1	0	1	1	2	1
1	0	0	2	2	1
1	1	1	1	1	1

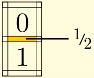
we need to pass between pixels...

Towards the 2nd key idea









we need to pass between pixels...

...and with many values

To progress correctly both in image space and in value space...

SECOND KEY IDEA

...we need a discrete image representation
that has some appropriate continuous properties!

THE (IMPOSSIBLE?) 2ND KEY IDEA

To progress correctly both in image space and in value space...

SECOND KEY IDEA

...we need a <u>discrete</u> image representation
that has some appropriate <u>continuous</u> properties!

To that aim a couple of tools are required:

- cubical complexes / Khalimsky grid
- set-valued maps

The nD space of cubical complexes:

$$H_0^1 = \{\, \{a\};\, a \in \mathbb{Z} \,\} \qquad H_1^1 = \{\, \{a, a+1\};\, a \in \mathbb{Z} \,\}$$

$$H^1 = H_0^1 \cup H_1^1 \qquad H^n = \times_n \, H^1$$

Cubical Complexes / Khalimsky's Grid

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$$H^1 = H_0^1 \cup H_1^1 \qquad H^n = \times_n\, H^1$$

Consider $h \in H^n$ being the $\times$ product of d elements of H_1^1 and $n - d$ elements of H_0^1

- we have $h \subset \mathbb{Z}^n$
- we say that h is a d-face and that d is the dimension of h

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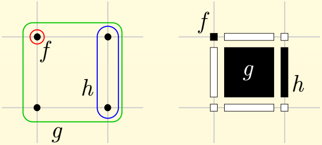

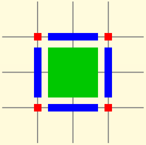

Two representations of a set of faces... ...and Khalimsky's grid.

With $h^{\uparrow} = \{ h' \in H^n \mid h \subseteq h' \}$ and $h^{\downarrow} = \{ h' \in H^n \mid h' \subseteq h \}$:

- the pair $(H^n, \subseteq)$ forms a poset,
- the set $\mathcal{U} = \{ U \subseteq H^n \mid \forall h \in U,\ h^{\uparrow} \subseteq U \}$ is a T0-Alexandroff topology on H^n.

Cubical Complexes / Khalimsky's Grid

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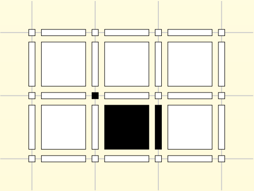
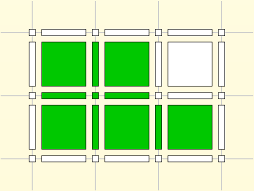
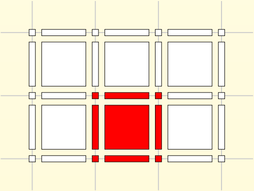
so we have some topological operators:



$E = \{f,\, g,\, h\}$



star: $E^{\uparrow}$



closure: $E^{\downarrow}$

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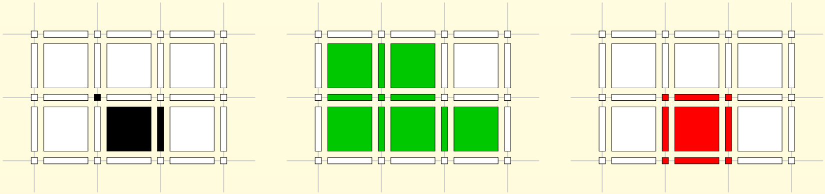

$E = \{f,\ g,\ h\}$ star: $E^{\uparrow}$ closure: $E^{\downarrow}$

and an easy and effective structure to work on...

Set-valued maps

A set-valued map U

$\text{U} : X \rightsquigarrow Y$ is characterized by its graph:

$$\mathrm{Gra}(\text{U}) = \{ (x, y) \in X \times Y \mid y \in \text{U}(x) \}$$

actually we have $\text{U} : X \to \mathcal{P}(Y)$

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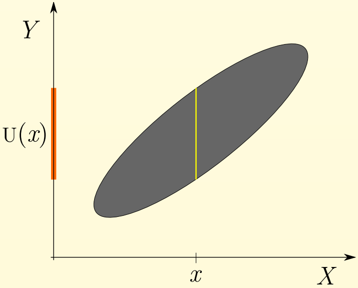

Set-valued maps

Continuity:

- when $\mathrm{U}(x)$ is compact, U is u.s.c. at x if

$$\forall \varepsilon > 0,\ \exists\, \eta > 0 \text{ such that } \forall\, x' \in B_X(x,\eta),\ \mathrm{U}(x') \subset B_Y(\mathrm{U}(x),\varepsilon).$$

- U is u.s.c. iif $\forall x \in X,\ \mathrm{U}$ is u.s.c. at x
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$$\text{the } \textit{core} \text{ of } M \subset Y \text{ by } \mathrm{U} \text{ is } \{\, x \in X \,|\, \mathrm{U}(x) \subset M \,\}$$

SET-VALUED MAPS

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Inverse:

the *core* of $M \subset Y$ by U is $\{\, x \in X \,|\, \text{U}(x) \subset M \,\}$

A continuity characterization:

U is U.S.C. if and only if the core of any open subset is open

RECAP

Recap:

- we want a continuous propagation in order to sort pixels
- we can rely
  - on cellular complexes
    for a propagation front to be able pass between pixels
  - and on set-valued maps
    to model that we can have several level lines passing between pixels

RECAP

Recap:

- we want a continuous propagation in order to sort pixels
- we can rely
  - on cellular complexes
    for a propagation front to be able pass between pixels
  - and on set-valued maps
    to model that we can have several level lines passing between pixels

At that point:

- we have to change u into $\mathrm{U} = \mathfrak{I}(u)$
  so that $\mathfrak{I}(u)$ *is suitable to compute the tree of shapes*
- we need to define <u>cuts of set-valued maps</u>
  so that we can have the notion of shapes of $\mathfrak{I}(u)$

CUTS OF SET-VALUED MAPS NEW!

definition of large cuts:

$$
\begin{aligned}
[\,\mathrm{U} \trianglelefteq \lambda\,] &= \{\, x \in X \mid \exists\, \mu \in \mathrm{U}(x),\ \mu \leq \lambda \,\} \\
[\,\mathrm{U} \trianglerighteq \lambda\,] &= \{\, x \in X \mid \exists\, \mu \in \mathrm{U}(x),\ \mu \geq \lambda \,\}
\end{aligned}
$$

Cuts of set-valued maps NEW!

definition of large cuts:

$$\begin{aligned} [\,\mathrm{U} \trianglelefteq \lambda\,] &= \{\, x \in X \,|\, \exists\, \mu \in \mathrm{U}(x),\ \mu \leq \lambda \,\} \\ [\,\mathrm{U} \trianglerighteq \lambda\,] &= \{\, x \in X \,|\, \exists\, \mu \in \mathrm{U}(x),\ \mu \geq \lambda \,\} \end{aligned}$$

by extension we define:

$$\begin{aligned} [\,\mathrm{U} \triangleleft \lambda\,] &= X \setminus [\,\mathrm{U} \trianglerighteq \lambda\,] \\ [\,\mathrm{U} \triangleright \lambda\,] &= X \setminus [\,\mathrm{U} \trianglelefteq \lambda\,] \\ [\,\mathrm{U} \,\square\, \lambda\,] &= [\,\mathrm{U} \trianglelefteq \lambda\,] \cap [\,\mathrm{U} \trianglerighteq \lambda\,] \\ [\,\mathrm{U} \boxslash \lambda\,] &= X \setminus [\,\mathrm{U} \,\square\, \lambda\,] \end{aligned}$$

Cuts of set-valued maps NEW!

definition of large cuts:

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by extension we define:

$$\begin{aligned} [\,\mathrm{U} \lhd \lambda\,] &= X \setminus [\,\mathrm{U} \trianglerighteq \lambda\,] \\ [\,\mathrm{U} \rhd \lambda\,] &= X \setminus [\,\mathrm{U} \trianglelefteq \lambda\,] \\ [\,\mathrm{U} \,\square\, \lambda\,] &= [\,\mathrm{U} \trianglelefteq \lambda\,] \cap [\,\mathrm{U} \trianglerighteq \lambda\,] \\ [\,\mathrm{U} \boxslash \lambda\,] &= X \setminus [\,\mathrm{U} \,\square\, \lambda\,] \end{aligned}$$

so we have:

$$\begin{aligned} [\,\mathrm{U} \lhd \lambda\,] &= \{\, x \in X \,|\, \forall\, \mu \in \mathrm{U}(x),\ \mu < \lambda \,\} \\ [\,\mathrm{U} \rhd \lambda\,] &= \{\, x \in X \,|\, \forall\, \mu \in \mathrm{U}(x),\ \mu > \lambda \,\} \\ [\,\mathrm{U} \,\square\, \lambda\,] &= \{\, x \in X \,|\, \lambda \in \mathrm{U}(x) \,\} \\ [\,\mathrm{U} \boxslash \lambda\,] &= \{\, x \in X \,|\, \lambda \notin \mathrm{U}(x) \,\} \end{aligned}$$

IMAGE REPRESENTATION NEW!

From u to $\mathfrak{I}(u)$:

4	2
1	8

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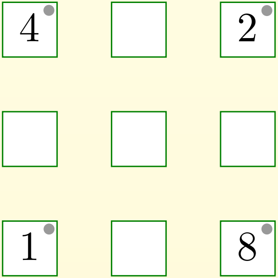
4	2
1	8



we subdivide $\mathbb{Z}^n$ into $\frac{1}{2}\mathbb{Z}^n$

From u to $\mathfrak{I}(u)$:

4	2
1	8

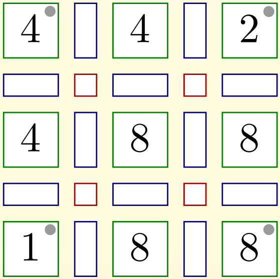
4	4	2
4	8	8
1	8	8

the `max` operator allows to “simulate” the c_{2n} / c_{3^n-1} connectivities

IMAGE REPRESENTATION NEW!

From u to $\mathfrak{I}(u)$:

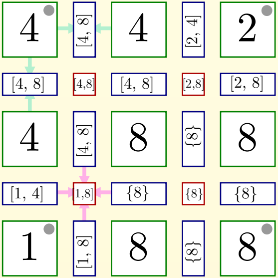
4	2
1	8



from $\frac{1}{2}\mathbb{Z}^n$ to $\frac{1}{2}H^n$ (introducing d-faces, $d < n$)

From u to $\mathfrak{I}(u)$:

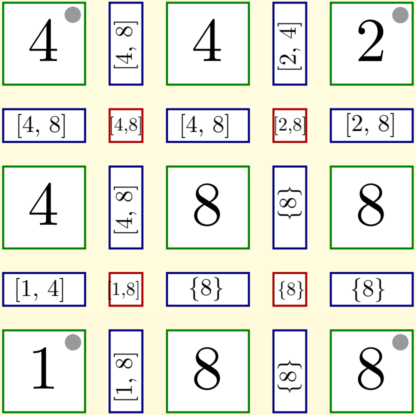
4	2
1	8



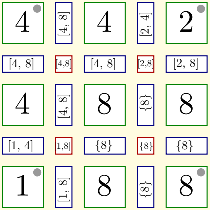
the span operator makes this set-valued map continuous

TESTING CONTINUITY

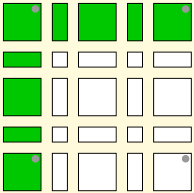
Get a pen, than test continuity of this image:



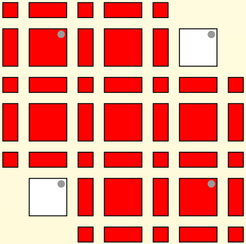
Testing cuts



$[\,\mathfrak{I}(u) \;\lhd\; 5\,]$



$[\,\mathfrak{I}(u) \;\trianglerighteq\; 3\,]$



with u

dual trees:

$$\mathcal{T}_{<}(u) \;=\; \{\, \Gamma \in \mathcal{CC}_{c_{2n}}([\, u < \lambda \,]) \,\}_{\lambda}$$

$$\mathcal{T}_{\geq}(u) \;=\; \{\, \Gamma \in \mathcal{CC}_{c_{3^n-1}}([\, u \geq \lambda \,]) \,\}_{\lambda}$$

shapes:

$$\mathcal{S}_{<}(u) \;=\; \{\, \mathrm{Sat}_{c_{3^n-1}}(\Gamma);\; \Gamma \in \mathcal{T}_{<}(u) \,\}$$

$$\mathcal{S}_{\geq}(u) \;=\; \{\, \mathrm{Sat}_{c_{2n}}(\Gamma);\; \Gamma \in \mathcal{T}_{\geq}(u) \,\}$$

tree of shapes:

$$\mathfrak{S}(u) \;=\; \mathcal{S}_{<}(u) \,\cup\, \mathcal{S}_{\geq}(u)$$

Analogies

with u | with $\mathrm{U} = \mathfrak{I}(u)$

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with u | with $\mathrm{U} = \mathfrak{I}(u)$

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$\mathfrak{S}(u) = \mathcal{S}_<(u) \cup \mathcal{S}_\geq(u)$ | $\mathfrak{S}(\mathrm{U}) = \mathcal{S}_\lhd(\mathrm{U}) \cup \mathcal{S}_\rhd(\mathrm{U})$

We have

- components and shapes of U are open sets
- for any tree, couples of components coincide: $\Gamma_\mathrm{U} \cap \mathbb{Z}^n = \Gamma_u$
- $\mathfrak{S}(u) = \mathfrak{S}(\mathrm{U})\,|_{\mathbb{Z}^n}$

Some direct consequences

Eventually we have:

$$\mathfrak{S}(u) \; = \; \{\, \Gamma \, \cap \, \mathbb{Z}^n \, ; \; \Gamma \in \mathfrak{S}(\mathfrak{I}(u)) \,\}$$

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- represent u by $\mathfrak{I}(u)$
- run a “continuous propagation” on $\mathfrak{I}(u)$ to <u>sort</u> faces (1st step)
- run a Union-Find-based computation of the tree (2nd step)
- clean-up the tree, i.e., keep $\mathbb{Z}^n$ elements only

SOME DIRECT CONSEQUENCES

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We have a proof of this algorithm.

ASCII ART: PROPAGATION / SORTING (1ST PASS)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
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```

ASCII art: propagation / sorting (1st pass)

```
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# | x |                               | x |       | x | #
# + - +                               + - +       + - + #
# | O |                               | x |       | O | #
# + - + - + - + - +                   + - +       + - + #
# | x | x | x | x |                   | x |       | x | #
# + - + - + - + - +                   + - +       + - + #
# | O |       | x |                   | x |       | O | #
# + - +       + - + - + - +   + - + - + - + - + - + - + #
# | x |       | x | x | x |   | x | x | x | x | x | x | #
# + - +       + - + - + - +   + - + - + - + - + - + - + #
# | O |               | x |   | x |   | x |       | O | #
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# | O |                               | x |       | O | #
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# | x |               | x | x | x | x | x | x | x | x | #
# + - +               + - + - + - + - + - + - + - + - + #
# | O |               | x |           | x |       | O | #
# + - +               + - +           + - +       + - + #
# | x |               | x |           | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
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```

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# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
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# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O | x |                           | x |       | O | #
# + - + - + - + - +                   + - +       + - + #
# | x | x | x | x |                   | x |       | x | #
# + - + - + - + - +                   + - +       + - + #
# | O |       | x |                   | x |       | O | #
# + - +       + - + - + - +   + - + - + - + - + - + - + #
# | x |       | x | x | x |   | x | x | x | x | x | x | #
# + - +       + - + - + - +   + - + - + - + - + - + - + #
# | O |               | x |   | x |   | x |   | x | O | #
# + - +               + - + - + - +   + - +   + - + - + #
# | x |               | x | x | x |   | x |   | x | x | #
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# | O |                               | x |   | x | O | #
# + - +               + - + - + - + - + - + - + - + - + #
# | x |               | x | x | x | x | x | x | x | x | #
# + - +               + - + - + - + - + - + - + - + - + #
# | O |               | x |           | x |       | O | #
# + - +               + - + - + - + - + - +       + - + #
# | x |               | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
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# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: propagation / sorting (1st pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O | x |       | O |               | x |       | O | #
# + - + - + - + - + - +               + - +       + - + #
# | x | x | x | x |                   | x |       | x | #
# + - + - + - + - +                   + - +       + - + #
# | O |       | x |                   | x |       | O | #
# + - +       + - + - + - +   + - + - + - + - + - + - + #
# | x |       | x | x | x |   | x | x | x | x | x | x | #
# + - +       + - + - + - +   + - + - + - + - + - + - + #
# | O |               | x |   | x |   | x |   | x | O | #
# + - +               + - + - + - +   + - +   + - + - + #
# | x |               | x | x | x |   | x |   | x | x | #
# + - +               + - + - + - +   + - +   + - + - + #
# | O |                               | x |   | x | O | #
# + - +               + - + - + - + - + - + - + - + - + #
# | x |               | x | x | x | x | x | x | x | x | #
# + - +               + - + - + - + - + - + - + - + - + #
# | O |               | x |           | x |       | O | #
# + - +               + - + - + - + - + - +       + - + #
# | x |               | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: propagation / sorting (1st pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O | x |   | x | O | x |           | x |       | O | #
# + - + - + - + - + - + - +           + - +       + - + #
# | x | x | x | x | x | x |           | x |       | x | #
# + - + - + - + - + - + - +           + - +       + - + #
# | O |       | x |                   | x |       | O | #
# + - +       + - + - + - +   + - + - + - + - + - + - + #
# | x |       | x | x | x |   | x | x | x | x | x | x | #
# + - +       + - + - + - +   + - + - + - + - + - + - + #
# | O |               | x |   | x |   | x |   | x | O | #
# + - +               + - + - + - +   + - +   + - + - + #
# | x |               | x | x | x |   | x |   | x | x | #
# + - +               + - + - + - +   + - +   + - + - + #
# | O |                               | x |   | x | O | #
# + - +               + - + - + - + - + - + - + - + - + #
# | x |               | x | x | x | x | x | x | x | x | #
# + - +               + - + - + - + - + - + - + - + - + #
# | O |               | x |           | x |       | O | #
# + - +               + - + - + - + - + - +       + - + #
# | x |               | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: propagation / sorting (1st pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O | x |   | x | O | x |           | x |       | O | #
# + - + - + - + - + - + - +           + - +       + - + #
# | x | x | x | x | x | x |           | x |       | x | #
# + - + - + - + - + - + - +           + - +       + - + #
# | O |       | x |                   | x |       | O | #
# + - +       + - + - + - +   + - + - + - + - + - + - + #
# | x |       | x | x | x |   | x | x | x | x | x | x | #
# + - +       + - + - + - +   + - + - + - + - + - + - + #
# | O |               | x |   | x |   | x |   | x | O | #
# + - +               + - + - + - +   + - +   + - + - + #
# | x |               | x | x | x |   | x |   | x | x | #
# + - +               + - + - + - +   + - +   + - + - + #
# | O |                               | x |   | x | O | #
# + - +               + - + - + - + - + - + - + - + - + #
# | x |               | x | x | x | x | x | x | x | x | #
# + - +               + - + - + - + - + - + - + - + - + #
# | O |               | x | O |       | x |       | O | #
# + - +               + - + - + - + - + - +       + - + #
# | x |               | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII ART: PROPAGATION / SORTING (1ST PASS)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O | x |   | x | O | x |           | x |       | O | #
# + - + - + - + - + - + - +           + - +       + - + #
# | x | x | x | x | x | x |           | x |       | x | #
# + - + - + - + - + - + - +           + - +       + - + #
# | O |       | x |                   | x |       | O | #
# + - +       + - + - + - +   + - + - + - + - + - + - + #
# | x |       | x | x | x |   | x | x | x | x | x | x | #
# + - +       + - + - + - +   + - + - + - + - + - + - + #
# | O |               | x |   | x |   | x |   | x | O | #
# + - +               + - + - + - +   + - +   + - + - + #
# | x |               | x | x | x |   | x |   | x | x | #
# + - +               + - + - + - +   + - +   + - + - + #
# | O |                               | x |   | x | O | #
# + - +               + - + - + - + - + - + - + - + - + #
# | x |               | x | x | x | x | x | x | x | x | #
# + - +               + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x |   | x |       | O | #
# + - +               + - + - + - + - + - +       + - + #
# | x |               | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: propagation / sorting (1st pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O | x | O | x | O | x |           | x |       | O | #
# + - + - + - + - + - + - +           + - +       + - + #
# | x | x | x | x | x | x |           | x |       | x | #
# + - + - + - + - + - + - +           + - +       + - + #
# | O |       | x | O |               | x |       | O | #
# + - +       + - + - + - +   + - + - + - + - + - + - + #
# | x |       | x | x | x |   | x | x | x | x | x | x | #
# + - +       + - + - + - +   + - + - + - + - + - + - + #
# | O |               | x |   | x |   | x |   | x | O | #
# + - +               + - + - + - +   + - +   + - + - + #
# | x |               | x | x | x |   | x |   | x | x | #
# + - +               + - + - + - +   + - +   + - + - + #
# | O |                               | x |   | x | O | #
# + - +               + - + - + - + - + - + - + - + - + #
# | x |               | x | x | x | x | x | x | x | x | #
# + - +               + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x |   | x |       | O | #
# + - +               + - + - + - + - + - +       + - + #
# | x |               | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: propagation / sorting (1st pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O | x | O | x | O | x |           | x |       | O | #
# + - + - + - + - + - + - +           + - +       + - + #
# | x | x | x | x | x | x |           | x |       | x | #
# + - + - + - + - + - + - +           + - +       + - + #
# | O |       | x | O | x |           | x |       | O | #
# + - +       + - + - + - +   + - + - + - + - + - + - + - + #
# | x |       | x | x | x |   | x | x | x | x | x | x | #
# + - +       + - + - + - +   + - + - + - + - + - + - + - + #
# | O |               | x |   | x |   | x |   | x | O | #
# + - +               + - + - + - +   + - +   + - + - + #
# | x |               | x | x | x |   | x |   | x | x | #
# + - +               + - + - + - +   + - +   + - + - + #
# | O |                               | x |   | x | O | #
# + - +               + - + - + - + - + - + - + - + - + #
# | x |               | x | x | x | x | x | x | x | x | #
# + - +               + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x |   | x |       | O | #
# + - +               + - + - + - + - + - +       + - + #
# | x |               | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: propagation / sorting (1st pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O | x | O | x | O | x |           | x |       | O | #
# + - + - + - + - + - + - +           + - +       + - + #
# | x | x | x | x | x | x |           | x |       | x | #
# + - + - + - + - + - + - +           + - +       + - + #
# | O |       | x | O | x |           | x |       | O | #
# + - +       + - + - + - +   + - + - + - + - + - + - + #
# | x |       | x | x | x |   | x | x | x | x | x | x | #
# + - +       + - + - + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x |   | x |   | x | O | #
# + - +               + - + - + - +   + - +   + - + - + #
# | x |               | x | x | x |   | x |   | x | x | #
# + - +               + - + - + - +   + - +   + - + - + #
# | O |                               | x |   | x | O | #
# + - +               + - + - + - + - + - + - + - + - + #
# | x |               | x | x | x | x | x | x | x | x | #
# + - +               + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x |   | x |       | O | #
# + - +               + - + - + - + - + - +       + - + #
# | x |               | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: propagation / sorting (1st pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O | x | O | x | O | x |           | x |       | O | #
# + - + - + - + - + - + - +           + - +       + - + #
# | x | x | x | x | x | x |           | x |       | x | #
# + - + - + - + - + - + - +           + - +       + - + #
# | O |       | x | O | x |           | x |       | O | #
# + - +       + - + - + - + - + - + - + - + - + - + - + #
# | x |       | x | x | x | x | x | x | x | x | x | x | #
# + - +       + - + - + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x |   | x |   | x | O | #
# + - +               + - + - + - +   + - +   + - + - + #
# | x |               | x | x | x |   | x |   | x | x | #
# + - +               + - + - + - +   + - +   + - + - + #
# | O |                               | x |   | x | O | #
# + - +               + - + - + - + - + - + - + - + - + #
# | x |               | x | x | x | x | x | x | x | x | #
# + - +               + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x |   | x |       | O | #
# + - +               + - + - + - + - + - +       + - + #
# | x |               | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII ART: PROPAGATION / SORTING (1ST PASS)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O | x | O | x | O | x |       | O | x |       | O | #
# + - + - + - + - + - + - +       + - + - +       + - + #
# | x | x | x | x | x | x |           | x |       | x | #
# + - + - + - + - + - + - +           + - +       + - + #
# | O |       | x | O | x |           | x |       | O | #
# + - +       + - + - + - + - + - + - + - + - + - + - + #
# | x |       | x | x | x | x | x | x | x | x | x | x | #
# + - +       + - + - + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x |   | x |   | x | O | #
# + - +               + - + - + - +   + - +   + - + - + #
# | x |               | x | x | x |   | x |   | x | x | #
# + - +               + - + - + - +   + - +   + - + - + #
# | O |                               | x |   | x | O | #
# + - +               + - + - + - + - + - + - + - + - + #
# | x |               | x | x | x | x | x | x | x | x | #
# + - +               + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x |   | x |       | O | #
# + - +               + - + - + - + - + - +       + - + #
# | x |               | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII ART: PROPAGATION / SORTING (1ST PASS)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O | x | O | x | O | x |   | x | O | x |       | O | #
# + - + - + - + - + - + - +   + - + - + - +       + - + #
# | x | x | x | x | x | x |   | x | x | x |       | x | #
# + - + - + - + - + - + - +   + - + - + - +       + - + #
# | O |       | x | O | x |           | x |       | O | #
# + - +       + - + - + - + - + - + - + - + - + - + - + #
# | x |       | x | x | x | x | x | x | x | x | x | x | #
# + - +       + - + - + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x |   | x |   | x | O | #
# + - +               + - + - + - +   + - +   + - + - + #
# | x |               | x | x | x |   | x |   | x | x | #
# + - +               + - + - + - +   + - +   + - + - + #
# | O |                               | x |   | x | O | #
# + - +               + - + - + - + - + - + - + - + - + #
# | x |               | x | x | x | x | x | x | x | x | #
# + - +               + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x |   | x |       | O | #
# + - +               + - + - + - + - + - +       + - + #
# | x |               | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O | x | O | x | O | x | O | x | O | x |       | O | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | x | x | x | x | x | x |   | x | x | x |       | x | #
# + - + - + - + - + - + - +   + - + - + - +       + - + #
# | O |       | x | O | x |           | x |       | O | #
# + - +       + - + - + - + - + - + - + - + - + - + - + #
# | x |       | x | x | x | x | x | x | x | x | x | x | #
# + - +       + - + - + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x |   | x |   | x | O | #
# + - +               + - + - + - +   + - +   + - + - + #
# | x |               | x | x | x |   | x |   | x | x | #
# + - +               + - + - + - +   + - +   + - + - + #
# | O |                               | x |   | x | O | #
# + - +               + - + - + - + - + - + - + - + - + #
# | x |               | x | x | x | x | x | x | x | x | #
# + - +               + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x |   | x |       | O | #
# + - +               + - + - + - + - + - +       + - + #
# | x |               | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: propagation / sorting (1st pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O | x | O | x | O | x | O | x | O | x |       | O | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O |       | x | O | x |           | x |       | O | #
# + - +       + - + - + - + - + - + - + - + - + - + - + #
# | x |       | x | x | x | x | x | x | x | x | x | x | #
# + - +       + - + - + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x |   | x |   | x | O | #
# + - +               + - + - + - +   + - +   + - + - + #
# | x |               | x | x | x |   | x |   | x | x | #
# + - +               + - + - + - +   + - +   + - + - + #
# | O |                               | x |   | x | O | #
# + - +               + - + - + - + - + - + - + - + - + #
# | x |               | x | x | x | x | x | x | x | x | #
# + - +               + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x |   | x |       | O | #
# + - +               + - + - + - + - + - +       + - + #
# | x |               | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O | x | O | x | O | x | O | x | O | x |       | O | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O |       | x | O | x |           | x |       | O | #
# + - +       + - + - + - + - + - + - + - + - + - + - + #
# | x |       | x | x | x | x | x | x | x | x | x | x | #
# + - +       + - + - + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x |   | x | O | x | O | #
# + - +               + - + - + - +   + - + - + - + - + #
# | x |               | x | x | x |   | x |   | x | x | #
# + - +               + - + - + - +   + - +   + - + - + #
# | O |                               | x |   | x | O | #
# + - +               + - + - + - + - + - + - + - + - + #
# | x |               | x | x | x | x | x | x | x | x | #
# + - +               + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x |   | x |       | O | #
# + - +               + - + - + - + - + - +       + - + #
# | x |               | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII ART: PROPAGATION / SORTING (1ST PASS)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O | x | O | x | O | x | O | x | O | x |       | O | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O |       | x | O | x |           | x |       | O | #
# + - +       + - + - + - + - + - + - + - + - + - + - + #
# | x |       | x | x | x | x | x | x | x | x | x | x | #
# + - +       + - + - + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x |   | x | O | x | O | #
# + - +               + - + - + - +   + - + - + - + - + #
# | x |               | x | x | x |   | x | x | x | x | #
# + - +               + - + - + - +   + - + - + - + - + #
# | O |                               | x |   | x | O | #
# + - +               + - + - + - + - + - + - + - + - + #
# | x |               | x | x | x | x | x | x | x | x | #
# + - +               + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x |   | x |       | O | #
# + - +               + - + - + - + - + - +       + - + #
# | x |               | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O | x | O | x | O | x | O | x | O | x |       | O | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O |       | x | O | x | O |       | x |       | O | #
# + - +       + - + - + - + - + - + - + - + - + - + - + #
# | x |       | x | x | x | x | x | x | x | x | x | x | #
# + - +       + - + - + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x |   | x | O | x | O | #
# + - +               + - + - + - +   + - + - + - + - + #
# | x |               | x | x | x |   | x | x | x | x | #
# + - +               + - + - + - +   + - + - + - + - + #
# | O |                               | x |   | x | O | #
# + - +               + - + - + - + - + - + - + - + - + #
# | x |               | x | x | x | x | x | x | x | x | #
# + - +               + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x |   | x |       | O | #
# + - +               + - + - + - + - + - +       + - + #
# | x |               | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII ART: PROPAGATION / SORTING (1ST PASS)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O | x | O | x | O | x | O | x | O | x |       | O | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O |       | x | O | x | O | x |   | x |       | O | #
# + - +       + - + - + - + - + - + - + - + - + - + - + #
# | x |       | x | x | x | x | x | x | x | x | x | x | #
# + - +       + - + - + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x |   | x | O | x | O | #
# + - +               + - + - + - +   + - + - + - + - + #
# | x |               | x | x | x |   | x | x | x | x | #
# + - +               + - + - + - +   + - + - + - + - + #
# | O |                               | x |   | x | O | #
# + - +               + - + - + - + - + - + - + - + - + #
# | x |               | x | x | x | x | x | x | x | x | #
# + - +               + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x |   | x |       | O | #
# + - +               + - + - + - + - + - +       + - + #
# | x |               | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII ART: PROPAGATION / SORTING (1ST PASS)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O | x | O | x | O | x | O | x | O | x |       | O | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O |       | x | O | x | O | x |   | x |       | O | #
# + - +       + - + - + - + - + - + - + - + - + - + - + #
# | x |       | x | x | x | x | x | x | x | x | x | x | #
# + - +       + - + - + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x |   | x | O | x | O | #
# + - +               + - + - + - +   + - + - + - + - + #
# | x |               | x | x | x |   | x | x | x | x | #
# + - +               + - + - + - +   + - + - + - + - + #
# | O |                               | x |   | x | O | #
# + - +               + - + - + - + - + - + - + - + - + #
# | x |               | x | x | x | x | x | x | x | x | #
# + - +               + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x | O | x |       | O | #
# + - +               + - + - + - + - + - +       + - + #
# | x |               | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: propagation / sorting (1st pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O | x | O | x | O | x | O | x | O | x |       | O | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O |       | x | O | x | O | x | O | x |       | O | #
# + - +       + - + - + - + - + - + - + - + - + - + - + #
# | x |       | x | x | x | x | x | x | x | x | x | x | #
# + - +       + - + - + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x |   | x | O | x | O | #
# + - +               + - + - + - +   + - + - + - + - + #
# | x |               | x | x | x |   | x | x | x | x | #
# + - +               + - + - + - +   + - + - + - + - + #
# | O |                               | x |   | x | O | #
# + - +               + - + - + - + - + - + - + - + - + #
# | x |               | x | x | x | x | x | x | x | x | #
# + - +               + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x | O | x |       | O | #
# + - +               + - + - + - + - + - +       + - + #
# | x |               | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O | x | O | x | O | x | O | x | O | x |       | O | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | x | x | x | x | x | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - +       + - + #
# | O |       | x | O | x | O | x | O | x |       | O | #
# + - +       + - + - + - + - + - + - + - + - + - + - + #
# | x |       | x | x | x | x | x | x | x | x | x | x | #
# + - +       + - + - + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x |   | x | O | x | O | #
# + - +               + - + - + - +   + - + - + - + - + #
# | x |               | x | x | x |   | x | x | x | x | #
# + - +               + - + - + - +   + - + - + - + - + #
# | O |                               | x | O | x | O | #
# + - +               + - + - + - + - + - + - + - + - + #
# | x |               | x | x | x | x | x | x | x | x | #
# + - +               + - + - + - + - + - + - + - + - + #
# | O |               | x | O | x | O | x |       | O | #
# + - +               + - + - + - + - + - +       + - + #
# | x |               | x | x | x | x | x |       | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII ART: PROPAGATION / SORTING (1ST PASS)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x |   | x | O | #
# + - + - + - + - + - + - + - + - + - + - +   + - + - + #
# | x | x | x | x | x | x | x | x | x | x |   | x | x | #
# + - + - + - + - + - + - + - + - + - + - +   + - + - + #
# | O | x |   | x | O | x | O | x | O | x |   | x | O | #
# + - + - +   + - + - + - + - + - + - + - + - + - + - + #
# | x | x |   | x | x | x | x | x | x | x | x | x | x | #
# + - + - +   + - + - + - + - + - + - + - + - + - + - + #
# | O | x |           | x | O | x |   | x | O | x | O | #
# + - + - +           + - + - + - +   + - + - + - + - + #
# | x | x |           | x | x | x |   | x | x | x | x | #
# + - + - +           + - + - + - +   + - + - + - + - + #
# | O | x |                           | x | O | x | O | #
# + - + - +           + - + - + - + - + - + - + - + - + #
# | x | x |           | x | x | x | x | x | x | x | x | #
# + - + - +           + - + - + - + - + - + - + - + - + #
# | O | x |           | x | O | x | O | x |   | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII ART: PROPAGATION / SORTING (1ST PASS)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x |   | x | O | #
# + - + - + - + - + - + - + - + - + - + - +   + - + - + #
# | x | x | x | x | x | x | x | x | x | x |   | x | x | #
# + - + - + - + - + - + - + - + - + - + - +   + - + - + #
# | O | x |   | x | O | x | O | x | O | x |   | x | O | #
# + - + - +   + - + - + - + - + - + - + - + - + - + - + #
# | x | x |   | x | x | x | x | x | x | x | x | x | x | #
# + - + - +   + - + - + - + - + - + - + - + - + - + - + #
# | O | x |           | x | O | x |   | x | O | x | O | #
# + - + - +           + - + - + - +   + - + - + - + - + #
# | x | x |           | x | x | x |   | x | x | x | x | #
# + - + - +           + - + - + - +   + - + - + - + - + #
# | O | x |               | O |       | x | O | x | O | #
# + - + - +           + - + - + - + - + - + - + - + - + #
# | x | x |           | x | x | x | x | x | x | x | x | #
# + - + - +           + - + - + - + - + - + - + - + - + #
# | O | x |           | x | O | x | O | x |   | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII ART: PROPAGATION / SORTING (1ST PASS)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x |   | x | O | #
# + - + - + - + - + - + - + - + - + - + - +   + - + - + #
# | x | x | x | x | x | x | x | x | x | x |   | x | x | #
# + - + - + - + - + - + - + - + - + - + - +   + - + - + #
# | O | x |   | x | O | x | O | x | O | x |   | x | O | #
# + - + - +   + - + - + - + - + - + - + - + - + - + - + #
# | x | x |   | x | x | x | x | x | x | x | x | x | x | #
# + - + - +   + - + - + - + - + - + - + - + - + - + - + #
# | O | x |           | x | O | x |   | x | O | x | O | #
# + - + - +           + - + - + - +   + - + - + - + - + #
# | x | x |           | x | x | x |   | x | x | x | x | #
# + - + - +           + - + - + - +   + - + - + - + - + #
# | O | x |           | x | O | x |   | x | O | x | O | #
# + - + - +           + - + - + - + - + - + - + - + - + #
# | x | x |           | x | x | x | x | x | x | x | x | #
# + - + - +           + - + - + - + - + - + - + - + - + #
# | O | x |           | x | O | x | O | x |   | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x |   | x | O | #
# + - + - + - + - + - + - + - + - + - + - +   + - + - + #
# | x | x | x | x | x | x | x | x | x | x |   | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x |   | x | O | x | O | x | O | x | O | x | O | #
# + - + - +   + - + - + - + - + - + - + - + - + - + - + #
# | x | x |   | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O |       | x | O | x |   | x | O | x | O | #
# + - + - + - +       + - + - + - +   + - + - + - + - + #
# | x | x |           | x | x | x |   | x | x | x | x | #
# + - + - +           + - + - + - +   + - + - + - + - + #
# | O | x |           | x | O | x |   | x | O | x | O | #
# + - + - +           + - + - + - + - + - + - + - + - + #
# | x | x |           | x | x | x | x | x | x | x | x | #
# + - + - +           + - + - + - + - + - + - + - + - + #
# | O | x |           | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: propagation / sorting (1st pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x |   | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x |   | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x |   | x | O | x |   | x | O | x | O | #
# + - + - + - + - +   + - + - + - +   + - + - + - + - + #
# | x | x | x | x |   | x | x | x |   | x | x | x | x | #
# + - + - + - + - +   + - + - + - +   + - + - + - + - + #
# | O | x |           | x | O | x |   | x | O | x | O | #
# + - + - +           + - + - + - + - + - + - + - + - + #
# | x | x |           | x | x | x | x | x | x | x | x | #
# + - + - +           + - + - + - + - + - + - + - + - + #
# | O | x |           | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII ART: PROPAGATION / SORTING (1ST PASS)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x |   | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x |   | x | O | x |   | x | O | x | O | #
# + - + - + - + - +   + - + - + - +   + - + - + - + - + #
# | x | x | x | x |   | x | x | x |   | x | x | x | x | #
# + - + - + - + - +   + - + - + - +   + - + - + - + - + #
# | O | x |         | x | O | x |   | x | O | x | O | #
# + - + - +         + - + - + - + - + - + - + - + - + #
# | x | x |         | x | x | x | x | x | x | x | x | #
# + - + - +         + - + - + - + - + - + - + - + - + #
# | O | x |         | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: propagation / sorting (1st pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x |   | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x |   | x | O | x |   | x | O | x | O | #
# + - + - + - + - +   + - + - + - +   + - + - + - + - + #
# | x | x | x | x |   | x | x | x |   | x | x | x | x | #
# + - + - + - + - +   + - + - + - + - + - + - + - + - + #
# | O | x |           | x | O | x | O | x | O | x | O | #
# + - + - +           + - + - + - + - + - + - + - + - + #
# | x | x |           | x | x | x | x | x | x | x | x | #
# + - + - +           + - + - + - + - + - + - + - + - + #
# | O | x |           | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x |   | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x |   | x | O | x |   | x | O | x | O | #
# + - + - + - + - +   + - + - + - + - + - + - + - + - + #
# | x | x | x | x |   | x | x | x | x | x | x | x | x | #
# + - + - + - + - +   + - + - + - + - + - + - + - + - + #
# | O | x |           | x | O | x | O | x | O | x | O | #
# + - + - +           + - + - + - + - + - + - + - + - + #
# | x | x |           | x | x | x | x | x | x | x | x | #
# + - + - +           + - + - + - + - + - + - + - + - + #
# | O | x |           | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x |   | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x |   | x | O | x | O | x | O | x | O | #
# + - + - + - + - +   + - + - + - + - + - + - + - + - + #
# | x | x | x | x |   | x | x | x | x | x | x | x | x | #
# + - + - + - + - +   + - + - + - + - + - + - + - + - + #
# | O | x |           | x | O | x | O | x | O | x | O | #
# + - + - +           + - + - + - + - + - + - + - + - + #
# | x | x |           | x | x | x | x | x | x | x | x | #
# + - + - +           + - + - + - + - + - + - + - + - + #
# | O | x |           | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII ART: PROPAGATION / SORTING (1ST PASS)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x |   | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x |   | x | O | x | O | x | O | x | O | #
# + - + - + - + - +   + - + - + - + - + - + - + - + - + #
# | x | x | x | x |   | x | x | x | x | x | x | x | x | #
# + - + - + - + - +   + - + - + - + - + - + - + - + - + #
# | O | x | O |       | x | O | x | O | x | O | x | O | #
# + - + - + - +       + - + - + - + - + - + - + - + - + #
# | x | x | x |       | x | x | x | x | x | x | x | x | #
# + - + - + - +       + - + - + - + - + - + - + - + - + #
# | O | x | O |       | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x |   | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x |   | x | O | x | O | x | O | x | O | #
# + - + - + - + - +   + - + - + - + - + - + - + - + - + #
# | x | x | x | x |   | x | x | x | x | x | x | x | x | #
# + - + - + - + - +   + - + - + - + - + - + - + - + - + #
# | O | x | O | x |   | x | O | x | O | x | O | x | O | #
# + - + - + - + - +   + - + - + - + - + - + - + - + - + #
# | x | x | x | x |   | x | x | x | x | x | x | x | x | #
# + - + - + - + - +   + - + - + - + - + - + - + - + - + #
# | O | x | O | x |   | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: propagation / sorting (1st pass)

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# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x |   | x | O | x | O | x | O | x | O | #
# + - + - + - + - +   + - + - + - + - + - + - + - + - + #
# | x | x | x | x |   | x | x | x | x | x | x | x | x | #
# + - + - + - + - +   + - + - + - + - + - + - + - + - + #
# | O | x | O | x |   | x | O | x | O | x | O | x | O | #
# + - + - + - + - +   + - + - + - + - + - + - + - + - + #
# | x | x | x | x |   | x | x | x | x | x | x | x | x | #
# + - + - + - + - +   + - + - + - + - + - + - + - + - + #
# | O | x | O | x |   | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
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ASCII ART: PROPAGATION / SORTING (1ST PASS)

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# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x |   | x | x | x | x | x | x | x | x | #
# + - + - + - + - +   + - + - + - + - + - + - + - + - + #
# | O | x | O | x |   | x | O | x | O | x | O | x | O | #
# + - + - + - + - +   + - + - + - + - + - + - + - + - + #
# | x | x | x | x |   | x | x | x | x | x | x | x | x | #
# + - + - + - + - +   + - + - + - + - + - + - + - + - + #
# | O | x | O | x |   | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
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ASCII ART: PROPAGATION / SORTING (1ST PASS)

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# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x |   | x | O | x | O | x | O | x | O | #
# + - + - + - + - +   + - + - + - + - + - + - + - + - + #
# | x | x | x | x |   | x | x | x | x | x | x | x | x | #
# + - + - + - + - +   + - + - + - + - + - + - + - + - + #
# | O | x | O | x |   | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
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ASCII art: propagation / sorting (1st pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x |   | x | x | x | x | x | x | x | x | #
# + - + - + - + - +   + - + - + - + - + - + - + - + - + #
# | O | x | O | x |   | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
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# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x |   | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII ART: PROPAGATION / SORTING (1ST PASS)

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# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
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ASCII ART: TREE COMPUTATION (2ND PASS)

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ASCII ART: TREE COMPUTATION (2ND PASS)

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ASCII ART: TREE COMPUTATION (2ND PASS)

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#               x | O                                   #
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ASCII ART: TREE COMPUTATION (2ND PASS)

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ASCII ART: TREE COMPUTATION (2ND PASS)

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# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: tree computation (2nd pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                                           O           #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                   O               O                   #
#                   -               -                   #
#                   x               x                   #
#                   -                                   #
#           O | x | O                                   #
#           - + - + -                                   #
#           x | x | x                                   #
#           - + - + -                                   #
#           O | x | O                                   #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII ART: TREE COMPUTATION (2ND PASS)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                                           O           #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                   O               O                   #
#                   -               -                   #
#                   x               x                   #
#                   -               -                   #
#           O | x | O               O                   #
#           - + - + -                                   #
#           x | x | x                                   #
#           - + - + -                                   #
#           O | x | O                                   #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII ART: TREE COMPUTATION (2ND PASS)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                                           O           #
#                                                       #
#                                                       #
#                                                       #
#           O                                           #
#                                                       #
#                                                       #
#                                                       #
#                   O               O                   #
#                   -               -                   #
#                   x               x                   #
#                   -               -                   #
#           O | x | O               O                   #
#           - + - + -                                   #
#           x | x | x                                   #
#           - + - + -                                   #
#           O | x | O                                   #
#                                                       #
#                                                       #
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# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                                           O           #
#                                           -           #
#                                           x           #
#                                                       #
#           O                                           #
#           -                                           #
#           x                                           #
#                                                       #
#               x | O               O                   #
#               - + -               -                   #
#           x | x | x               x                   #
#           - + - + -               -                   #
#           O | x | O               O                   #
#           - + - + -                                   #
#           x | x | x                                   #
#           - + - + -                                   #
#           O | x | O                                   #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                                           O           #
#                                           -           #
#                                           x           #
#                                           -           #
#           O                               O           #
#           -                                           #
#           x                                           #
#           -                                           #
#           O | x | O               O                   #
#           - + - + -               -                   #
#           x | x | x               x                   #
#           - + - + -               -                   #
#           O | x | O               O                   #
#           - + - + -                                   #
#           x | x | x                                   #
#           - + - + -                                   #
#           O | x | O                       O           #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: tree computation (2nd pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                                           O           #
#                                           -           #
#                                           x           #
#                                           -           #
#           O                               O           #
#           -                                           #
#           x                                           #
#           -                                           #
#           O | x | O               O                   #
#           - + - + -               -                   #
#           x | x | x               x                   #
#           - + - + -               -                   #
#           O | x | O | x       x | O                   #
#           - + - + -                                   #
#           x | x | x                                   #
#           - + - + -                                   #
#           O | x | O                       O           #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII ART: TREE COMPUTATION (2ND PASS)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                                           O           #
#                                           -           #
#                                           x           #
#                                           -           #
#           O                               O           #
#           -                                           #
#           x                                           #
#           -                                           #
#           O | x | O               O                   #
#           - + - + -               -                   #
#           x | x | x               x                   #
#           - + - + -               -                   #
#           O | x | O | x | O | x | O                   #
#           - + - + -                                   #
#           x | x | x                                   #
#           - + - + -                                   #
#           O | x | O                       O           #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
#                                                       #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#                                           x | x       #
#                                           - + -       #
#                                           O | x       #
#                                           - + -       #
#                                           x | x       #
#                                           - + -       #
#       x | O                               O | x       #
#       - + -                                           #
#       x | x                                           #
#       - + -                                           #
#       x | O | x | O               O                   #
#       - + - + - + -               -                   #
#       x | x | x | x               x                   #
#       - + - + - + -               -                   #
#       x | O | x | O | x | O | x | O                   #
#       - + - + - + -                                   #
#       x | x | x | x                                   #
#       - + - + - + -                                   #
#       x | O | x | O                       O | x       #
#       - + - + - + -                       - + -       #
#       x | x | x | x                       x | x       #
#                                                       #
#                                                       #
#                                                       #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: tree computation (2nd pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#                                           x | x       #
#                                           - + -       #
#                                           O | x       #
#                                           - + -       #
#                                           x | x       #
#                                           - + -       #
#       x | O                               O | x       #
#       - + -                                           #
#       x | x                                           #
#       - + -                                           #
#       x | O | x | O               O                   #
#       - + - + - + -               -                   #
#       x | x | x | x               x                   #
#       - + - + - + -               -                   #
#       x | O | x | O | x | O | x | O       O           #
#       - + - + - + -                                   #
#       x | x | x | x                                   #
#       - + - + - + -                                   #
#       x | O | x | O                       O | x       #
#       - + - + - + -                       - + -       #
#       x | x | x | x                       x | x       #
#                                                       #
#                                                       #
#                                                       #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII ART: TREE COMPUTATION (2ND PASS)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#                                           x | x       #
#                                           - + -       #
#                                           O | x       #
#                                           - + -       #
#                                           x | x       #
#                                           - + -       #
#       x | O                       O       O | x       #
#       - + -                                           #
#       x | x                                           #
#       - + -                                           #
#       x | O | x | O               O                   #
#       - + - + - + -               -                   #
#       x | x | x | x               x                   #
#       - + - + - + -               -                   #
#       x | O | x | O | x | O | x | O       O           #
#       - + - + - + -                                   #
#       x | x | x | x                                   #
#       - + - + - + -                                   #
#       x | O | x | O                       O | x       #
#       - + - + - + -                       - + -       #
#       x | x | x | x                       x | x       #
#                                                       #
#                                                       #
#                                                       #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#                                           x | x       #
#                                           - + -       #
#                                           O | x       #
#                                           - + -       #
#                                           x | x       #
#                                           - + -       #
#       x | O                       O       O | x       #
#       - + -                                           #
#       x | x                                           #
#       - + -                                           #
#       x | O | x | O               O                   #
#       - + - + - + -               -                   #
#       x | x | x | x               x                   #
#       - + - + - + -               -                   #
#       x | O | x | O | x | O | x | O       O           #
#       - + - + - + -                                   #
#       x | x | x | x                                   #
#       - + - + - + -                                   #
#       x | O | x | O               O       O | x       #
#       - + - + - + -                       - + -       #
#       x | x | x | x                       x | x       #
#                                                       #
#                                                       #
#                                                       #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: tree computation (2nd pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#                                           x | x       #
#                                           - + -       #
#                                           O | x       #
#                                           - + -       #
#                                           x | x       #
#                                           - + -       #
#       x | O                   x | O       O | x       #
#       - + -                                           #
#       x | x                                           #
#       - + -                                           #
#       x | O | x | O               O                   #
#       - + - + - + -               -                   #
#       x | x | x | x               x                   #
#       - + - + - + -               -                   #
#       x | O | x | O | x | O | x | O       O           #
#       - + - + - + -                                   #
#       x | x | x | x                                   #
#       - + - + - + -                                   #
#       x | O | x | O               O       O | x       #
#       - + - + - + -                       - + -       #
#       x | x | x | x                       x | x       #
#                                                       #
#                                                       #
#                                                       #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII ART: TREE COMPUTATION (2ND PASS)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#                                           x | x       #
#                                           - + -       #
#                                           O | x       #
#                                           - + -       #
#                                           x | x       #
#                                           - + -       #
#       x | O               O | x | O       O | x       #
#       - + -                                           #
#       x | x                                           #
#       - + -                                           #
#       x | O | x | O               O                   #
#       - + - + - + -               -                   #
#       x | x | x | x               x                   #
#       - + - + - + -               -                   #
#       x | O | x | O | x | O | x | O       O           #
#       - + - + - + -                                   #
#       x | x | x | x                                   #
#       - + - + - + -                                   #
#       x | O | x | O               O       O | x       #
#       - + - + - + -                       - + -       #
#       x | x | x | x                       x | x       #
#                                                       #
#                                                       #
#                                                       #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#                                           x | x       #
#                                           - + -       #
#                                           O | x       #
#                                           - + -       #
#                                           x | x       #
#                                           - + -       #
#       x | O               O | x | O       O | x       #
#       - + -                                           #
#       x | x                                           #
#       - + -                                           #
#       x | O | x | O               O                   #
#       - + - + - + -               -                   #
#       x | x | x | x               x       x           #
#       - + - + - + -               -       -           #
#       x | O | x | O | x | O | x | O       O           #
#       - + - + - + -                                   #
#       x | x | x | x                                   #
#       - + - + - + -                                   #
#       x | O | x | O               O       O | x       #
#       - + - + - + -                       - + -       #
#       x | x | x | x                       x | x       #
#                                                       #
#                                                       #
#                                                       #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: tree computation (2nd pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#                                           x | x       #
#                                           - + -       #
#                                           O | x       #
#                                           - + -       #
#                                           x | x       #
#                                           - + -       #
#       x | O               O | x | O       O | x       #
#       - + -                                           #
#       x | x                                           #
#       - + -                                           #
#       x | O | x | O               O       O           #
#       - + - + - + -               -       -           #
#       x | x | x | x               x       x           #
#       - + - + - + -               -       -           #
#       x | O | x | O | x | O | x | O       O           #
#       - + - + - + -                                   #
#       x | x | x | x                                   #
#       - + - + - + -                                   #
#       x | O | x | O               O       O | x       #
#       - + - + - + -                       - + -       #
#       x | x | x | x                       x | x       #
#                                                       #
#                                                       #
#                                                       #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII ART: TREE COMPUTATION (2ND PASS)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#                                           x | x       #
#                                           - + -       #
#                                           O | x       #
#                                           - + -       #
#                           x               x | x       #
#                           -               - + -       #
#       x | O               O | x | O       O | x       #
#       - + -                                           #
#       x | x                                           #
#       - + -                                           #
#       x | O | x | O               O       O           #
#       - + - + - + -               -       -           #
#       x | x | x | x               x       x           #
#       - + - + - + -               -       -           #
#       x | O | x | O | x | O | x | O       O           #
#       - + - + - + -                                   #
#       x | x | x | x                                   #
#       - + - + - + -                                   #
#       x | O | x | O               O       O | x       #
#       - + - + - + -                       - + -       #
#       x | x | x | x                       x | x       #
#                                                       #
#                                                       #
#                                                       #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: tree computation (2nd pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#                                           x | x       #
#                                           - + -       #
#                           O               O | x       #
#                           -               - + -       #
#                           x               x | x       #
#                           -               - + -       #
#       x | O               O | x | O       O | x       #
#       - + -                                           #
#       x | x                                           #
#       - + -                                           #
#       x | O | x | O               O       O           #
#       - + - + - + -               -       -           #
#       x | x | x | x               x       x           #
#       - + - + - + -               -       -           #
#       x | O | x | O | x | O | x | O       O           #
#       - + - + - + -                                   #
#       x | x | x | x                                   #
#       - + - + - + -                                   #
#       x | O | x | O               O       O | x       #
#       - + - + - + -                       - + -       #
#       x | x | x | x                       x | x       #
#                                                       #
#                                                       #
#                                                       #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: tree computation (2nd pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#                                           x | x       #
#                                           - + -       #
#                               O | x       O | x       #
#                               - + -       - + -       #
#                               x | x | x   x | x       #
#                               - + - + -   - + -       #
#       x | O                   O | x | O   O | x       #
#       - + -                                           #
#       x | x                                           #
#       - + -                                           #
#       x | O | x | O                   O   O           #
#       - + - + - + -                   -   -           #
#       x | x | x | x                   x   x           #
#       - + - + - + -                   -   -           #
#       x | O | x | O | x | O | x | O   O               #
#       - + - + - + -                                   #
#       x | x | x | x                                   #
#       - + - + - + -                                   #
#       x | O | x | O                   O   O | x       #
#       - + - + - + -                       - + -       #
#       x | x | x | x                       x | x       #
#                                                       #
#                                                       #
#                                                       #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: tree computation (2nd pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#                                           x | x       #
#                                           - + -       #
#                           O | x | O       O | x       #
#                           - + - + -       - + -       #
#                           x | x | x       x | x       #
#                           - + - + -       - + -       #
#       x | O               O | x | O       O | x       #
#       - + -                                           #
#       x | x                                           #
#       - + -                                           #
#       x | O | x | O               O       O           #
#       - + - + - + -               -       -           #
#       x | x | x | x               x       x           #
#       - + - + - + -               -       -           #
#       x | O | x | O | x | O | x | O       O           #
#       - + - + - + -                                   #
#       x | x | x | x                                   #
#       - + - + - + -                                   #
#       x | O | x | O               O       O | x       #
#       - + - + - + -                       - + -       #
#       x | x | x | x                       x | x       #
#                                                       #
#                                                       #
#                                                       #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: tree computation (2nd pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#                                           x | x       #
#                                           - + -       #
#                           O | x | O       O | x       #
#                           - + - + -       - + -       #
#                           x | x | x       x | x       #
#                           - + - + -       - + -       #
#       x | O               O | x | O       O | x       #
#       - + -               -                           #
#       x | x               x                           #
#       - + -                                           #
#       x | O | x | O               O       O           #
#       - + - + - + -               -       -           #
#       x | x | x | x               x       x           #
#       - + - + - + -               -       -           #
#       x | O | x | O | x | O | x | O       O           #
#       - + - + - + -                                   #
#       x | x | x | x                                   #
#       - + - + - + -                                   #
#       x | O | x | O               O       O | x       #
#       - + - + - + -                       - + -       #
#       x | x | x | x                       x | x       #
#                                                       #
#                                                       #
#                                                       #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: tree computation (2nd pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#                                           x | x       #
#                                           - + -       #
#                           O | x | O       O | x       #
#                           - + - + -       - + -       #
#                           x | x | x       x | x       #
#                           - + - + -       - + -       #
#       x | O               O | x | O       O | x       #
#       - + -               -                           #
#       x | x               x                           #
#       - + -               -                           #
#       x | O | x | O       O       O       O           #
#       - + - + - + -               -       -           #
#       x | x | x | x               x       x           #
#       - + - + - + -               -       -           #
#       x | O | x | O | x | O | x | O       O           #
#       - + - + - + -                                   #
#       x | x | x | x                                   #
#       - + - + - + -                                   #
#       x | O | x | O               O       O | x       #
#       - + - + - + -                       - + -       #
#       x | x | x | x                       x | x       #
#                                                       #
#                                                       #
#                                                       #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: tree computation (2nd pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#                                           x | x       #
#                                           - + -       #
#                           O | x | O       O | x       #
#                           - + - + -       - + -       #
#                           x | x | x       x | x       #
#                           - + - + -       - + -       #
#       x | O           x | O | x | O       O | x       #
#       - + -               -                           #
#       x | x               x                           #
#       - + -               -                           #
#       x | O | x | O       O       O       O           #
#       - + - + - + -               -       -           #
#       x | x | x | x               x       x           #
#       - + - + - + -               -       -           #
#       x | O | x | O | x | O | x | O       O           #
#       - + - + - + -                                   #
#       x | x | x | x                                   #
#       - + - + - + -                                   #
#       x | O | x | O               O       O | x       #
#       - + - + - + -                       - + -       #
#       x | x | x | x                       x | x       #
#                                                       #
#                                                       #
#                                                       #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: tree computation (2nd pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                         #
#                                                         #
#                                                         #
#                                             x | x       #
#                                             - + -       #
#           O               O | x | O         O | x       #
#                           - + - + -         - + -       #
#                           x | x | x         x | x       #
#                           - + - + -         - + -       #
#       x | O       O | x | O | x | O         O | x       #
#       - + -               -                             #
#       x | x               x                             #
#       - + -               -                             #
#       x | O | x | O       O       O         O           #
#       - + - + - + -               -         -           #
#       x | x | x | x               x         x           #
#       - + - + - + -               -         -           #
#       x | O | x | O | x | O | x | O         O           #
#       - + - + - + -                                     #
#       x | x | x | x                                     #
#       - + - + - + -                                     #
#       x | O | x | O               O         O | x       #
#       - + - + - + -                         - + -       #
#       x | x | x | x                         x | x       #
#                                                         #
#                                                         #
#                                                         #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: tree computation (2nd pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#                                           x | x       #
#                                           - + -       #
#           O               O | x | O       O | x       #
#                           - + - + -       - + -       #
#                           x | x | x       x | x       #
#                           - + - + -       - + -       #
#       x | O       O | x | O | x | O       O | x       #
#       - + -               -                           #
#       x | x               x                           #
#       - + -               -                           #
#       x | O | x | O       O       O       O           #
#       - + - + - + -               -       -           #
#       x | x | x | x               x       x           #
#       - + - + - + -               -       -           #
#       x | O | x | O | x | O | x | O       O           #
#       - + - + - + -                                   #
#       x | x | x | x                                   #
#       - + - + - + -                                   #
#       x | O | x | O           x | O       O | x       #
#       - + - + - + -                       - + -       #
#       x | x | x | x                       x | x       #
#                                                       #
#                                                       #
#                                                       #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: tree computation (2nd pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#                                           x | x       #
#                                           - + -       #
#           O               O | x | O       O | x       #
#                           - + - + -       - + -       #
#                           x | x | x       x | x       #
#                           - + - + -       - + -       #
#       x | O       O | x | O | x | O       O | x       #
#       - + -               -                           #
#       x | x               x                           #
#       - + -               -                           #
#       x | O | x | O       O       O       O           #
#       - + - + - + -               -       -           #
#       x | x | x | x               x       x           #
#       - + - + - + -               -       -           #
#       x | O | x | O | x | O | x | O       O           #
#       - + - + - + -                                   #
#       x | x | x | x                                   #
#       - + - + - + -                                   #
#       x | O | x | O       O | x | O       O | x       #
#       - + - + - + -                       - + -       #
#       x | x | x | x                       x | x       #
#                                                       #
#                                                       #
#                                                       #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#                                           x | x       #
#                                           - + -       #
#           O | x       x | O | x | O       O | x       #
#                       - + - + - + -       - + -       #
#                   x | x | x | x | x       x | x       #
#                   - + - + - + - + -       - + -       #
#       x | O       O | x | O | x | O       O | x       #
#       - + -               -                           #
#       x | x               x                           #
#       - + -               -                           #
#       x | O | x | O       O       O       O           #
#       - + - + - + -               -       -           #
#       x | x | x | x               x       x           #
#       - + - + - + -               -       -           #
#       x | O | x | O | x | O | x | O       O           #
#       - + - + - + -                                   #
#       x | x | x | x                                   #
#       - + - + - + -                                   #
#       x | O | x | O       O | x | O       O | x       #
#       - + - + - + -                       - + -       #
#       x | x | x | x                       x | x       #
#                                                       #
#                                                       #
#                                                       #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII ART: TREE COMPUTATION (2ND PASS)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#                                           x | x       #
#                                           - + -       #
#           O | x | O | x | O | x | O       O | x       #
#                   - + - + - + - + -       - + -       #
#                   x | x | x | x | x       x | x       #
#                   - + - + - + - + -       - + -       #
#       x | O       O | x | O | x | O       O | x       #
#       - + -               -                           #
#       x | x               x                           #
#       - + -               -                           #
#       x | O | x | O       O       O       O           #
#       - + - + - + -               -       -           #
#       x | x | x | x               x       x           #
#       - + - + - + -               -       -           #
#       x | O | x | O | x | O | x | O       O           #
#       - + - + - + -                                   #
#       x | x | x | x                                   #
#       - + - + - + -                                   #
#       x | O | x | O       O | x | O       O | x       #
#       - + - + - + -                       - + -       #
#       x | x | x | x                       x | x       #
#                                                       #
#                                                       #
#                                                       #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII art: tree computation (2nd pass)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
#                                                       #
#                                                       #
#                                                       #
#       x | x | x | x | x | x | x | x       x | x       #
#       - + - + - + - + - + - + - + -       - + -       #
#       x | O | x | O | x | O | x | O       O | x       #
#                   - + - + - + - + -       - + -       #
#                   x | x | x | x | x       x | x       #
#                   - + - + - + - + -       - + -       #
#       x | O       O | x | O | x | O       O | x       #
#       - + -               -                           #
#       x | x               x                           #
#       - + -               -                           #
#       x | O | x | O       O       O       O | x       #
#       - + - + - + -               -       - + -       #
#       x | x | x | x               x       x | x       #
#       - + - + - + -               -       - + -       #
#       x | O | x | O | x | O | x | O       O | x       #
#       - + - + - + -                                   #
#       x | x | x | x                                   #
#       - + - + - + -                                   #
#       x | O | x | O       O | x | O       O | x       #
#       - + - + - + -       - + - + -       - + -       #
#       x | x | x | x       x | x | x       x | x       #
#                                                       #
#                                                       #
#                                                       #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

ASCII ART: TREE COMPUTATION (2ND PASS)

```
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | x | x | x | x | x | x | x | x | x | x | x | x | x | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# | O | x | O | x | O | x | O | x | O | x | O | x | O | #
# + - + - + - + - + - + - + - + - + - + - + - + - + - + #
# # # # # # # # # # # # # # # # # # # # # # # # # # # # #
```

Source code available in the next release of Milena:

- generic and efficient image processing C++ library
- free software under the GPL v2 licence
- including many pieces of mathematical morphology in it...

Try Milena!

- *"Since I use Milena, I feel different,"* Steve Jobs
- *"Google is named after Milena, yet we change some letters,"* L. Page and S. Brin
- *"Milena just leaves me speechless,"* M. Wilkinson

http://olena.lrde.epita.fr

Conclusion

Contributions:

- a new algorithm to compute the tree of shapes
    - working on nD images
    - quasi-linear for low quantized data
    - very easy to implement
- a new image representation

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  - based on cellular complexes and set-valued maps
  - discrete yet "continuous"
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WHAT IS *not* PRESENTED HERE

In a journal paper (in preparation):

- a formal proof of our algorithm
- a discussion about propagation initialization (root node)
- how to reduce the memory bloat $(\times 4^n$ from u to $\mathfrak{I}(u))$
- a comparison of execution times of existing algorithms
- parallization strategy

Contributions

	Thierry Géraud:	funny ideas and lead
	Edwin Carlinet:	algorithm design, seminal implementation
	Sébastien Crozet:	tests, algorithm proof, parallelization
	Laurent Najman:	good ideas and references

THE END

Yet another killing app: food detection

Thanks for your attention. Any question?

STEP 1: "UNION-FIND"-BASED TREE COMPUTATION

```
FIND_ROOT(zpar, x) : P
begin
    if zpar (x) = x then
        return x
    else
        zpar(x) ← FIND_ROOT(zpar, zpar(x))
        return zpar(x)
```

```
UNION_FIND(R) : T
begin
    for all p do
        zpar(p) ← undef
    for i ← N − 1 to 0 do
        p ← R[i]
        parent(p) ← p
        zpar(p) ← p
        for all n ∈ N(p) such aszpar(n) ≠ undef do
            r ← FIND_ROOT(zpar, n)
            if r ≠ p then
                parent(r) ← p
                zpar(r) ← p
    return parent
```

Step 1: sorting ("continuous" propagation)

PRIORITY_PUSH(q, h, U, ℓ)
/* modify q */
begin
- $[lower, upper] \leftarrow U(h)$ **if** $lower > \ell$ **then**
  - $\ell' \leftarrow lower$
- **else if** $upper < \ell$ **then**
  - $\ell' \leftarrow upper$
- **else**
  - $\ell' \leftarrow \ell$
- PUSH($q[\ell'], h$)

PRIORITY_POP(q, ℓ) : H
/* modify q, and sometimes ℓ */
begin
- **if** $q[\ell]$ is empty **then**
  - $\ell' \leftarrow$ level next to ℓ such as $q[\ell']$ is not empty
  - $\ell \leftarrow \ell'$
- **return** POP($q[\ell]$)

SORT(U) : Pair(Array[H], Image)
begin
- **for all** h **do**
  - $deja\_vu(h) \leftarrow$ false
- $i \leftarrow 0$
- PUSH($q[\ell_\infty], p_\infty$)
- $deja\_vu(p_\infty) \leftarrow$ true
- $\ell \leftarrow \ell_\infty$ /* start from root level */
- **while** q is not empty **do**
  - $h \leftarrow$ PRIORITY_POP(q, ℓ)
  - $u^\flat(h) \leftarrow \ell$
  - $\mathcal{R}[i] \leftarrow h$
  - **for all** $n \in \mathcal{N}(h)$ **such as** $deja\_vu(n) =$ false **do**
    - PRIORITY_PUSH(q, n, U, ℓ)
    - $deja\_vu(n) \leftarrow$ true
  - $i \leftarrow i + 1$
  - /* if $q[\ell]$ is empty we are done with level ℓ */
- **return** $(\mathcal{R}, u^\flat)$

Tree of shape computation

COMPUTE_TREE_OF_SHAPES (u) : Pair(Array[P], T)
begin
  $U \leftarrow$ INTERPOLATE(u)
  $(\mathcal{R}, u^\flat) \leftarrow$ SORT(U)
  parent $\leftarrow$ UNION_FIND($\mathcal{R}$)
  CANONICALIZE_TREE($u^\flat$, $\mathcal{R}$, *parent*)
  return UN-INTERPOLATE($\mathcal{R}$, *parent*)

End of the slide set…

.

ANSWER TO MICHAEL'S QUESTION

low quantized data:

- appropriate data structure = hierarchical queue
- time complexity $O(N)$
- K (number of quanta) is a multiplicative constant

oddity:

- having high-bit-depth data...
- ...and using a hierarchical queue (unappropriate data structure!)
- time complexity is quasi-polynomial: $O(K\,N)$
- nobody wants that!

high-bit-depth data:

- appropriate data structure = Y-fast trie
- time complexity $O(N \log K)$
- space complexity $O(N)$