On making *n*D images well-composed by a self-dual local interpolation

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Outline

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Forewords

Classical issues in digital topology:

- the set of connected components depends on the chosen connectivity!
- Jordan Curve Theorem does not work anymore! (*Latecki 1995 CVIU*)

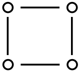
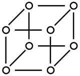
Forewords

Having a well-composed image is great:

- $3^n - 1$ and $2n$−connectivities in images are equivalent (*Rosenfeld 1970 JACM*)
- the Jordan Separation Theorem holds
- topological properties are conserved by rigid transforms (image registration and warping) (*Ngo 2014 ITIP*)
- thinning algorithms are simplified (*Marchadier 2004 PRL*)
- graph structures resulting from skeleton algorithms are simplified (*Latecki 1995 CVIU*)
- the Tree of Shapes (ToS) is unique (*Najman 2013 ISMM, Geraud 2013 ISMM*)

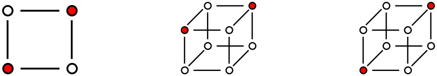
Outline

Well-composedness in nD

- Blocks of $\mathbb{Z}^n$:  (1D),  (2D),  (3D), ...

- Two points are said antagonist of a block iff they are as far from each other as it is possible in the block

- A critical configuration is a set of two points which are antagonist in a block of dimension k, $k \in [2, n]$.



$k = 2, n = 2$ $k = 2, n = 3$ $k = 3, n = 3$

- A set $X \subseteq \mathbb{Z}^n$ is well-composed iff there is no critical configuration in X or X^c.

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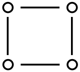
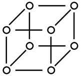
- Blocks of $\mathbb{Z}^n$:  (1D),  (2D),  (3D), ...
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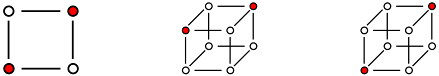

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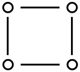
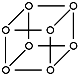
- Blocks of $\mathbb{Z}^n$:  (1D),  (2D),  (3D), ...
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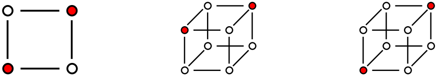

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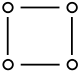
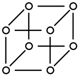
- Blocks of $\mathbb{Z}^n$:  (1D),  (2D),  (3D), ...
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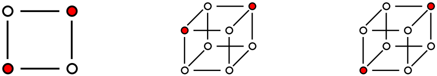

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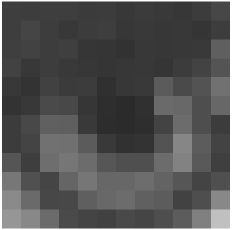
- A set $X \subseteq \mathbb{Z}^n$ is well-composed iff there is no critical configuration in X or X^c.

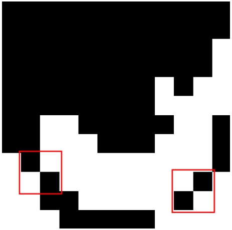
Well-composedness for nD Images

Let $\mathcal{D} \subseteq \mathbb{Z}^n$ be the domain of the image u.

- For any $\lambda \in \mathbb{R}$, we call strict upper threshold set and strict lower threshold set the sets $[u > \lambda] = \{m \in \mathcal{D} | u(m) > \lambda\}$ and $[u < \lambda] = \{m \in \mathcal{D} | u(m) < \lambda\}$.
- An image $u : \mathcal{D} \mapsto \mathbb{Z}$ is said well-composed iff all its threshold sets are well-composed.



One eye of Lena ...



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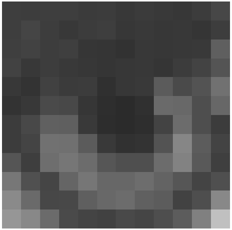

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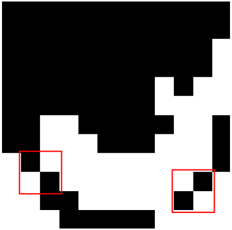

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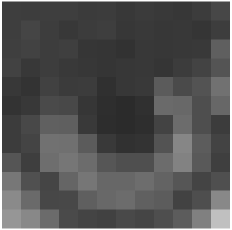

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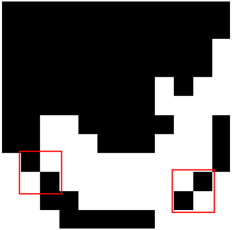

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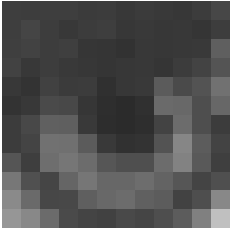

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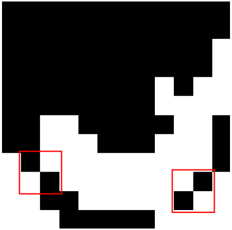

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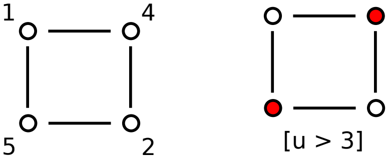
2D Characterization of Latecki

An image $u : \mathbb{Z}^2 \mapsto \mathbb{Z}$ is well-composed iff $\forall z \in \mathbb{Z}^2$:

$$\mathrm{intvl}(u(z_1, z_2), u(z_1 + 1, z_2 + 1)) \cap \mathrm{intvl}(u(z_1 + 1, z_2), u(z_1, z_2 + 1)) \neq \emptyset$$

Counter-example:



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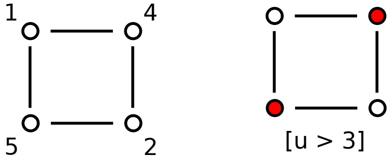
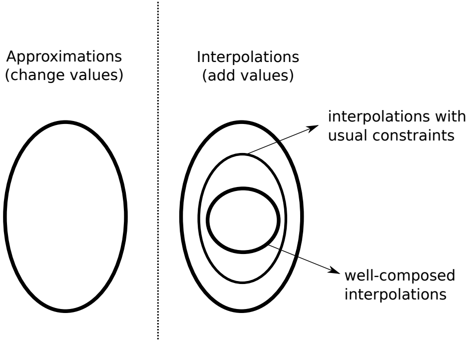


Outline

Interpolation VS Approximations

How to make an image well-composed:



Usual Constraints (1)

- One Subdivision: to limit the necessary amount of memory.
- Invariances: by translations, $\pi/2$-rotations, and axial symmetries.
- Ordered: First we set the values at the centers of the edges, then at the centers of the squares, and so on ...

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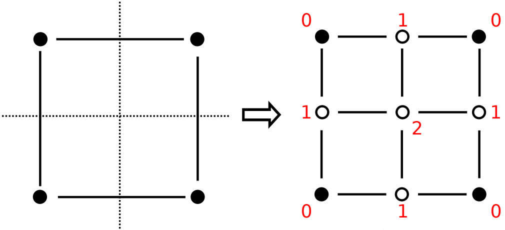
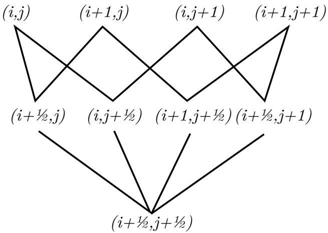
$\left(\frac{\mathbb{Z}}{2}\right)^n$ as a poset (1)

We subdivide the space $\mathbb{Z}^n$: we obtain $(\mathbb{Z}/2)^n$!



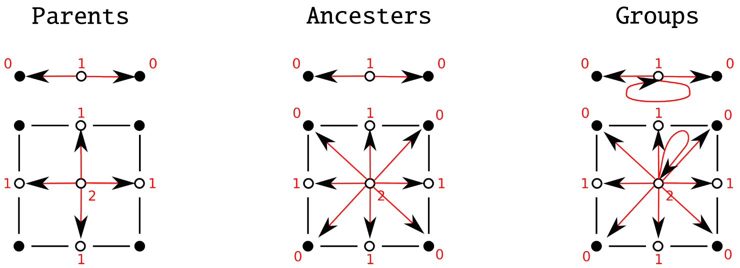
Order of a point in $\left(\frac{\mathbb{Z}}{2}\right)^n$



$\left(\frac{\mathbb{Z}}{2}\right)^n$ as a partially ordered set.

$\mathbb{E}_0$ are the original points, $\mathbb{E}_1$ the centers of the subdivided edges, $\mathbb{E}_2$ the centers of the subdivided squares, ...

$\left(\frac{\mathbb{Z}}{2}\right)^n$ as a poset (2)



Formulation (1)

Lemma

Any interpolation $\Im : u \mapsto u'$ verifying locality, orderedness, and invariance by translations and rotations can be characterized by a set of functions $\{f_k\}_{k \in [1,n]}$ such that:

$$\forall z \in \left(\frac{\mathbb{Z}}{2}\right)^n, \; u'(z) = \begin{cases} u(z) & \text{if } z \in \mathbb{E}_0 \\ f_k(u|_{\mathbb{A}(z)}) & \text{if } z \in \mathbb{E}_k, \; k \in [1,n] \end{cases}$$

u' at z depends only on u at the ancesters of z

We have a set of functions $\{f_1, f_2, f_3, \ldots\}$ such that:

- f_1 interpolates at the centers of the subdivided edges,
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Formulation (2): Which function f_1 ?

- f_1 has to be self-dual, symmetrical, and in-between.
- We choose one usual function satisfying these constraints:

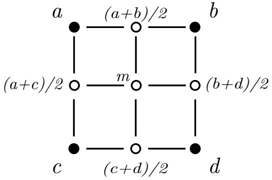
  the mean function $f_1(a,b) = (a+b)/2$.
- *N.B.:* There exists some other functions (e.g., $\operatorname{med}(a,b,\frac{1}{2})$).

Formulation (3): And about f_2?

f_2, f_3, ... must choose a value $u'(z)$ such as u' is well-composed on the group of z! (necessary condition)

With $m = f_2(a,b,c,d)$:

$$\begin{aligned}
\text{intvl}(a,m) \cap \text{intvl}((a+b)/2,(a+c)/2) &\neq \emptyset, \quad (1)\\
\text{intvl}((a+b)/2,(b+d)/2) \cap \text{intvl}(m,b) &\neq \emptyset, \quad (2)\\
\text{intvl}((a+c)/2,(c+d)/2) \cap \text{intvl}(m,c) &\neq \emptyset, \quad (3)\\
\text{intvl}(m,d) \cap \text{intvl}((c+d)/2,(b+d)/2) &\neq \emptyset. \quad (4)
\end{aligned}$$



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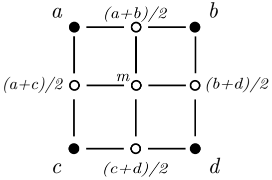

Formulation (4): Resulting f_2

We obtain finally that f_2 must satisfy:

Theorem

$$
\begin{array}{lll}
f_2(u|_{\mathbb{A}(z)}) & = \mathrm{med}\{u|_{\mathbb{A}(z)}\} & \text{if } u|_{\mathbb{A}(z)} \text{ is not W.C.,} \\
f_2(u|_{\mathbb{A}(z)}) & \text{is in-between} & \text{otherwise.}
\end{array}
$$

9	11	15
7	1	13
3	5	3

Original Image

9	10	11	13	15
8	7	6	10	14
7	4	1	7	13
5	4	3	4	8
3	4	5	4	3

Mean/Median (*Latecki*)

9	10	11	13	15
8	8	6	12	14
7	4	1	7	13
5	4	3	4	8
3	4	5	4	3

Median (*Geraud*)

Formulation (5): And about f_3?

- We have: a definition of 3D well-composed images (*Geraud, GT GéoDis, June 2013*).
- We need: a characterization (to study f_3).

Outline

3D Well-Composed Images (1)

Characterization

A gray-valued 3D image $u : \mathcal{D} \mapsto \mathbb{R}$ is well-composed on $\mathcal{D}$ iff on any block $S \subseteq \mathcal{D}$, $u\big|_S$ satisfies the properties $(P_i), i \in [1, 10]$.

These ten constraints allow to search for a 3D well-composed interpolation.

Characterization (2)

- LEMMA 1: The threshold sets $[u > \lambda]$ and $[u < \lambda]$, $\lambda \in \mathbb{R}$, of a gray-valued image u defined on a block S do not contain any critical configurations of type 1 iff the six following properties hold:

Lemma (1)

$$\begin{aligned}
&\mathrm{intvl}(a,d) \bigcap \mathrm{intvl}(b,c) \neq \emptyset && (P1),\\
&\mathrm{intvl}(e,h) \bigcap \mathrm{intvl}(g,f) \neq \emptyset && (P2),\\
&\mathrm{intvl}(a,f) \bigcap \mathrm{intvl}(b,e) \neq \emptyset && (P3),\\
&\mathrm{intvl}(c,h) \bigcap \mathrm{intvl}(g,d) \neq \emptyset && (P4),\\
&\mathrm{intvl}(a,g) \bigcap \mathrm{intvl}(e,c) \neq \emptyset && (P5),\\
&\mathrm{intvl}(b,h) \bigcap \mathrm{intvl}(f,d) \neq \emptyset && (P6).
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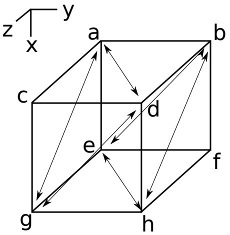
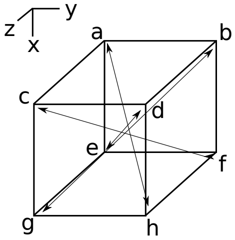


Characterization (3)

LEMMA 2: The threshold sets $[u > \lambda]$ and $[u < \lambda]$, $\lambda \in \mathbb{R}$, of a gray-valued image u do not contain any critical configurations of type 2 iff the six following properties hold:

Lemma (2)

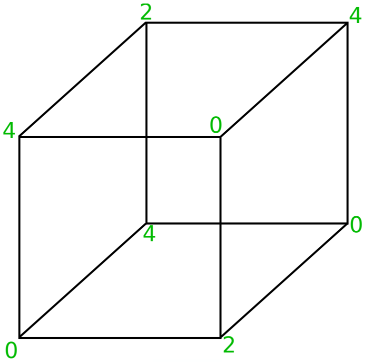
$$\text{intvl}(a, h) \bigcap \text{span}\{b, c, d, e, f, g\} \neq \emptyset \quad (P7)$$
$$\text{intvl}(b, g) \bigcap \text{span}\{a, c, d, e, f, h\} \neq \emptyset \quad (P8)$$
$$\text{intvl}(c, f) \bigcap \text{span}\{a, b, d, e, g, h\} \neq \emptyset \quad (P9)$$
$$\text{intvl}(d, e) \bigcap \text{span}\{a, b, c, f, g, h\} \neq \emptyset \quad (P10)$$



Outline

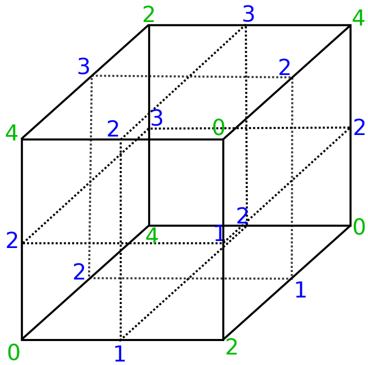
- We have chosen f_1.
- We have the equations of f_2.
- We have (a part of) the equations of f_3.
- We can test our interpolation on an example.

Equations of f_3 (1)



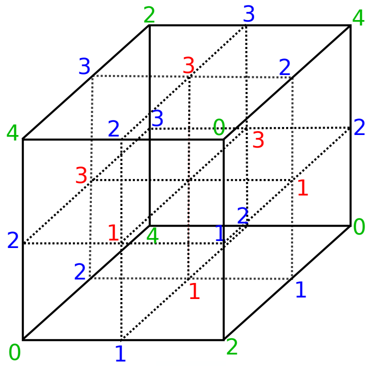
Initial value of the $3D$ image.

Equations of f_3 (2)



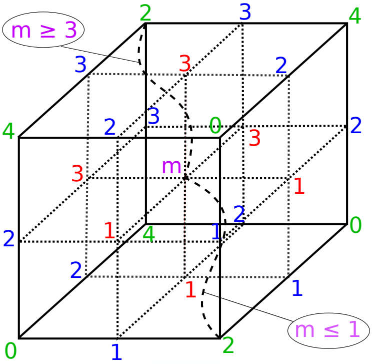
Applying the mean function f_1.

Equations of f_3 (3)



u is not well-composed on its faces! $\Rightarrow$ f_2 is the median function.

Equations of f_3 (4)



2 incompatible constraints ⇒ no solution.

Outline

Our contributions

- We extended the characterization formula of a 2D well-composed gray-valued image to 3D.
- We proposed a formulation, and then a model, for an *usual* local interpolation scheme in nD.
- We provided a counter-example showing that this reviewed scheme cannot succeed.

Future Works

We have to remove some constraints : the locality !

$\Rightarrow$ Proposed solution: a front propagation algorithm!

Thanks for your attention!

Questions? :D