# Well-Composedness in Alexandrov Spaces implies Digital Well-Composedness in $\mathbb{Z}^n$

Nicolas Boutry[1,2] Thierry Géraud[1] Laurent Najman[2]

$\hookrightarrow$ nicolas.boutry@lrde.epita.fr

[1] EPITA Research and Development Laboratory (LRDE), France

[2] Université Paris-Est, LIGM, Équipe A3SI, ESIEE, France

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# Outline

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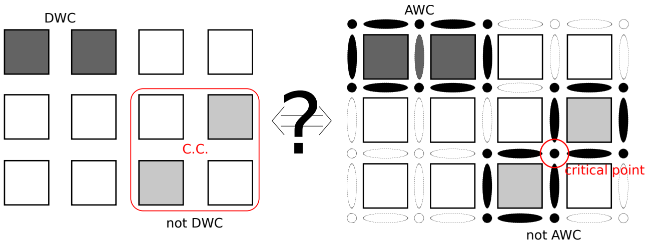
# Context

DWC ≡ Digital Well-Composed:
⇝ a set $X \subset \mathbb{Z}^n$ is DWC iff it does not contain any critical configuration

AWC ≡ Alexandrov Well-Composed:
⇝ a set $X \subset \mathbb{Z}^n$ is AWC iff the boundary of its immersion is made of a disjoint union of discrete $(n-1)$-surfaces,

**Are AWCness and DWCness related ?**



# Why is DWCness so important?

For DWC sets:

- no ambiguity in matter of connectivities (locally),
  - $\rightsquigarrow$ no “hole problem” using the Marching Cubes (2D/3D),
- no ambiguity in matter of connectivities (globally),
  - $\rightsquigarrow$ the tree of shapes is well-defined (Géraud and Najman ISMM 2013),

If AWCness implies DWCness,
AWC sets benefit from these strong properties.

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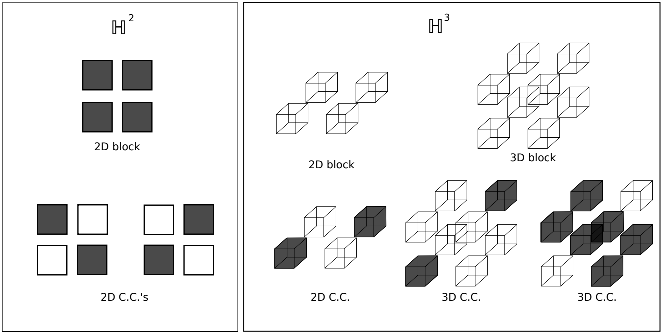
# Digital Well-composedness



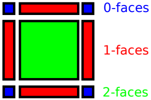


A $n$D set $X$ is said DWC iff $X$ does NOT contain any critical configuration.

# Khalimsky grids

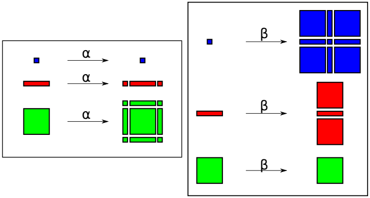
Let $\mathbb{H}^n$ be the Khalimsky grids of dimension $n$.

Faces of dimension $k$ ($k$-faces):



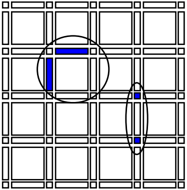

Topological operators:

- $\alpha(f) = \{f' \in \mathbb{H}^n \; ; \; f' \leq \; f\}$
- $\beta(f) = \{f' \in \mathbb{H}^n \; ; \; f \leq f'\}$
- $\theta(f) = \alpha(f) \cup \beta(f)$ ("neighborhood")



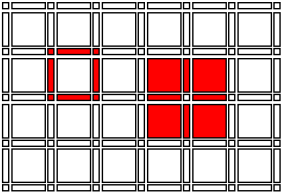
# $n$-D discrete surfaces

0-surfaces

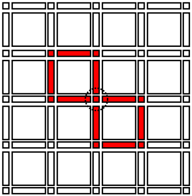


Examples of 0-surfaces :

1-surfaces



not a 1-surface

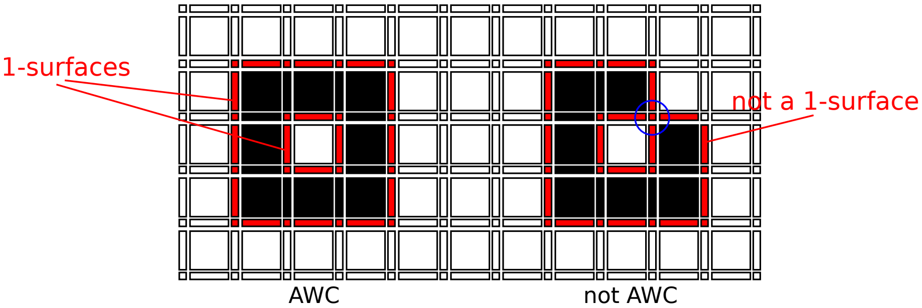


Examples of 1-surfaces :

$|N|$ is a $n$-surface iff:

- $N = \{a, b\}$ such that $a \notin \theta(b)$ when $n = 0$,
- $|N|$ is connected, non empty, and if $\forall z \in N, |\theta^{\square}_N(z)| = \theta(z) \cap N \setminus \{z\}$ is a $(n-1)$-surface when $n \geq 1$.

# Alexandrov Well-Composedness




An AWC set $X$ is a set such as the components of the boundary of its immersion $\mathcal{X}$ in $\mathbb{H}^n$ are $(n-1)$-surfaces.

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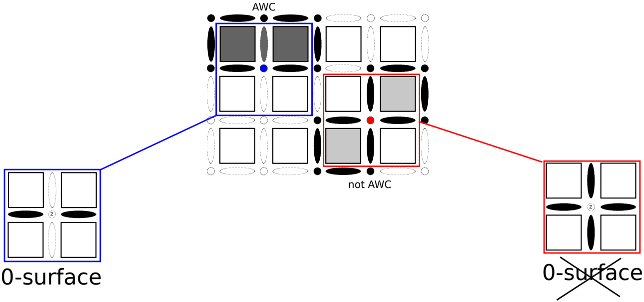
# Reformulating the problem

"$X$ is not AWC" can be expressed in a **local** way:

> $X$ is AWC iff $\forall z \in N$, the subspace $|\beta_N^{\Box}(z)|$ is a $\xi(z)$-surface,

where $N$ is the boundary of the immersion $\mathcal{X}$ of $X$, and
$\xi(z) \equiv (n - 2 - dim(z))$.



# Key idea of proof

Summarily, our aim is then to prove that:

"If $X$ contains a critical configuration,

$\exists z^* \in N,$ s.t. the subspace $|\beta_N^\square(z^*)|$ is NOT a $\xi(z^*)$-surface"

Hint: for $k \geq 0$, the disjoint union of two $k$-surfaces is NOT a $k$-surface.

# Proof

Schematically, we obtain that if $X$ is not DWC,

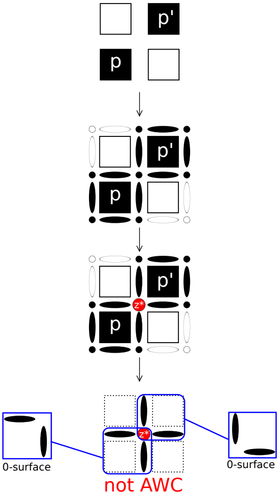
$\exists S \in \mathcal{B}(\mathbb{Z}^n)$, $X \cap S = \{p, p'\}$ or $S \setminus X = \{p, p'\}$ with $p' = \mathrm{antag}_S(p)$,

$\Rightarrow$ the infimum $z^* = p \wedge p'$ between $p$ and $p'$ satisfies:

$$|\beta_N^{\square}(z^*)| = |\alpha(p) \wedge \beta(z^*)| \cup |\alpha(p') \wedge \beta(z^*)|$$

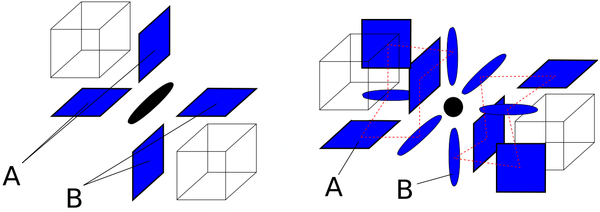
Since $|\alpha(p) \wedge \beta(z^*)|$ and $|\alpha(p') \wedge \beta(z^*)|$ are disjoint $\xi(z^*)$-surfaces, $|\beta_N^{\square}(z^*)|$ is NOT a $\xi(z^*)$-surface.

Then $X$ is not AWC.



# Examples in 2D and in 3D

3D examples:




$|\beta_N^{\square}(z^*)|$ is the union of two 0-surfaces (2D C.C.) or of two 1-surfaces (3D C.C.), and then is NOT a discrete surface.

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# Conclusion and perspectives

Conclusion:
By cross-section topology, we easily extend our result to functions.

Perspective:
"Does DWCness implies AWCness in $n$-D?"

Recall: it is well-known to be true in 2D and in 3D.

Thanks for your attention ! :)