An equivalence relation between morphological dynamics and persistent homology in $1D$

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Outline

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Dynamics vs. persistence

Domains:

- Dynamics $\in$ Mathematical Morphology
- Topological persistence $\in$ Persistent Homology.

Practical uses:

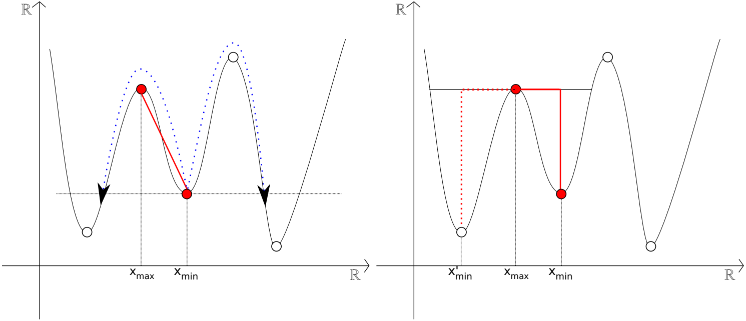
- dynamics $\rightsquigarrow$ markers $\rightsquigarrow$ watersheds $\rightsquigarrow$ segmentation $\rightsquigarrow$ image analysis,
- persistence $\rightsquigarrow$ pairings $\rightsquigarrow$ cancelations $\rightsquigarrow$ simplification $\rightsquigarrow$ image visualisation

Differences:

- dynamics $\rightsquigarrow$ correspond to regional minima,
- persistence $\rightsquigarrow$ is related to gradient vector fields (Morse-Smale complexes).

Both encode topological information of functions.

An example



$\rightsquigarrow$ same pairings but different definitions!

Outline

Morse functions

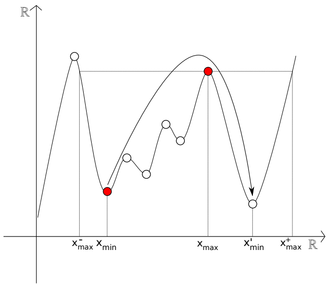
Morse functions (general definition):

- $f \in \mathcal{C}^2(\mathcal{D})$ and the Hessian matrix is not degenerated at the critical points,
- is does not have any plateau,
- for pairings, we need critical values to be unique.

Dynamics I

- $f : \mathbb{R} \to \mathbb{R}$ continuous, and $x_{\min}$ a local minimum of f,
- γ a path following the graph of f from $\gamma(0) := x_{\min}$ to $\gamma(1)$ s.t. $f(\gamma(1)) < f(x_{\min})$,

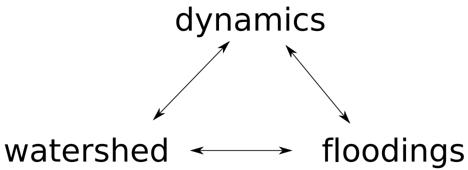
$$\mathrm{dyn}(x_{\min}) := \min_{\forall\ \gamma} \max_{s \in [0,1]} f(\gamma(s)) - f(x_{\min}), \quad \text{[Grimaud (1992)]}$$

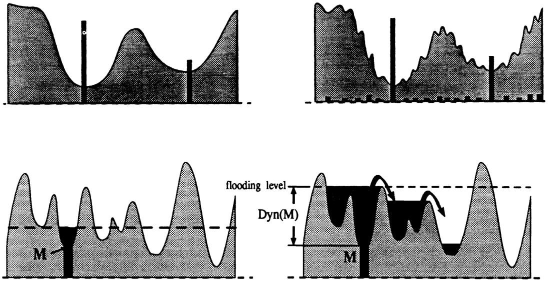


Equivalently, $\mathrm{dyn}(x_{\min}) = f(x_{\max}) - f(x_{\min})$ with $x_{\max}$ the local max of f corresponding to the minimal effort.

Dynamics II

An interesting relation in MM:





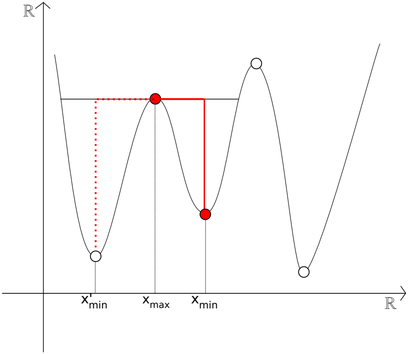
Topological persistence I

Pairing by persistence

- $f : \mathbb{R} \to \mathbb{R}$ a Morse function,
- $x_{\max}$ a local maximum of f,
- $x^-_{\max}, x^+_{\max}$ s.t.
  $[x^-_{\max}, x^+_{\max}] = \mathcal{CC}([f \leq f(x_{\max})], x_{\max})$,
- $\mathcal{CC}_1 := [x^-_{\max}, x_{\max}]$ and $\mathcal{CC}_2 := [x_{\max}, x^+_{\max}]$,
- $\text{rep}_i := \arg\min_{x \in \mathcal{CC}_i} f(x), \ \forall i \in \{1, 2\}$,
- $x_{\min} := \arg\max_{x \in \{rep_1, rep_2\}} f(x)$,

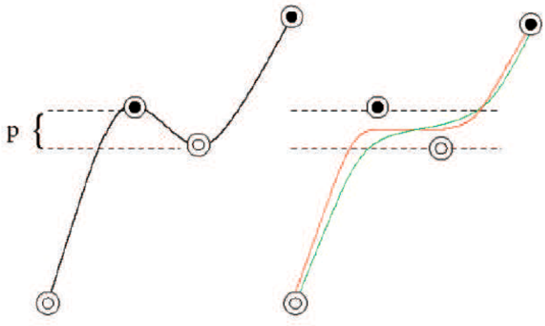
Then, $x_{\max}$ is paired with $x_{\min}$ by persistence.

$\Rightarrow$ Topological persistence $:= f(x_{\max}) - f(x_{\min})$.



Topological persistence II

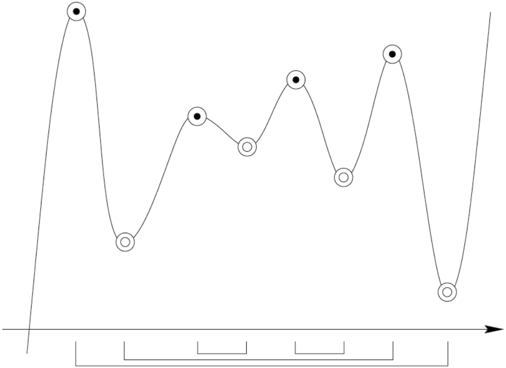
Example of topological simplification in 1D:



We remove critical points but keep the main topological information.

Topological persistence III

Procedure: we remove pairs of points by increasing persistence.

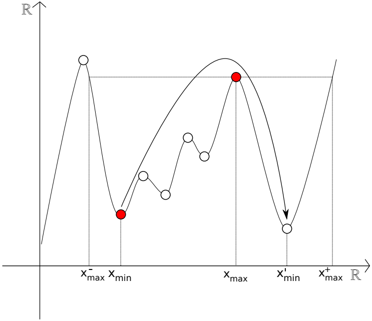


Nested relationship of intervals (pairing by persistence) ⇒ hierarchy of simplifications.

Outline

Hypothesis

- $f : \mathbb{R} \to \mathbb{R}$ *a Morse function,*
- x_{min} *a local minimum of* f*,*
- x_{min} *paired with* x_{max} *by dynamics,*
- $x_{\text{max}} > x_{\text{min}}$,
- $[x^-_{\text{max}}, x^+_{\text{max}}] = CC([f \leq f(x_{\text{max}})], x_{\text{max}})$.



Property $(P1)$

$x_{\text{min}} = \arg\min_{x \in [x^-_{\text{max}}, x_{\text{max}}]} f(x)$

Property $(P2)$

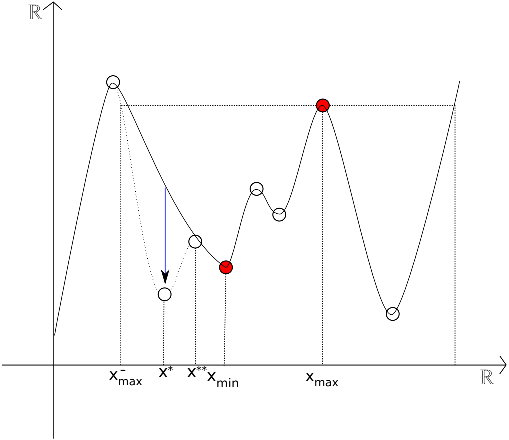
Under some constraints, $x'_{\text{min}} := \arg\min_{x \in [x_{\text{max}}, x^+_{\text{max}}]} f(x)$ *satisfies* $f(x'_{\text{min}}) < f(x_{\text{min}})$.

Theorem

x_{max} *is paired with* x_{min} *by persistence.*

Property $(P1)$

$x_{\min} = \arg\min_{x \in [x^-_{\max}, x_{\max}]} f(x)$



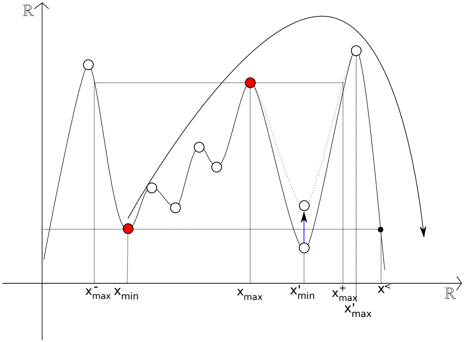
Intuition: if there exists $x^* \in [x^-_{\max}, x_{\max}]$ s.t. $f(x^*)$ is lower than $f(x_{\min})$, $\mathrm{dyn}(x_{\min}) < f(x_{\max}) - f(x_{\min}) \rightsquigarrow$ contradiction.

Property ($P2$)

When we have:

- $[x^-_{\max}, x^+_{\max}] = CC([f \leq f(x_{\max}), x_{\max}])$,
- $x^+_{\max}$ *is finite,*

Then $x'_{\min} := \arg\min_{x \in [x_{\max}, x^+_{\max}]} f(x)$ *satisfies* $f(x'_{\min}) < f(x_{\min})$.



Intuition: if we increase $f(x'_{\min})$ enough, $\mathrm{dyn}(x_{\min})$ increases too $\rightsquigarrow$ contradiction.

First main result of this paper:

Theorem

When f is a 1D Morse function and $x_{\min}$ and $x_{\max}$ are paired by dynamics, then $x_{\max}$ and $x_{\min}$ are paired by persistence too.

We have proved that pairing by dynamics implies pairing by persistence. Let us prove the converse.

Outline

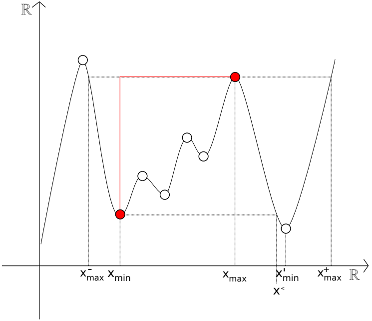
Pairing by persistence implies pairing by dynamics I

Hypothesis

- *$f : \mathbb{R} \to \mathbb{R}$ a Morse function with x_{max} a local maximum of f,*
- *x_{max} and x_{min} are paired by persistence with $x_{min} < x_{max}$.*
- $[x^-_{max}, x^+_{max}] = CC([f \leq f(x_{max})], x_{max})$,
- $x'_{min} := \arg\min_{x \in [x_{max}, x^+_{max}]} f(x)$

Property

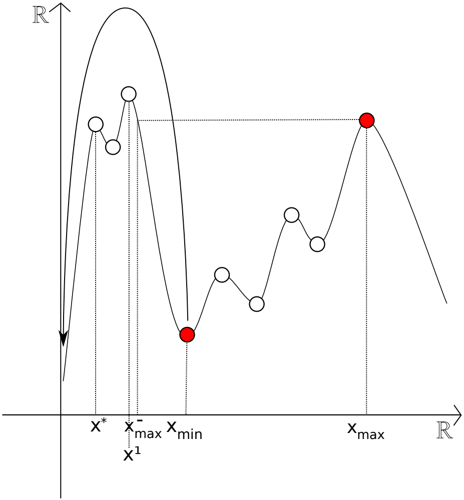
1. *$\exists\, \gamma$ from x_{min} to x'_{min} following the graph of f corresponding to an "effort" of $f(x_{max}) - f(x_{min})$,*
2. *then $\mathrm{dyn}(x_{min}) \leq f(x_{max}) - f(x_{min})$,*



Pairing by persistence implies pairing by dynamics II

Property

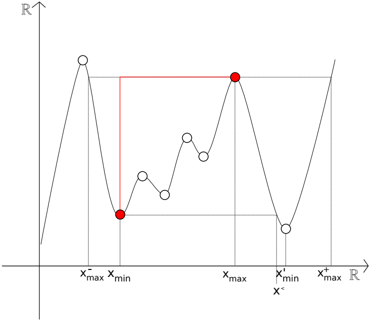
3. *if* $x_{\min}$ *is paired with* $x^* < x_{\min}$ *by dynamics, then* $\mathrm{dyn}(x_{\min}) > f(x_{\max}) - f(x_{\min})$ *(Contradiction).*
4. *then* $x_{\min}$ *is paired with an abscissa* $x^* > x_{\min}$ *by dynamics,*



Pairing by persistence implies pairing by dynamics III

Property

5. *to "reach" a level lower than $f(x_{\min})$ on f, we need to go through $x_{\max}$, then* $\mathrm{dyn}(x_{\min}) \geq f(x_{\max}) - f(x_{\min})$,
6. *then* $\mathrm{dyn}(x_{\min}) = f(x_{\max}) - f(x_{\min})$,
7. *the only local extremum satisfying* (6) *is* $x_{\max}$.



Second main result of this paper:

Theorem

When f is a 1D Morse function and $x_{\min}$ and $x_{\max}$ are paired by persistence, then $x_{\max}$ and $x_{\min}$ are paired by dynamics too.

Outline

Summary

- on Morse functions, pairings by persistence and by dynamics are equivalent,
- persistence and dynamics values are then equal,
- another relation between MM and MT:

$$WS(f) \cup WS(-f) = MS(f),$$

- finally, we reinforced the relation between MM and MT!

Future works

- 2D extension of dynamics $\Leftrightarrow$ persistence,
- extension to discrete Morse functions (Forman),
- investigate if algorithms of MM can be used in PH and conversely,

Questions

Is this a Morse function?