Self-duality and discrete topology:

Links between the morphological tree of shapes and well-composed gray-level images

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This talk is about...

Objective

a self-dual representation of gray-level images without topological issues

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Motivation

- get very strong topological properties
- ensure a “pure” self-duality
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Keypoint

- one connectedness relationship
- i.e., a unique topological structure

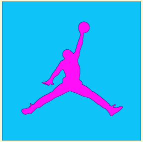
REMINDER

Let's start by reviewing some basic things about:

- digital topology
- self-duality
- mathematical morphology

Jordan Curve Theorem

A simple closed curve divides the plane into two regions ("interior" and "exterior").



Digital Topology and Connectivities

Jordan Curve Theorem

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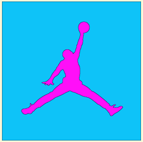

in discrete topology, a "Jordan pair" of connectivities $(c_\alpha,\ c_\beta)$ are required:
one for the interior, the other for the exterior

for instance: $(c_4,\ c_8)$ in 2D, $(c_6,\ c_{18})$ or $(c_6,\ c_{26})$ in 3D, $(c_{2n},\ c_{3^n-1})$ in nD.

Practially, given a set X:

- choose either c_α or c_β for the "object" X
- choose the other one for the "background", i.e., $\complement X$
- so there is no topological paradox

Connectivities and Sets

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in this talk:

- $X \subset \mathbb{Z}^n$
- so we follows the path of Bhattacharya, Eckart, Latecki, Rosenfeld, and Wang...

SELF-DUALITY

Imagine that you process an image u:

$$u \xrightarrow{\text{processing}} \varphi(u)$$

Self-duality

Imagine that you apply the same processing to $\complement u$:

$$
\begin{array}{ccc}
u & \xrightarrow{\text{processing}} & \varphi(u) \\
\text{complementation} \downarrow & & \\
\complement u & \xrightarrow{\text{processing}} & \varphi(\complement u)
\end{array}
$$

Self-duality

You may want a self-dual behavior:

$$
\begin{tikzcd}[column sep=huge]
u \arrow[r, "\text{processing}"] \arrow[d, "\text{complementation}"'] & \varphi(u) \arrow[d, "\text{complementation}"] \\
\complement u \arrow[r, "\text{processing}"] & \varphi(\complement u) \; = \; \complement\, \varphi(u)
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that is, <u>you process the same way the image contents whatever the contrast</u>

(i.e., light objects over dark background versus dark objects over light background)

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that is, <u>you process the same way the image contents whatever the contrast</u>

(i.e., light objects over dark background versus dark objects over light background)

sometimes:

- we <u>cannot</u> make an assumption about contrast
- we <u>do not want</u> to make such an assumption
  because “object” $\neq$ “subject” $\rightsquigarrow$

Mathematical Morphology (MM)

A very particular way to define MM

- a gray-scale image is considered as a landscape; gray values are elevations
- to process an image is to modify the landscape, i.e., its topography
- we transform the shape of the landscape

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A possible taxonomy of MM:

- with a structuring element or without s.e.
- on sets (binary images) or on functions (gray-level images)
- dual operators or self-dual operators
- connected operators or not

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The context of this work;

- the powerful subset of MM emphasized above
- this subset relies on component trees

Two Morphological Dual Trees...

A Couple of Dual Sets

Given a nD image $u : \mathbb{Z}^n \to \mathbb{Z}$,

$$\begin{aligned} \text{lower level sets:} \quad & [\, u < \lambda \,] = \{\, x \in X \mid u(x) < \lambda \,\} \\ \text{upper level sets:} \quad & [\, u \geq \lambda \,] = \{\, x \in X \mid u(x) \geq \lambda \,\} \end{aligned}$$

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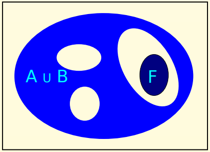
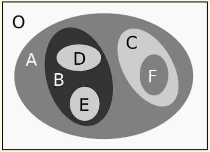
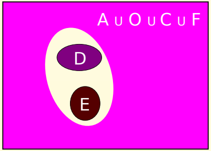


a lower level set



u



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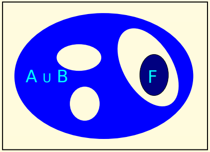

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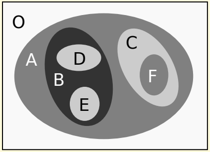

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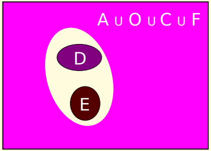

a upper level set

$\rightsquigarrow$ we focus on the connected component of the lower and upper level sets

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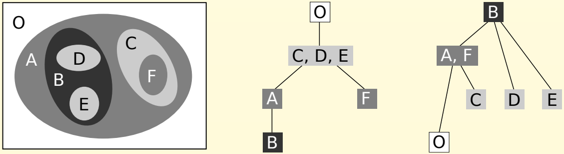

...AND A SELF-DUAL TREE

SHAPES

With the cavity-fill-in operator Sat:

$$\begin{array}{lrcl} \text{lower shapes:} & \mathcal{S}_{<}(u) & = & \{\, \mathrm{Sat}(\Gamma);\ \Gamma \in \mathcal{T}_{<}(u) \,\} \\ \text{upper shapes:} & \mathcal{S}_{\geq}(u) & = & \{\, \mathrm{Sat}(\Gamma);\ \Gamma \in \mathcal{T}_{\geq}(u) \,\} \end{array}$$

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A self-dual tree

$$\rightsquigarrow \quad \text{tree of shapes:} \quad \mathfrak{S}(u) \;=\; \mathcal{S}_<(u) \;\cup\; \mathcal{S}_\geq(u)$$

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A SELF-DUAL TREE

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PROPERTY = SELF-DUALITY

we *"almost"* have: $\mathfrak{S}(\complement u) \; = \; \mathfrak{S}(u)$

—that contrats with the duality of the min- and max- trees: $\mathcal{T}_\geq(\complement u) \; = \; \mathcal{T}_<(u)$—

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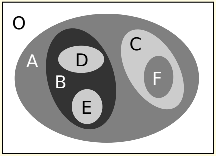
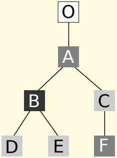
Schematic Example

image

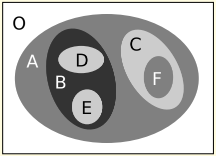


tree of shapes

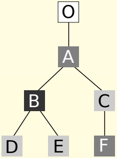


SCHEMATIC EXAMPLE

image



tree of shapes

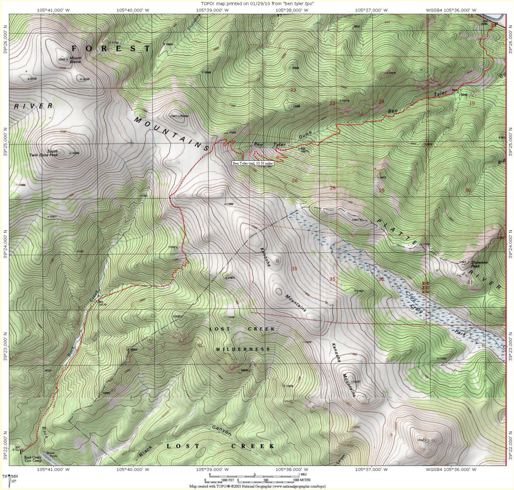


ALT. DEFINITIONS OF SHAPES

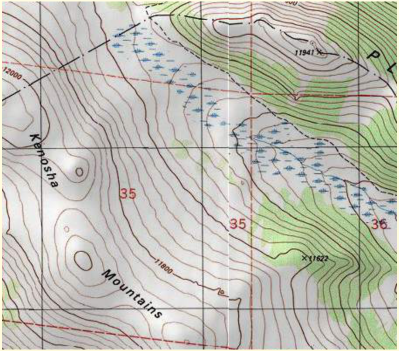
the shapes are

- the cavities of upper and lower level sets
- the interior regions of level lines.

A SELF-DUAL TOPOGRAPHIC TREE-BASED REPRESENTATION



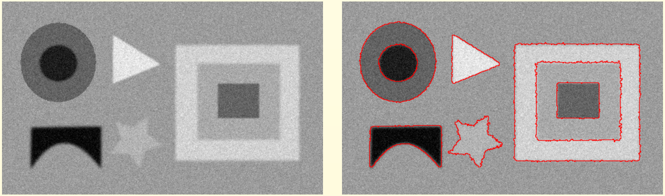
A Self-Dual Topographic Tree-Based Representation



excerpt

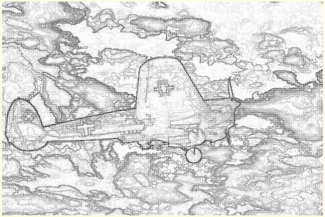
grain filter

Some Applications

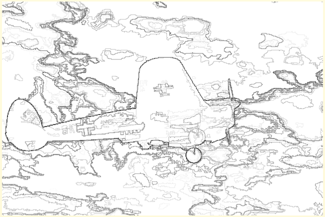


object detection

Some Applications



input $\rightsquigarrow$ contour saliency



extinction $\rightsquigarrow$ (hierarchical) segmentation

Some Applications

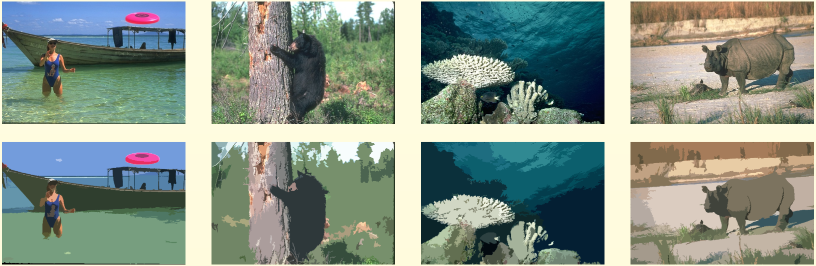
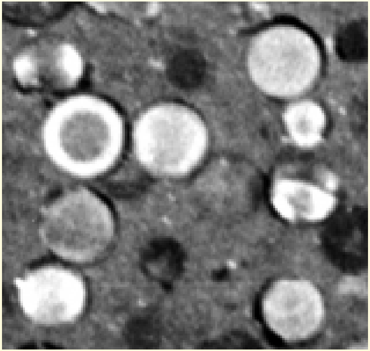
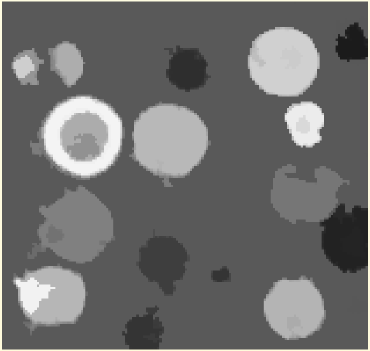


image simplification

Some Applications





morphological shapings

Some Applications

local feature detection

Dealing with Dual Connectivities

whatever a connectivity c (with $-c$ denoting its "dual"), and a relation $\mathcal{R}$

from a set of components, we can have a set of shapes:

$$\mathcal{T}_{(\mathcal{R},c)} \;=\; \{\, \Gamma \in \mathcal{CC}_c([u\,\mathcal{R}\,\lambda])\,\}_\lambda \;\longrightarrow\; \mathcal{S}_{(\mathcal{R},c)}(u) \;=\; \{\, \mathrm{Sat}_{-c}(\Gamma);\; \Gamma \in \mathcal{T}_{(\mathcal{R},c)}(u)\,\}$$

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and derive a "properly" defined tree of shapes:

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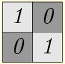
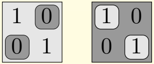
For instance:

$$\mathfrak{S}_{(<,\,c_4)}(\complement u) \;=\; \mathfrak{S}_{(\leq,\,c_8)}(u)$$

CONSEQUENCES 1/2

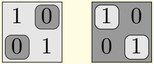
We have an arbitrary choice between $\mathfrak{S}_{(<,\,c_\alpha)}(u)$ and $\mathfrak{S}_{(>,\,c_\alpha)}(u)$:

$u =$ 

$\longrightarrow$ two possible trees: 

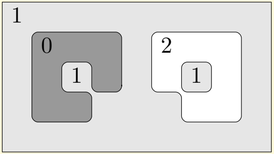
We have an arbitrary choice between $\mathfrak{S}_{(<,\, c_\alpha)}(u)$ and $\mathfrak{S}_{(>,\, c_\alpha)}(u)$:

$$u = \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \longrightarrow$$ two possible trees:



When choosing $\mathfrak{S}_{(<,c_\alpha)}(u)$, lower and upper shapes are resp. c_α and c_β:

$$u = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&1&1&1 \\ \hline 1&0&0&0&1&2&2&2&1 \\ \hline 1&0&1&0&1&2&1&2&1 \\ \hline 1&0&0&1&1&1&2&2&1 \\ \hline 1&1&1&1&1&1&1&1&1 \\ \hline \end{array} \longrightarrow \mathfrak{S}_{(<,\, c_\alpha)}(u) =$$



Consequences 2/2

If u is continuous (or discrete with some continuity property†):

the different types of shapes do **not** have the same topology!

for instance:
in $\mathfrak{S}_{(<,\, c_\alpha)}(u)$, lower shapes are open sets v. upper shapes are closed sets.

† T. Géraud, E. Carlinet, S. Crozet, L. Najman. *A quasi-linear algorithm to compute the tree of shapes of nD images*. In Proc. of the 11th Intl. Symp. on Mathematical Morphology (ISMM), 2013.

L. Najman, T. Géraud. *Discrete set-valued continuity and interpolation*. In Proc. of the 11th Intl. Symp. on Mathematical Morphology (ISMM), 2013.

THE GRAAL

we want:

A PURELY SELF-DUAL TREE

$$\mathfrak{S}(\complement u) \;=\; \mathfrak{S}(u)$$

The Graal

we want:

A purely self-dual tree

$$\mathfrak{S}(\complement u) \;=\; \mathfrak{S}(u)$$

that starts with:

A first requirement

a single connectivity relation for both lower and upper shapes

Dummy Examples

with c_4 for both types of shapes (so Sat_{c_8}), we have those two shapes:

2	2	2	2	2	2
2	2	0	0	0	2
2	0	1	2	0	2
2	2	0	0	2	2
2	2	2	2	2	2

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with c_8 for both types of shapes (so Sat_{c_4}), we have those two shapes:

1	1	1	1	1	1
1	1	2	0	1	1
1	2	0	2	0	1
1	1	2	0	1	1
1	1	1	1	1	1

1	1	1	1	1	1
1	1	2	0	1	1
1	2	0	2	0	1
1	1	2	0	1	1
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in both cases, we do **not** have: $S_1 \cap S_2 \neq \emptyset \;\Rightarrow\; (\, S_1 \subset S_2 \text{ or } S_2 \subset S_1 \,)$

$\rightsquigarrow$ the set of shapes is **not** a tree / it is a lattice since $(\mathfrak{S}_c, \subset)$ is a poset

The SLIDE!

given any gray-level image u

- $\mathfrak{S}_c(u)$ is a lattice
- taking $c = c_\alpha$ or $c = c_\beta$ is equivalent when an image is *well-composed* (...)
- we can have an interpolation $\mathfrak{I}(u)$ of u that is a well-composed image
- we expect $\mathfrak{S}_{c_\alpha}(\mathfrak{I}(u))$ to be a perfectly self-dual tree of shapes

under constraints

- the interpolation $\mathfrak{I}$ has to be self-dual, i.e., $\mathfrak{I}(\complement u) = \complement\mathfrak{I}(u)$
- $\mathfrak{I}(u)$ can be considered as a rasterization equivalent to u
- we shall stick to the "morphological way":

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- extend the notion of "well-composedness" to nD images on a cubical grid
- prove that:

  if a gray-level nD image v is WC then $\mathfrak{S}_c(v)$ is a tree

- study how to make a 2D image well-composed, that is:
- see if we can do it the same way for $\mathfrak{I}^{3D}$
- deal with the nD case

What's Up Now...

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2D Well-Composed (WC) Sets (Latecki, CVIU, 1995)

WCness for 2D Sets

Definitions:

- a set is weakly well-composed if any 8-component of this set is a 4-component
- a set is well-composed if both this set and its complement are weakly WC

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Local Characterization

- a set X is locally 4-connected if $\forall\, p \in X,\ \mathcal{N}_8(p) \cap X$ is 4-connected
- X is locally 4-connected $\Leftrightarrow$ X is well-composed

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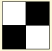
Critical Configurations

It is equivalent to:

a set is WC if the configurations  and  do not appear

WC for Gray-Level Pictures (Latecki, CVIU, 1995)

Extension to Gray-Levels

A gray-level image u is well-composed if any set $[u \geq \lambda]$ is well-composed.

Example of an image (left) whose interpolation (right) is well-composed:

3	2
1	8

3	2	2
2	2	5
1	4	8

$\rightsquigarrow$ for every blocks

a	d
c	b

we should have: $\text{intvl}(a, b) \cap \text{intvl}(c, d) \neq \emptyset$

where $\text{intvl}(v, w) = [\![\min(v, w), \max(v, w)]\!]$

WC for 3D Sets (Latecki, GMIP, 1997)

Definition

a set X is well-composed if ∂X is a 2D manifold *in the continuous analog*

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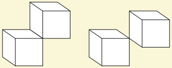
- the configurations  do not appear in X
- every component of ∂X is a simple closed surface

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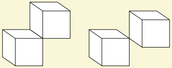
- the configurations  do not appear in X
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About Jordan-Bouwer Theorem

if X is WC, then, $\forall\, S \in \mathcal{CC}(\partial X)$, $\mathbb{R}^3 \setminus S$ has precisely 2 connected components of which S is the common boundary

BREATHE!

2, 3, ... then n

WCNESS IN nD NEW!

DEFINITION

a nD set X is <u>well-composed</u> if ∂X is a nD manifold *in the continuous analog*

WCness in nD NEW!

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Logical Equivalences

It is equivalent to:

- X is locally c_{2n}-connected, i.e., $\forall\, p \in X,\ \mathcal{N}_{c_{3^n-1}}(p) \cap X$ is c_{2n}-connected
- ∂X is a discrete n-surface *in the cellular complex*
- the restriction of X to any hyperplane of $\mathbb{Z}^n$ is well-composed (in $\mathbb{Z}^{n-1}$) and the critical configuration based on c_{3^n-1} does not appear

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SAME EXTENSION TO GRAY-LEVELS

A gray-level nD image u is well-composed if any set $[u \geq \lambda]$ is well-composed.

Link Between WC and ToS NEW!

The Return of the Tree of Shapes

if a gray-level nD image u is well-composed, then $\mathfrak{S}(u)$ is a tree

The Return of the Tree of Shapes

if a gray-level nD image u is well-composed, then $\mathfrak{S}(u)$ is a tree

it is a sufficient condition (not a necessary one):

with $u = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 1 & 0 & 2 & 1 \\ \hline 1 & 2 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array}$, $\mathfrak{S}(u)$ is a tree, while u is not well-composed.

Sketch of the Proof

With $\mathfrak{T}(u) = \mathcal{T}_{<}(u) \cup \mathcal{T}_{\geq}(u)$, consider $A \in \mathfrak{T}(u)$ and $B \in \mathfrak{T}(u)$.

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- if $A \cap B = \emptyset$
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    this Lemma is proven in the book from Caselles & Monasse (LNCS vol. 1984, 2009)

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- otherwise $A \cap B \neq \emptyset$
  - case "A and B with the same type" (e.g., $A \in \mathcal{CC}([u < \lambda])$ and $B \in \mathcal{CC}([u < \mu])$:
    since $A \cap B \neq \emptyset$, we have either $A \subseteq B$ or $B \subseteq A$
    since Sat is increasing, $\text{Sat}(A)$ and $\text{Sat}(B)$ are nested
  - case "A and B with different types" (e.g., $A \in \mathcal{CC}([u \geq \lambda])$ and $B \in \mathcal{CC}([u < \mu])$:
    with $x \in A \cap B$, $\lambda \leq u(x) < \mu \Rightarrow \lambda < \mu$
    let $\Delta B = \{ q \in \mathcal{N}_{c_{2n}}(p) \mid p \in B,\ q \notin B \}$, so we have $\Delta B \subseteq [u \geq \mu] \subseteq [u \geq \lambda]$
    $\rightsquigarrow$ cont'd next slide

Sketch of the Proof

we can split $\Delta B = E \cup C$ where

- C is the part of ΔB included in cavities of B
- E is the other part ($\approx$ E *is the "external" boundary of* B *w.r.t.* c_{2n})

we have:

unicoherency <u>and</u> well-composedness $\Rightarrow$ E is a connected component

b	e
e'	?

$\Rightarrow$

b	e
e'	e"

which is crucial for the following!

look there $\downarrow$

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we have:

- $\text{Sat}(\Delta B) = \text{Sat}(E)$
- a component $F \in \mathcal{CC}([u \geq \lambda])$ exists such as $E \subseteq F$ $\leftarrow$ here!

so:

- either $F \cap A = \emptyset$ then $A \subseteq \text{Sat}(B)$ so $\text{Sat}(A) \subseteq \text{Sat}(B)$
- or $F \cap A \neq \emptyset$ then $F \subseteq A$ thus
  $\text{Sat}(B) \subseteq \text{Sat}(\Delta B) = \text{Sat}(E) \subseteq \text{Sat}(F) \subseteq \text{Sat}(A)$

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unicoherency <u>and</u> well-composedness $\Rightarrow$ E is a connected component

$$\begin{array}{|c|c|}\hline b & e \\ \hline e' & ? \\ \hline\end{array} \quad \Rightarrow \quad \begin{array}{|c|c|}\hline b & e \\ \hline e' & e'' \\ \hline\end{array}$$

which is crucial for the following!

look there $\downarrow$

we have:

- $\mathrm{Sat}(\Delta B) = \mathrm{Sat}(E)$
- a component $F \in \mathcal{CC}([u \geq \lambda])$ exists such as $E \subseteq F$ $\leftarrow$ here!

so:

- either $F \cap A = \emptyset$ then $A \subseteq \mathrm{Sat}(B)$ so $\mathrm{Sat}(A) \subseteq \mathrm{Sat}(B)$
- or $F \cap A \neq \emptyset$ then $F \subseteq A$ thus
  $\mathrm{Sat}(B) \subseteq \mathrm{Sat}(\Delta B) = \mathrm{Sat}(E) \subseteq \mathrm{Sat}(F) \subseteq \mathrm{Sat}(A)$

$\square$

Key-Point of the Proof

we expect B to be included in the saturation of a component such as A:

u

2	2	2	2	2	2
2	2	0	0	0	2
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2	2	2	2	2	2

B

2	2	2	2	2	2
2	2	0	0	0	2
2	0	1	2	0	2
2	2	0	0	2	2
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A

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2	2	0	0	2	2
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yet

- E may **not** be a connected component if the image is not WC
- so we may **not** have a component $F \in \mathfrak{T}$ such as B is in a cavity of F
- here the candidate is A and we don't have $\mathrm{Sat}(B) \subseteq \mathrm{Sat}(A)$

RECAP

a gray-level image v is well-composed $\Rightarrow$ $\mathfrak{S}(v)$ is a purely self-dual tree

Recap

a gray-level image v is well-composed $\Rightarrow$ $\mathfrak{S}(v)$ is a purely self-dual tree

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when done, $\mathfrak{I}(u)$ is a self-dual representation of u with a perfect tree of shapes

we thus have to find $\mathfrak{I}$ ... let's start with the 2D case!

Making a 2D Image WC: The Constraints (1/2)

$$u = \begin{array}{|c|c|} \hline a & d \\ \hline c & b \\ \hline \end{array} \quad \longrightarrow \quad \mathfrak{I}_{2D} = \begin{array}{|c|c|c|} \hline a & ? & d \\ \hline ? & ? & ? \\ \hline c & ? & b \\ \hline \end{array}$$

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Constraints:

- determinism
  - an increasing function f exists such as $ad = f(a, d)$, $ac = f(a, c)$, and so on
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  - $f(v,w) = f(w,v)$
  - $g(a,b,c,d) = g(a,b,d,c)$, and the other symmetries
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  - $g(a, b, c, d) = g(c, d, b, a)$, and the other rotations
- no new extremum
  - $f(v, w) \in \text{intvl}(v, w)$
  - $m \in \text{intvl}(ac, bd)$ and $m \in \text{intvl}(ad, bc)$

Making a 2D Image WC: The Constraints (2/2)

$$u = \begin{array}{|c|c|} \hline a & d \\ \hline c & b \\ \hline \end{array} \quad \longrightarrow \quad \mathfrak{I}_{2D} = \begin{array}{|c|c|c|} \hline a & ad & d \\ \hline ac & m & bd \\ \hline c & bc & b \\ \hline \end{array}$$

Other constraints:

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  - $\text{intvl}(a, m) \cap \text{intvl}(ac, ad) \neq \emptyset$ (top left)
  - likewise for the 3 other 2×2 parts

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  - $f(\complement v, \complement w) = \complement f(v, w)$
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- extra and <u>optional</u>
  - h exists such as $g(a, b, c, d) = h(f(a, c), f(b, d)) = h(f(a, d), f(b, c))$
  - so we have $h(\complement v,\ \complement w) = \complement\, h(v,\ w)$
  - and $h(v, w) = h(w, v) \in \text{intvl}(v, w)$

Making a 2D Image WC: First Attempts

$$u = \begin{array}{|c|c|} \hline a & d \\ \hline c & b \\ \hline \end{array} \quad \longrightarrow \quad \mathfrak{I}_{2D} = \begin{array}{|c|c|c|} \hline a & f(a,d) & d \\ \hline f(a,c) & g(a,b,c,d) & f(b,d) \\ \hline c & f(b,c) & b \\ \hline \end{array}$$

Nota bene: if f is bisymmetrical, i.e., $f(\,f(a,c), f(b,d)\,) = f(\,f(a,d), f(b,c)\,)$ then g is just applying f twice (like in the bilinear interpolation)

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Idea #1: f is either min or max

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- satisfy all constraints except self-duality, since $\min(\complement v, \complement w) = \complement \max(v,w)$

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Idea #2: f is a mean (i.e., $\min \leq f \leq \max$, $f \neq \min$, $f \neq \max$, $f(v,w) = f(w,v)$, f increasing)

- some well-known bisymmetrical means: $2\,x\,y/(x+y)$, $(x+y)/2$, $\sqrt{x\,y}$, $\sqrt{(x^2+y^2)/2}$
- yet they fail with self-duality and/or WCness!

Making a 2D Image WC: an How-To

- consider a 3x3 part of $\mathfrak{I}(u)$ and a threshold set X
- notation: $\bullet \in X$, $\bullet \in \complement X$, and $\circ$ when we do not know
- it yields to 4 cases (modulo symmetries, rotations, and $\complement$ation)
- using only the "no new extremum" constraint, we have:

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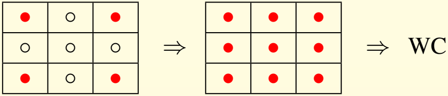

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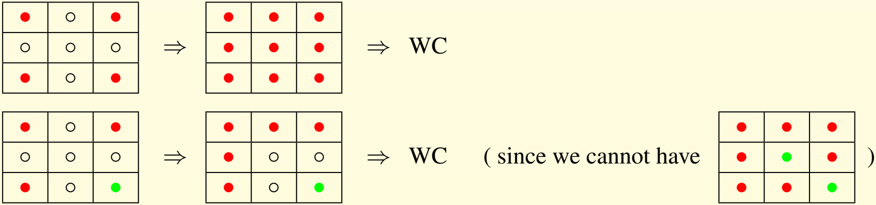

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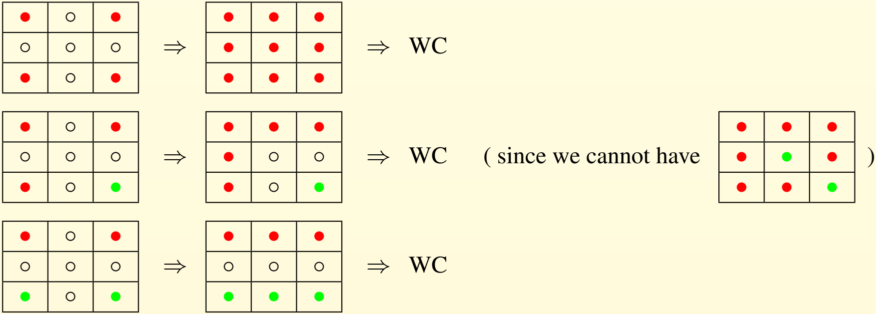

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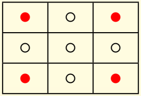
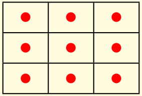
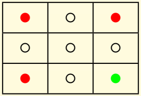
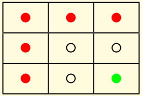
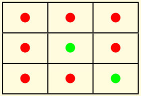
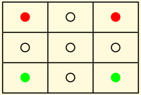
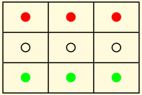
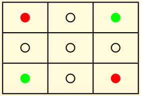
 $\Rightarrow$  $\Rightarrow$ WC

 $\Rightarrow$  $\Rightarrow$ WC (since we cannot have )

 $\Rightarrow$  $\Rightarrow$ WC

 $\Rightarrow$? so we have to study this case...

THE SADDLE-POINT CASE

a	ad	d
ac	m	bd
c	bc	b

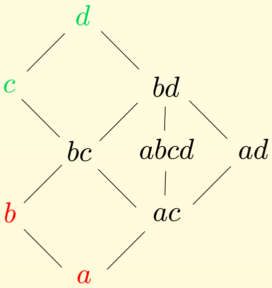
$\rightsquigarrow$

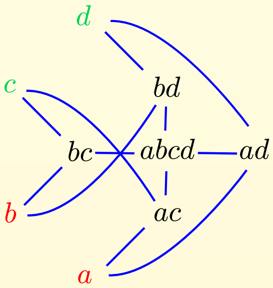
●	○	●
○	○	○
●	○	●

let us assume that $a < b < c < d$, *nota bene*: below $abcd = m$

just remark that $v < w \Rightarrow v < vw < w$ (the "no new extr." constraint)

so we have the following Hasse diagram (left) and depicted with 4-adjacencies (right):

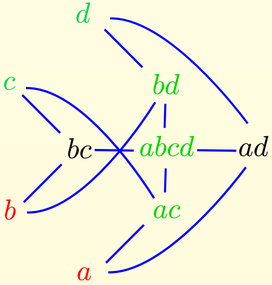




THE SADDLE-POINT CASE

Assume that the point of value *ac* is in X (so depicted in green)

we thus have:



so:

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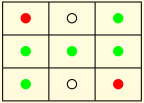
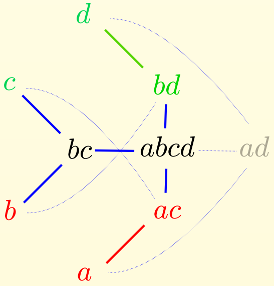
$\rightsquigarrow$



$\Rightarrow$ WC

The same goes when assuming that the point of value *bd* is in $\complement X$ (red).

The remaining case is therefore:



so:

a	ad	d
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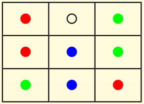
$\rightsquigarrow$



$\Rightarrow$

$$\text{WC} \quad \text{iff} \quad m = bc \quad \text{i.e.,} \quad \text{iff} \quad g(a,b,c,d) = f(b,c)$$

THE CONCLUSION FOR 2D

in the morphology setting, we want <u>an operator</u> so:

$$(\text{ WC} \quad \text{iff} \quad \mathrm{op}(\{a,b,c,d\}) \;=\; \mathrm{op}(\{b,c\})\,) \quad \Rightarrow \quad \mathrm{op} \ \text{ is } \underline{\text{a}} \text{ median}$$

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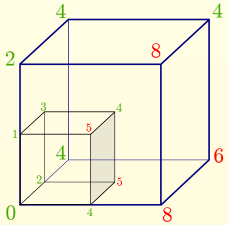
...what about in 3D?

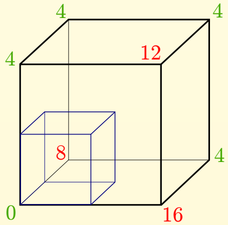
What About 3D?

consider $X = [\, u \leq 4 \,]$ with $u =$



one median subdivision with a critical configuration, so: $\mathfrak{I}^{3D}_{\text{med}} \not\Rightarrow$ WC

consider $X = [\, u' \leq 4 \,]$ with $u' =$



a first subdivision gives the blue cube above, so: $\mathfrak{I}^{3D}_{\text{med}} \circ \mathfrak{I}^{3D}_{\text{med}} \not\Rightarrow$ WC

THE CONCLUSION FOR 3D AND nD

CONJECTURE

for $n > 2$

there is no self-dual nD interpolation *operator* (i.e., writable without "if")
that makes well-composed an gray-level image defined on $\mathbb{Z}^n$

whatever the number of subdivisions of the space

CONCLUSION (1/2)

- we can get rid of topological paradoxes thanks to
  - a single connectivity relationship *and/or* self-duality
  - the notion of well-composedness of nD gray-level images
  - the how-to in 2D: a local interpolation with the median operator

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  - strong topological properties
  - a purely self-dual representation of 2D images, that is, the tree of shapes

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we have explored the links between the notion of well-composedness
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we have some new (interesting?) results and proofs

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QUASI-LINEAR ALGORITHM

A quasi-linear algorithm to compute the tree of shapes of nD images.
T. Géraud, E. Carlinet, S. Crozet, and L. Najman.
ISMM, 2013.

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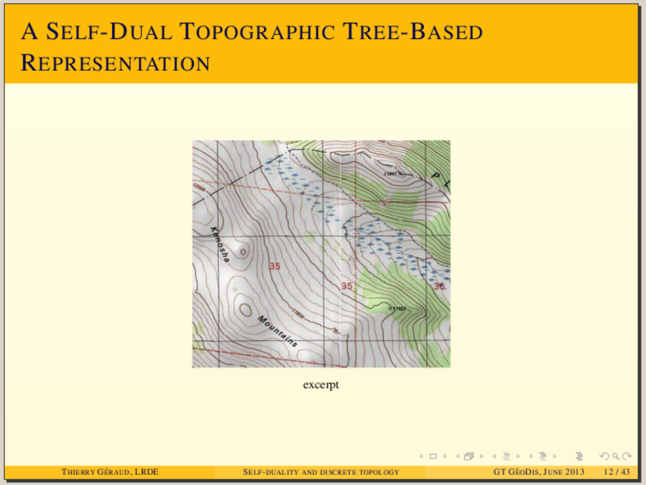
A Discrete **yet** Continuous Representation

Discrete set-valued continuity and interpolation.
L. Najman and T. Géraud.
ISMM, 2013.

remember that slide:



Left as an Exercise to the Reader...

draw the level lines of respective levels 1 and 3 for this image:

6	6	6	6	6
6	4	4	2	6
6	4	0	4	6
6	0	4	4	6
6	6	6	6	6

so what!?

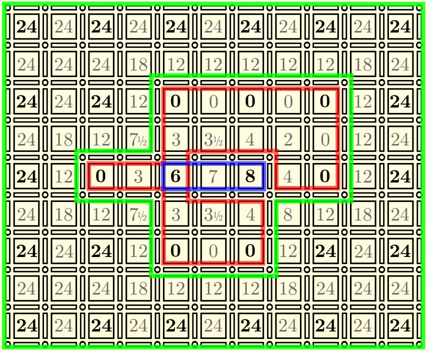
some "past-the-end" slides

To be Interpolated

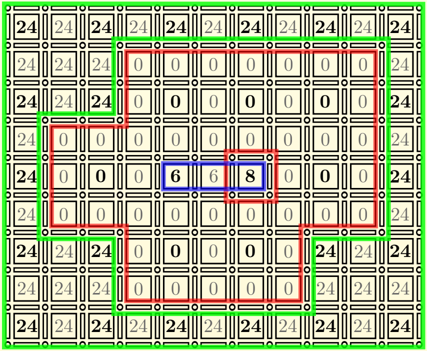
u

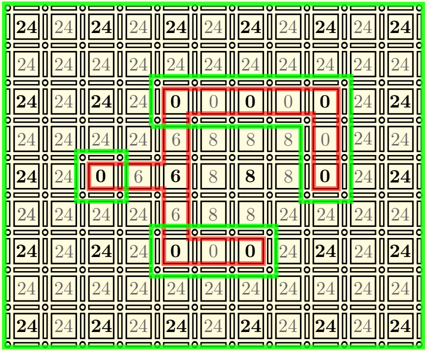
24	24	24	24	24	24
24	24	0	0	0	24
24	0	6	8	0	24
24	24	0	0	24	24
24	24	24	24	24	24

$$\mathfrak{I}_{\text{mean}}(u) \rightsquigarrow \text{poset}\,(\mathfrak{S}_c(u), \subset)$$



$$\mathfrak{I}_{\min}(u) \;\rightsquigarrow\; \mathfrak{S}_{(>,\,c_\alpha)}(u) \qquad \text{and} \qquad \mathfrak{I}_{\max}(u) \;\rightsquigarrow\; \mathfrak{S}_{(<,\,c_\alpha)}(u)$$





Self-Dual Interpolation

$$\mathfrak{I}_{\text{med}}(u) \;\rightsquigarrow\; \mathfrak{S}(u)$$