Introducing the Dahu Pseudo-Distance

Que la montagne de pixels est belle. Jean Serrat.

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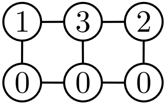
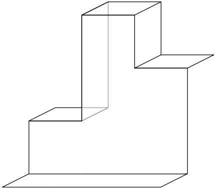
About image representations

1	3	2
0	0	0

a 2D array



a graph



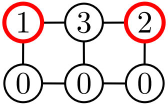
a surface

L. Najman and J. Cousty, "A graph-based mathematical morphology reader," *Pattern Recognition Letters*, vol. 47, pp. 3-17, Oct. 2014. [PDF]

The Minimum Barriere (MB) Distance

MB distance

minimal interval of gray-level values
in an image along a path between two points,
where the image is considered as a vertex-valued graph



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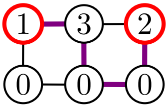

pink path values $= \langle 1, 3, 0, 0, 2 \rangle \rightsquigarrow$ interval $= [0, 3] \rightsquigarrow$ barrier $= 3$

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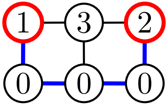

$$\text{blue path values} = \langle 1, 0, 0, 0, 2 \rangle \rightsquigarrow \text{interval} = [0, 2] \rightsquigarrow \text{barrier} = 2$$

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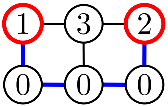

$$\text{blue path values} = \langle 1, 0, 0, 0, 2 \rangle \rightsquigarrow \text{interval} = [0, 2] \rightsquigarrow \text{barrier} = 2$$

$$\rightsquigarrow \text{distance } d^{\text{MB}} = 2$$

Formally

MB distance

Barrier of a path π in a gray-level image u:

$$\tau_u(\pi) \;=\; \max_{\pi_i \in \pi} u(\pi_i) \;-\; \min_{\pi_i \in \pi} u(\pi_i).$$

Minimum barrier distance between x and x' in u:

$$d_u^{\mathrm{MB}}(x, x') \;=\; \min_{\pi \in \Pi(x,x')} \tau_u(\pi).$$

This is a *pseudo*-distance:

- $d_u^{\mathrm{MB}}(x) \geq 0$ (non-negativity)
- $d_u^{\mathrm{MB}}(x, x) = 0$ (identity)
- $d_u^{\mathrm{MB}}(x, x') = d_u^{\mathrm{MB}}(x', x)$ (symmetry)
- $d_u^{\mathrm{MB}}(x, x'') \leq d_u^{\mathrm{MB}}(x, x') + d_u^{\mathrm{MB}}(x', x'')$ (subadditivity)
- ~~$x' \neq x \Rightarrow d_u^{\mathrm{MB}}(x, x') > 0$ (positivity)~~

An important distance

- relying on function dynamics
  (so not a “classical” path-length distance)
- related to mathematical morphology!

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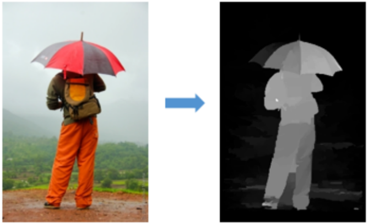
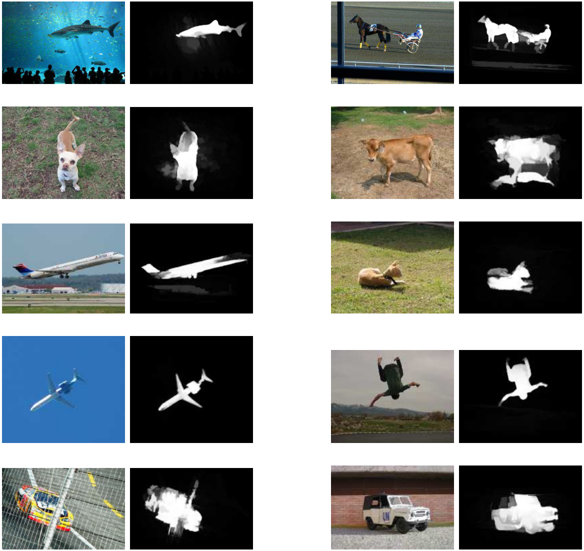
  (so not a “classical” path-length distance)
- related to mathematical morphology!
- effective for segmentation tasks...



Distance maps from the image border



References

R. Strand, K.C. Ciesielski, F. Malmberg, and P.K. Saha, "The minimum barrier distance," *Computer Vision and Image Understanding*, vol. 117, pp. 429-437, 2013. [PDF]

K.C. Ciesielski, R. Strand, F. Malmberg, and P.K. Saha, "Efficient Algorithm for Finding the Exact Minimum Barrier Distance," *Computer Vision and Image Understanding*, vol. 123, pp. 53–64, 2014. [PDF]

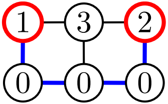
J. Zhang, S. Sclaroff, Z. Lin, X. Shen, B. Price, and R. Mech, "Minimum barrier salient object detection at 80 FPS," *in: Proc. of ICCV*, pp. 1404–1412, 2015. [PDF]

W.C. Tu, S. He, Q. Yang, and S.Y. Chien, "Real-time salient object detection with a minimum spanning tree," *in: Proc. of IEEE CVPR*, pp. 2334–2342, 2016. [PDF]

J. Zhang, S. Sclaroff, "Exploiting Surroundedness for Saliency Detection: A Boolean Map Approach," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 38, num. 5, pp. 889–902, 2016. [PDF]

The glitch!

In the graph world:



the MB distance is **2**

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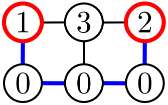

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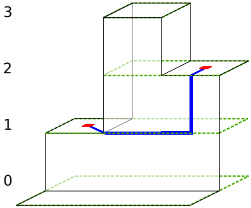
In the continuous world:



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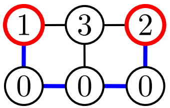

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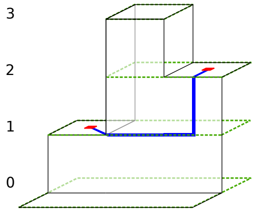

the MB distance should be **1**!

The glitch!

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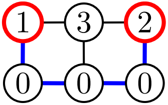

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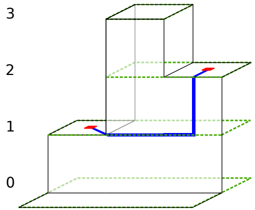
In the continuous world:



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⇒ **we need a new definition...**

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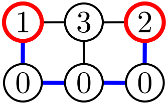

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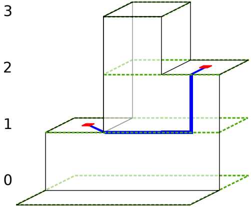
In the continuous world:



the MB distance should be **1**!

⇒ **we need a new definition...**

This talk is only about this definition and about its computation.

A $\approx$ new representation...

Given a scalar image $u : \mathbb{Z}^n \rightarrow Y$, we use two tools:

- cubical complexes: $\mathbb{Z}^n$ is replaced by $\mathbb{H}^n$
- set-valued maps: Y is replaced by $\mathbb{I}_Y$

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$\Rightarrow$ a *continuous* (and *discrete*!) representation of images

T. Géraud, E. Carlinet, S. Crozet, and L. Najman, "A quasi-linear algorithm to compute the tree of shapes of *n*-D images," *in: Proc. of ISMM*, LNCS, vol. 7883, pp. 98–110, Springer, 2013. [PDF]

L. Najman and T. Géraud, "Discrete set-valued continuity and interpolation," *in: Proc. of ISMM*, LNCS, vol. 7883, pp. 37–48, Springer, 2013. [PDF]

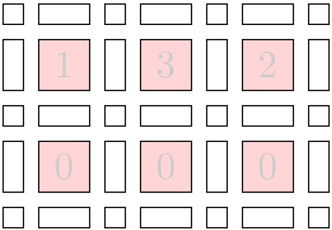
A both discrete and continuous representation

discrete point $x \in \mathbb{Z}^n$ $\rightsquigarrow$ n-face $h_x \in \mathbb{H}^n$

domain $\mathcal{D} \subset \mathbb{Z}^n$ $\rightsquigarrow$ $\mathcal{D}_H = cl(\{h_x;\ x \in \mathcal{D}\}) \subset \mathbb{H}^n$

1	3	2
0	0	0

from a scalar image u...



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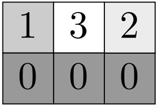
domain $\mathcal{D} \subset \mathbb{Z}^n$	$\rightsquigarrow$	$\mathcal{D}_H = cl(\{h_x;\ x \in \mathcal{D}\}) \subset \mathbb{H}^n$
scalar image $u : \mathcal{D} \subset \mathbb{Z}^n \to Y$	$\rightsquigarrow$	interval-valued map $\widetilde{u} : \mathcal{D}_H \subset \mathbb{H}^n \to \mathbb{I}_Y$



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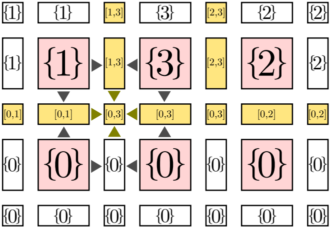

to an interval-valued image $\widetilde{u}$

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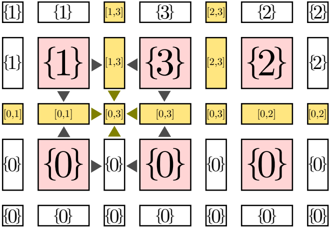


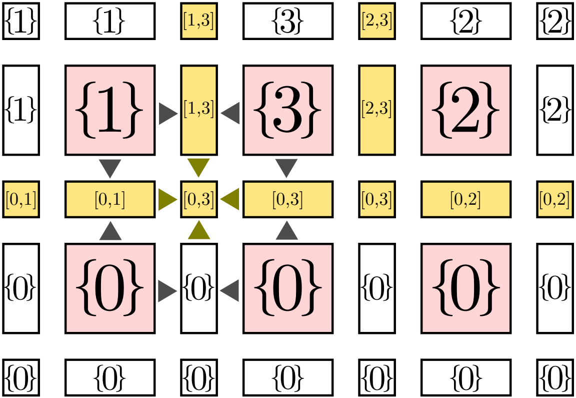
to an interval-valued image $\widetilde{u}$

We set:

$$\forall\, h \in \mathcal{D}_H,\ \ \widetilde{u}(h) \;=\; \mathrm{span}\{\, u(x);\ x \in \mathcal{D} \text{ and } h \subset h_x \,\}.$$

A both discrete and continuous representation

zoomed in:



$\widetilde{u}$

how huge!

A both discrete and continuous representation

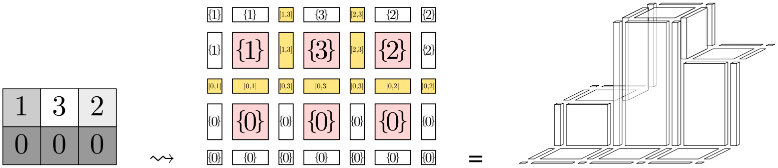


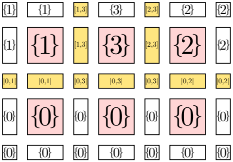
image u $\rightsquigarrow$ set-valued image $\widetilde{u}$ = $\widetilde{u}$ in 3D

A both discrete and continuous representation

1	3	2
0	0	0

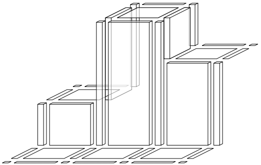
image u

$\rightsquigarrow$



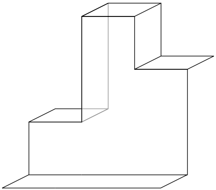
set-valued image $\widetilde{u}$

=



$\widetilde{u}$ in 3D

$\Updownarrow$



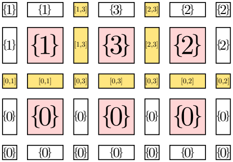
3D version of u in $\mathbb{R}^3$

A both discrete and continuous representation

1	3	2
0	0	0

image u

$\rightsquigarrow$



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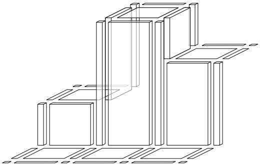
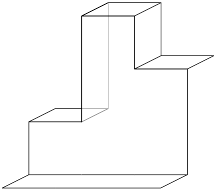


$\widetilde{u}$ in 3D

$\Updownarrow$

continuity! $\longrightarrow$



3D version of u in $\mathbb{R}^3$

A both discrete and continuous representation

we have a representation for the image surface

$\leadsto$ we want to express the “continuous” distance...

Notation

Inclusion

with u a scalar image, and U a set-valued image:

$$u \lessdot U \quad \Leftrightarrow \quad \forall\, x \in X,\;\; u(x) \in U(x)$$

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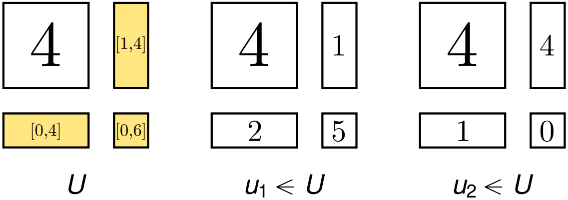
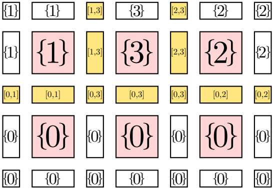
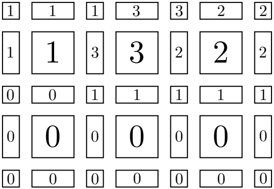


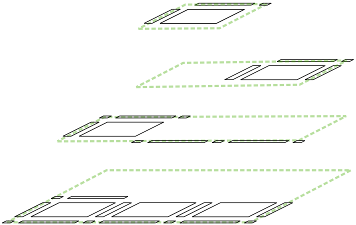
Finding the continuous MB distance



interval-valued image $\widetilde{u}$

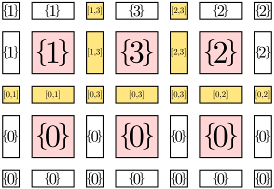


a scalar image $\overline{u} \lessdot \widetilde{u}$



3D version of $\overline{u}$

Finding the continuous MB distance



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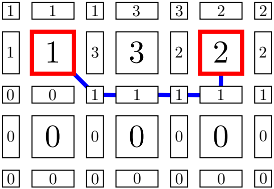
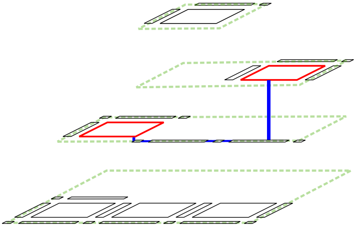
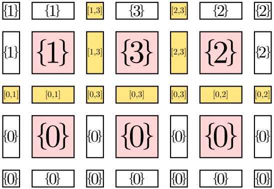


a minimal path in a $\overline{u} < \widetilde{u}$



its 3D version

Finding the continuous MB distance



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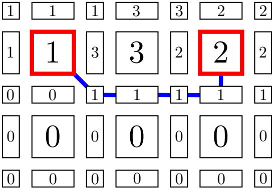

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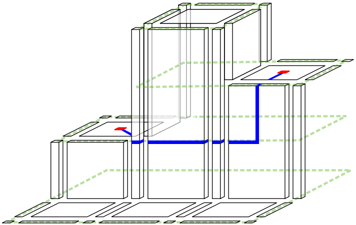

3D version of the path in $\widetilde{u}$

The naive Dahu distance

The "naive" Dahu distance:

$$D_u^{\text{naive}}(x, x') \;=\; \min_{\overline{u} \lessdot \widetilde{u}} \; \underbrace{\min_{\pi \in \Pi(h_x, h_{x'})} \Big(\overbrace{\max_{\pi_i \in \pi} \overline{u}(\pi_i) \;-\; \min_{\pi_i \in \pi} \overline{u}(\pi_i)}^{\text{barrier } \tau_{\overline{u}}(\pi)} \Big)}_{\text{minimum barrier distance } d_{\overline{u}}^{\mathrm{MB}}(h_x, h_{x'})}$$

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it looks like we have added an extra combinatorial complexity w.r.t. the original MB distance...

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...actually it can be computed **exactly** and **efficiently** with: the *morphological tree of shapes*!!!

The morphological tree of shapes (ToS)

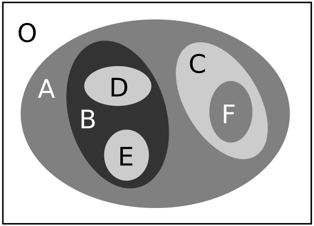
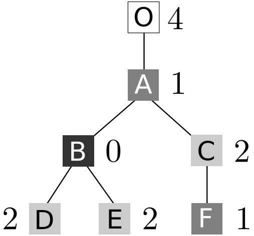


image u



its tree of shapes $\mathfrak{S}(u)$

this is a morphological representation of an image
based on the components of its level sets

The morphological tree of shapes (ToS)

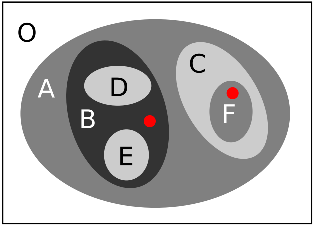
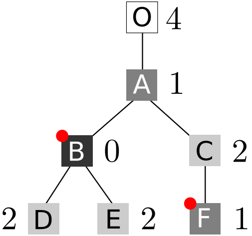


image u



its tree of shapes $\mathfrak{S}(u)$

let us consider a couple of points of the image:
each point belongs to a particular ToS node

The morphological tree of shapes (ToS)

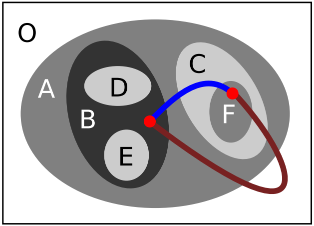
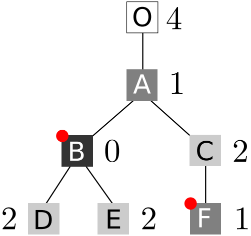


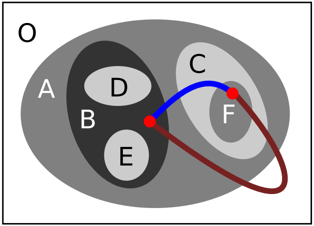
image u



its tree of shapes $\mathfrak{S}(u)$

finding a minimal path in the image is straightforward:
all paths **have to** go through regions A and C.

The morphological tree of shapes (ToS)



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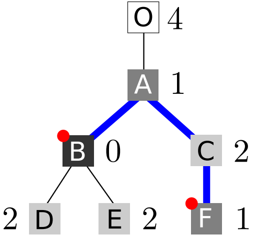
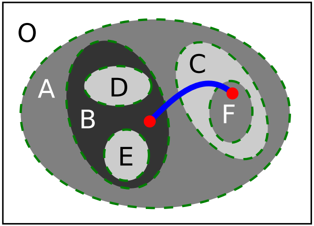


its tree of shapes $\mathfrak{S}(u)$

$\leadsto$ a minimal path *in the image* only goes through the minimal set of regions and it can be "**read**" on the ToS!

The morphological tree of shapes (ToS)



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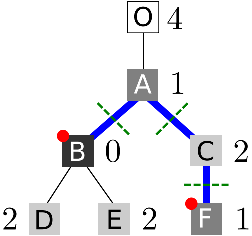

its tree of shapes $\mathfrak{S}(u)$

and this minimal path crosses the image level lines
(so they have to be well formed...)

Recap

We have a continuous-like definition of the MB distance
and it can be computed efficiently thanks to the tree of shapes

$\rightsquigarrow$ but we have to fix a digital topology issue
and to re-express the distance *on* the tree...

About digital topology

Digital topology implies:

- use of dual connectivities for object/background
- dual connectivities for lower/upper level sets $\Rightarrow$ the ToS exists

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Issues with *two* connectivities:

- it would be painful to consider paths [...]
- we would have some inconsistent results in distance computation [...]

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- dual connectivities for lower/upper level sets $\Rightarrow$ the ToS exists

Issues with *two* connectivities:

- it would be painful to consider paths [...]
- we would have some inconsistent results in distance computation [...]

An important class of images: *digitally well-composed* (DWC) images

- connectivities are equivalent for all components of level sets
- boundaries of level sets do not have pinches
- if an image is DWC $\Rightarrow$ its ToS and the level lines are well defined

T. Géraud, E. Carlinet, S. Crozet, "Self-Duality and Discrete Topology: Links Between the Morphological Tree of Shapes and Well-Composed Gray-Level Images," *in: Proc. of ISMM*, LNCS, vol. 9082, pp. 573–584, Springer, 2015. [PDF]

About digital topology

An image can be made DWC by subdivision + interpolation:

- using the median operator in 2D,
- using a non-local process in *n*D.

N. Boutry, T. Géraud, and L. Najman, "How to make *n*D functions well-composed in a self-dual way," *in: Proc. of ISMM*, LNCS, vol. 9082, pp. 561–572, Springer, 2015. [PDF]

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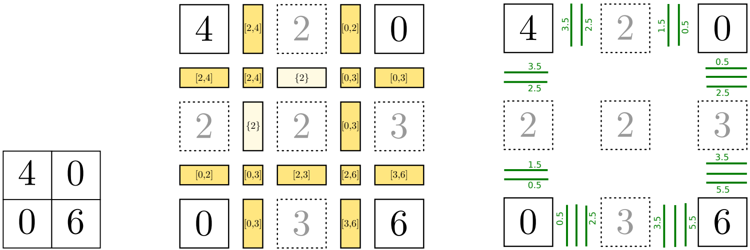

u $\widetilde{u}_{\text{med}}$ level lines of $\widetilde{u}_{\text{med}}$

what are the level lines?
(make the chunks connect...)

N. Boutry, T. Géraud, and L. Najman, "How to make *n*D functions well-composed in a self-dual way," *in: Proc. of ISMM*, LNCS, vol. 9082, pp. 561–572, Springer, 2015. [PDF]

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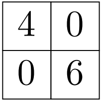
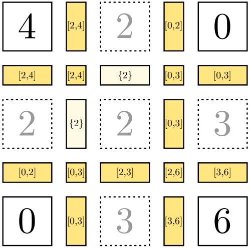
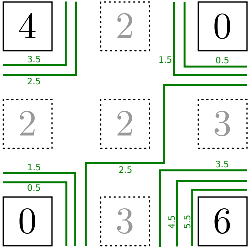


u



$\widetilde{u}_{med}$



level lines of $\widetilde{u}_{med}$

u_{med} is DWC $\Rightarrow$ there is only *one way* to arrange level lines (thus shapes) into an inclusion tree :−)

N. Boutry, T. Géraud, and L. Najman, "How to make *n*D functions well-composed in a self-dual way," *in: Proc. of ISMM*, LNCS, vol. 9082, pp. 561–572, Springer, 2015. [PDF]

A flawless definition

NAIVE definition of the Dahu distance:

$$D_u(x, x') \;=\; \min_{\overline{u} \lessdot \widetilde{u}} \; d^{\text{MB}}_{\overline{u}}(h_x, h_{x'})$$

A flawless definition

scalar image		DWC interpolated		interval-valued
$(u : \mathbb{Z}^n \to Y)$	$\xrightarrow{\text{step 1}}$	$(u_\square : (\frac{\mathbb{Z}}{2})^n \to Y')$	$\xrightarrow{\text{step 2}}$	$(\widetilde{u_\square} : (\frac{\mathbb{H}}{2})^n \to \mathbb{I}_{Y'})$

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actually, the interpolation does not introduce a bias in the distance values;

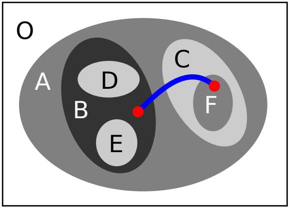
it just makes their definition and computation sound and consistent :-)

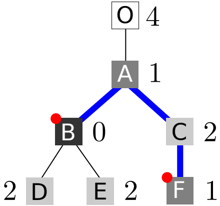
A flawless definition

we have a sound definition for a continuous-like distance

$\rightsquigarrow$ we now want to compute distances on $\mathfrak{S}(\widetilde{u_\square})$...

Mapping the Dahu distance on the tree





Notations:

t	node of a tree
t_x	node that corresponds to $x \in \mathbb{Z}^n$
$\text{parent}(t)$	the parent node of t in the tree
$\text{lca}(t, t')$	the lowest common ancestor of the nodes t and t'
$\mu(t)$	gray level of the node in the image

We have:

- $t_A = \text{lca}(t_B, t_F)$
- $\langle t_B, t_A, t_C, t_F \rangle$ is the "minimal" path on the tree for the two red points

Mapping the Dahu distance on the tree

The **NEW** definition of the Dahu distance becomes:

$$D_u(x, x') \;=\; \max_{t \,\in\, \pi_{\mathfrak{S}(u)}(t_x, t_{x'})} \mu(t) \;-\; \min_{t \,\in\, \pi_{\mathfrak{S}(u)}(t_x, t_{x'})} \mu(t)$$

Mapping the Dahu distance on the tree

The **NEW** definition of the Dahu distance becomes:

$$D_u(x, x') = \max_{t \in \pi_{\mathfrak{S}(u)}(t_x, t_{x'})} \mu(t) - \min_{t \in \pi_{\mathfrak{S}(u)}(t_x, t_{x'})} \mu(t)$$

The how-to:

1. pre-compute the ToS (...)
2. then get distances very efficiently for many couples (x, x').

E. Carlinet and T. Géraud, "A Comparative Review of Component Tree Computation Algorithms," *IEEE Transactions on Image Processing*, vol. 23, num. 9, pp. 3885–3895, 2014. [PDF]

Conclusion / Take-home messages

Reminder:

- the MB distance is **great for computer vision**!

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What we have done:

- introduce a new distance,
  that fits with a *continuous (yet discrete) representation* of images
- formalize it,
  and relate it to the morphological tree of shapes
- provide an efficient solution to compute distances.

Conclusion / Take-home messages

Reminder:

- the MB distance is **great for computer vision**!

What we have done:

- introduce a new distance,
  that fits with a *continuous (yet discrete) representation* of images
- formalize it,
  and relate it to the morphological tree of shapes
- provide an efficient solution to compute distances.

What we have skipped:

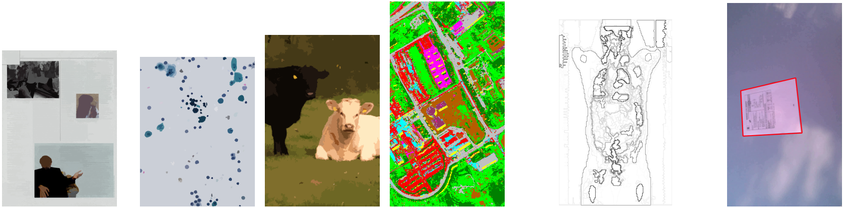
- actually many things...

A perspective:

- adapt the distance to color images ▶

using the multivariate tree of shapes (MToS)...

grain-like filtering | shaping | simplification | classification | saliency | obj. detection



E. Carlinet and T. Géraud, "MToS: A tree of shapes for multivariate images," *IEEE Transactions on Image Processing*, vol. 24, num. 12, pp. 5330–5342, 2015. [PDF]

That's all folks!

Thanks for your attention. Any questions?

Dahu descentius frontalis
(La Pointe Perce, 1895)

Dahu ascentius frontalis
(Le Charvin, 1901)

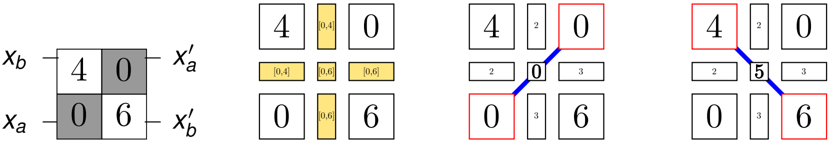
Dahu dextrogyre
(Col de la Colombire, 1904)

Young dahu lévogyre
(La Tournette, 1910)

[BACKUP SLIDE]

...

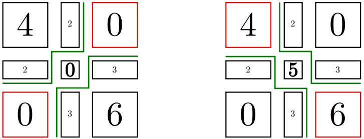
[BACKUP SLIDE] The issue with Digital Topology



u $\quad$ $\widetilde{u}$ $\quad$ $\overline{u}_a < \widetilde{u}$ $\quad$ $\overline{u}_b < \widetilde{u}$

$D_u^{\text{naive}}(x_a, x'_a) = 0$ $\quad$ $D_u^{\text{naive}}(x_b, x'_b) = 2$

this saddle case in 2D is a symptom of a discrete topology issue with $\widetilde{u}$



level lines $\lambda = 0.5$ $\quad$ level lines $\lambda = 3.5$

The morphological tree of shapes (ToS)

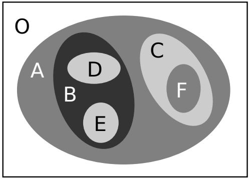
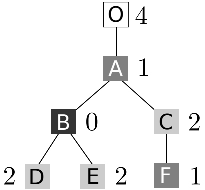


image u



its tree of shapes $\mathfrak{S}(u)$

- lowel level sets: $[u < \lambda] \;=\; \{\, x \in X;\; u(x) < \lambda \,\}$
- upper level sets: $[u \geq \lambda] \;=\; \{\, x \in X;\; u(x) \geq \lambda \,\}$
- tree of shapes: $\mathfrak{S}(u) \;=\; \{\, \mathrm{Sat}(\Gamma);\; \Gamma \in \mathcal{CC}([u < \lambda]) \cup \mathcal{CC}([u \geq \lambda]) \,\}_\lambda$

  an element of $\mathfrak{S}(u)$ is a shape of u
- level lines: $\{\, \partial\Gamma;\; \Gamma \in \mathfrak{S}(u) \,\}$

  if u is a well-composed image, level lines are Jordan curves
- level of a line: μ

  indicated on the tree, for every node

[BACKUP SLIDE] Cubical complex

The nD space of cubical complexes:

$$H_0^1 = \{\{a\};\ a \in \mathbb{Z}\} \qquad H_1^1 = \{\{a, a+1\};\ a \in \mathbb{Z}\}$$

$$\mathbb{H}^1 = H_0^1 \cup H_1^1 \qquad \mathbb{H}^n = \times_n H^1$$

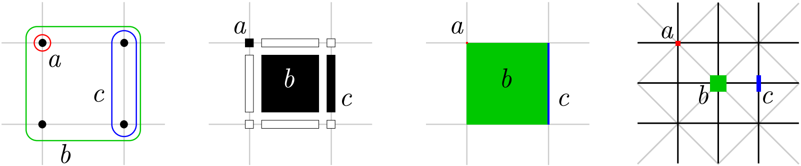
$h \in \mathbb{H}^n$: $\times$ product of d elements of H_1^1 and $n - d$ elements of H_0^1

- we have $h \subset \mathbb{Z}^n$
- h is a d-face
- d is the dimension of h

[BACKUP SLIDE] Cubical complex

Three faces of $\mathbb{H}^2$:

$a = \{0\} \times \{1\}$	0-face	closed
$b = \{0, 1\} \times \{0, 1\}$	2-face	open
$c = \{1\} \times \{0, 1\}$	1-face	clopen



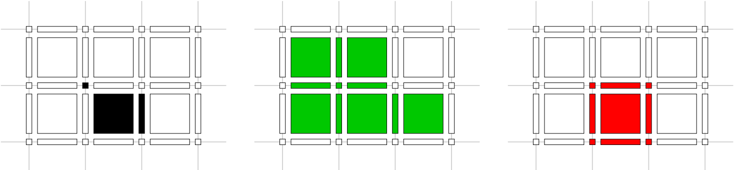
subsets of $\mathbb{Z}^2$ | elements of the cellular complex | geometrical objects (parts of $\mathbb{R}^2$) | vertices of the Khalimsky grid

Cubical complex

With $h^{\uparrow} = \{ h' \in \mathbb{H}^n \mid h \subseteq h' \}$ and $h^{\downarrow} = \{ h' \in \mathbb{H}^n \mid h' \subseteq h \}$:

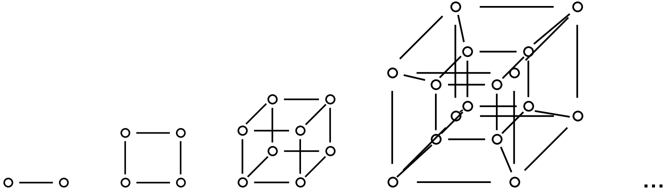
- $(\mathbb{H}^n, \subseteq)$
  is a poset,
- $\mathcal{U} = \{ U \subseteq \mathbb{H}^n \mid \forall h \in U,\ h^{\uparrow} \subseteq U \}$
  is a T0-Alexandroff topology on $\mathbb{H}^n$.

Topological operators:

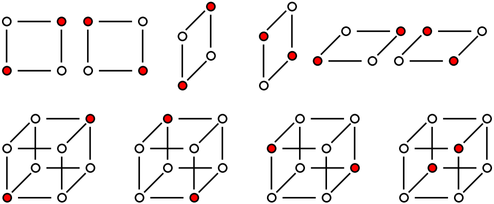


$E = \{ a, b, c \}$ — star: $E^{\uparrow}$ — closure: $E^{\downarrow}$

*n*D blocks:

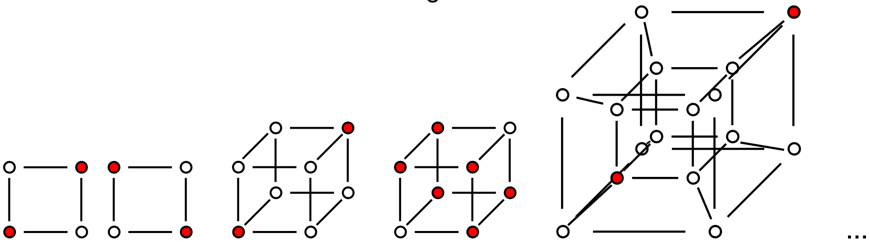


Antagonists in 3D:



DWC images

Critical configurations:

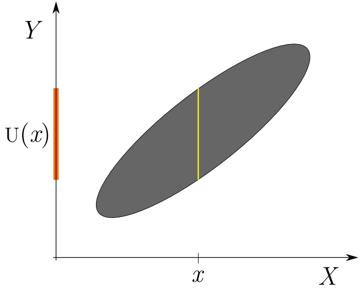


- A digital set $S \subset \mathbb{Z}^n$ is *digitally well-composed* (DWC) iff it does not contain any critical configuration
- A digital image $u : \mathbb{Z}^n \to Y$ is DWC iff its levels sets are DWC

Set-valued analysis

A set-valued map $U : X \to \mathcal{P}(Y)$ is characterized by its graph:

$$\mathrm{Gra}(U) \;=\; \{\, (x, y) \in X \times Y \mid y \in U(x) \,\}.$$



Set-valued analysis

Continuity:

- when $U(x)$ is compact, U is USC at x if

  $\forall \varepsilon > 0, \exists \eta > 0$ such that $\forall x' \in B_X(x, \eta),\ U(x') \subset B_Y(U(x), \varepsilon)$.

- U is USC iif $\forall x \in X$, U is USC at x
- this is the "natural" extension of the *continuity* of a scalar function.

Inverse:

the *core* of $M \subset Y$ by U is $U^{\ominus}(M) = \{ x \in X \mid U(x) \subset M \}$

A continuity characterization:

U is USC iff *the core of any open subset is open.*

Set-valued thresholds

Threshold sets:

$$[\,U \lhd \lambda\,] \;=\; \{\, x \in X \mid \forall\, \mu \in U(x),\; \mu < \lambda \,\}$$

$$[\,U \rhd \lambda\,] \;=\; \{\, x \in X \mid \forall\, \mu \in U(x),\; \mu > \lambda \,\}$$

The “large” versions:

$$\begin{aligned}[\,U \unlhd \lambda\,] &= X \setminus [\,U \rhd \lambda\,] \\ &= \{\, x \in X \mid \exists\, \mu \in U(x),\; \mu \leq \lambda \,\}\end{aligned}$$

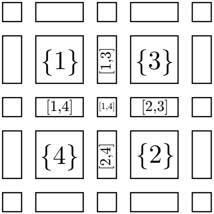
$$\begin{aligned}[\,U \unrhd \lambda\,] &= X \setminus [\,U \lhd \lambda\,] \\ &= \{\, x \in X \mid \exists\, \mu \in U(x),\; \mu \geq \lambda \,\}\end{aligned}$$

Iso-set:

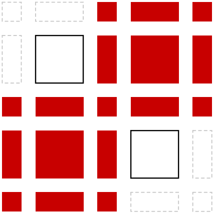
$$\begin{aligned}[\,U \,\square\, \lambda\,] &= [\,U \unlhd \lambda\,] \cap [\,U \unrhd \lambda\,] \\ &= \{\, x \in X \mid \lambda \in U(x) \,\}\end{aligned}$$

T. Géraud, E. Carlinet, S. Crozet, and L. Najman, “A quasi-linear algorithm to compute the tree of shapes of n-D images,” *in: Proc. of ISMM*, LNCS, vol. 7883, pp. 98–110, Springer, 2013. [PDF]

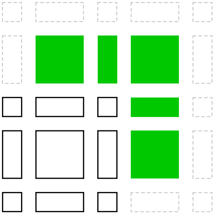
[BACKUP SLIDE] Set-valued thresholds



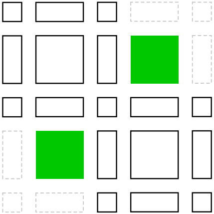
U



$[\,U \trianglerighteq 3\,] \;=\; cl(\,[\,U \triangleright 3-\iota\,]\,)$



$[\,U \triangleleft 4\,]$



$[\,U \triangleright 3-\iota\,]$

ToS of set-valued maps

dual trees:

$$\mathcal{T}_{\triangleleft}(U) \;=\; \{\, \Gamma \in \mathcal{CC}([\, U \triangleleft \lambda\,])\,\}_{\lambda} \quad \text{(min-tree)}$$

$$\mathcal{T}_{\triangleright}(U) \;=\; \{\, \Gamma \in \mathcal{CC}([\, U \triangleright \lambda\,])\,\}_{\lambda} \quad \text{(max-tree)}$$

shapes:

$$\mathcal{S}_{\triangleleft}(U) \;=\; \{\, \mathrm{Sat}(\Gamma);\; \Gamma \in \mathcal{T}_{\triangleleft}(U)\,\} \quad \text{(lower)}$$

$$\mathcal{S}_{\triangleright}(U) \;=\; \{\, \mathrm{Sat}(\Gamma);\; \Gamma \in \mathcal{T}_{\triangleright}(U)\,\} \quad \text{(upper)}$$

tree of shapes:

$$\mathfrak{S}(U) \;=\; \mathcal{S}_{\triangleleft}(U) \,\cup\, \mathcal{S}_{\triangleright}(U)$$

A well-defined ToS

If $u_{\square}$ is DWC then $\mathfrak{S}(u_{\square})$ is well defined.

New definition of the ToS of scalar functions

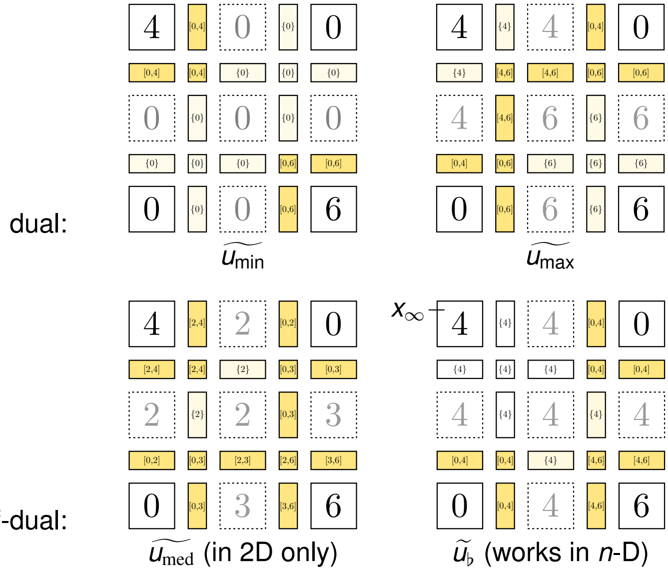
$$\mathfrak{S}^{\text{NEW}}(u) \;:=\; \mathfrak{S}(u_{\square})\,|_{\mathbb{Z}^n} \;\subset\; \mathfrak{S}(\widetilde{u_{\square}})\,|_{\mathbb{H}^n_n}$$

where $\mathbb{H}^n_n = \times_n H^1_1 \subset \mathbb{H}^n$ is the set of n-faces

A consequence:

- CCs of shape boundaries are continuous discrete manifold
- in 2D, they are Jordan curves.

Some well-composed representations



Some extra references

S. Crozet and T. Géraud, "A first parallel algorithm to compute the morphological tree of shapes of *n*D images," *in: Proc. of ICIP*, pp. 2933–2937, 2014. [PDF]

Y. Xu, T. Géraud, and L. Najman, "Connected filtering on tree-based shape-spaces," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 38, num. 6, pp. 1126–1140, 2016. [PDF]